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Stochastic models of tumour development and related mesoscopic equations

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Abstract. We consider different mathematical models inspired by the problems of medicine, in particular, the tumour growth and the related topics. We demonstrate how to starting from an individual-based (microscopic) description, which characterizes cells' behaviour, derive the so-called kinetic (mesoscopic) equations, which describe the approximate system density. Properties of the solutions to the mesoscopic equations (in particular, their long-time behaviour) reflect statistical characteristics of the whole system and demonstrate the corresponding dependence on the system parameters.

1 Introduction

1.1 Mathematical description

Within the microscopic description of cells, the framework of interacting particle systems in continuum and their possibility of deriving rigorously a kinetic description, also called mesoscopic description, in terms of non-local and in general non-linear equations plays a crucial role. Here we start from some (simple) stochastic microscopic (heuristic) description of a cell model and derive from that rigorously the kinetic equations for the density of this system. Such approach can be interpreted similarly to the mean field limit in Physics, where one scales the dynamics in a proper way and obtains from that in the limit a deterministic equation for the density of the system. We assume in general, that each cell is determined uniquely by its position and no other properties will be tracked. Note that it is also possible to introduce marks within such de-

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scriptions and extend the microscopic stochastic dynamics to a situation, where more complicated effects may be covered. For instance one can introduce age-dependence of cells, i.e. each cell has an individual age, which influences the microscopic interactions. Here we always assume that the number of cell-types is small compared to the number of cells within each type. Therefore such stochastic treatment is adequate.

The main difference to other cell biological models is, that we start with a stochastic description, which incorporates individual cell behaviour stochastically. The choice of the individual stochastic behaviour incorporates cell intern effects and can be used to model a wide class of cells. Ignoring the internal structure necessary leads to randomness, but also leads to new methods describing such large interacting systems.

Heuristically the evolution of a system is described via its elementary Markov events like birth, death and jumps of cells. In this framework the evolution is assumed to be Markovian, which is a reasonable approximation of reality. Note, that this approach could also be extended to non-Markovian structures in order to include dependencies on cell intern processes like aging or may lead to some sort of cell-memory. Nevertheless assuming Markovian behaviour already leads to many non-trivial examples and effects, which have to be studied more intensively.

The microscopic description and its analysis can answer questions about asymptotic clustering of the system, invariant states and ergodicity of the system. The precise formulation of clustering will be explained more extended in the next part of the article. Usually it is not possible to measure the microscopic quantities to full extend, so in order to have a practically useful description it is necessary to describe the system also via mesoscopic or even macroscopic quantities. Thus it is reasonable to seek for an effective description with practically measurable quantities. Similar to Thermodynamic limits, one tries to rescale the system and obtain another description, here for the density of the system. As a consequence the new description will not contain all information about the microscopic behaviour, for instance it does not contain information about asymptotic clustering, which will be explained lateron, and individual trajectories of the Markov process. Such Mesoscopic, i.e. kinetic description, describes instead of microscopic quantities the density of the system via a closed system of equations. Typical for such systems of equations is their non-linear structure and the appearance of convolutions of the density with the potentials involved in the interactions of cells. The analysis of such equations is a topic of applied mathematics and is studied intensively since the last 30 years, c.f. [10, 12, 20].

This kinetic description can give information about the long-time behaviour, invariant and stationary states, asymptotic speed of growth, front wave propagation and several other effects. Its analysis should be realized separately for each model. Let us outline the general approach and motivate the scaling used to derive the kinetic description. In general one suppresses the interactions of cells via a factor $\varepsilon > 0$. In the same way the density of the system is increased. Such attempt will suppress all correlations between the cells within the system and therefore a kinetic description will not contain this information. In the last step we will seek for a limiting description of this system and will arrive in a reduced description of the microscopic model. This reduced description will have not Markovian structure but is still a linear stochastic description involving infinitely many correlation functions. To get a closed equation for the density of the system, remember that all correlations between the cells are suppressed. Thus it is not surprisingly that starting from poissonan statistics, the evolution of the system will preserve this statistics. This property is know as the Chaos preservation principle.

In the following we will first outline a more detailed description of both approaches, introduce all necessary quantities and afterwards state the results for several biological important models of tumour growth, cell division, mortality etc. The last section contains all mathematical details, which are necessary for the analysis of such models.

The aim of this section is to motivate and explain this approach to scientists working in biological or medical research fields. The precise mathematical description will be given and proved afterwards separately.

1.2 One-component models

Let us first outline the necessary structures in the simpler case, where we consider only one type of cells. Since the cells are distinguished only by their positions, we will denote their positions by $x_1, \dots, x_n, \dots \in \mathbb{R}^d$ or more simple as a collection of positions $\gamma = \{x_1, \dots, x_n, \dots\} \subset \mathbb{R}^d$. In reality it is clear, that each organism has only a finite but very large number of cells. For such finite microscopic systems the existence of a Markov process is known. Moreover in [2], [3] asymptotic properties and conditions for explosions respectively non-explosion can be stated. Nevertheless it is still not understood how to derive rigorously, i.e. in the sense of convergence of the corresponding densities, the mesoscopic description. In contrast to infinite systems, i.e. $\gamma \subset \mathbb{R}^d$ contains infinitely many points, behave from the analytical point of view quite different. Here for many models it is already known that the density of the rescaled system will converge to the solution of the kinetic description. In this work we will mainly focus on infinite systems having in mind, that in the kinetic description the initial density should in addition be chosen to be integrable, and hence represents a system consisting only of finitely many cells.

Similar to limits from thermodynamics, some effects like asymptotic clustering or pattern formation can be captured simpler in the limit of infinite particles. Simulations suggest and for some dynamical systems it can be shown, that finite systems with a large number of cells behave in their interior like infinite systems. Finite systems can describe the growth of the system, whereas infinite systems capture the properties of the interior behaviour of cell patterns and their properties. Since we deal with a very large number of cells ($\approx 10^{10}$) it is justified to allow the cell number to be even infinity, so we will use a description which includes both finite and infinite systems.

In this case we have to assume, that locally the number of cells is still finite, i.e. for every bounded volume $\Lambda \subset \mathbb{R}^d$ the number of cells within Λ is finite: $|\gamma \cap \Lambda| < \infty$. This assumption implies, that the local density of the system (also other observables) are locally finite and thus can be measured/observed on each finite volume. Altogether our phase space (configuration space) is

$$\Gamma = \{\gamma \subset \mathbb{R}^d : \quad |\gamma \cap \Lambda| < \infty, \quad \forall \Lambda \subset \mathbb{R}^d \text{ bounded}\}.$$

Clearly the space of finite configurations

$$\Gamma_0 = \{\eta \subset \mathbb{R}^d : \quad |\eta| < \infty\}$$

is a subspace of Γ, i.e. $\Gamma_0 \subset \Gamma$. Heuristically, starting from some configuration $\gamma \in \Gamma$ we would like to describe a Markov process $X_t^\gamma \in \Gamma$ starting at γ, which incorporates all microscopic phenomena we would like to describe. For finite configurations such problem is well understood, c.f. [11] and references therein. The (Markov) dynamics is described via elementary events as birth, death and jumps of cells. A cell located at $x \in \mathbb{R}^d$ can die, i.e. the configuration changes as $\gamma \to \gamma\backslash x$, a cell can jump from x to $y \in \mathbb{R}^d$, i.e. $\gamma \to \gamma\backslash x \cup y$ and finally a new cell at location $y \in \mathbb{R}^d$ may appear, i.e. $\gamma \to \gamma \cup y$. All such events have certain intensities, which will depend on the positions x, y and on the configuration of cells γ. The probability of the new location $y \in \mathbb{R}^d$ is usually described via a probability distribution.

Mathematically a Markov process X_t^γ starting from a configuration of cells $\gamma \in \Gamma$ can be described completely in terms of the corresponding Markov generator L, c.f [18], which acts on functions F called observables. Therefore in order to describe the model it is enough to write down the expression for this Markov operator. For our models all terms contained in the operator have a simple interpretation, e.g. the general form of such generator is simply

$$(LF)(\gamma) = \sum_{x\in\gamma} d(x,\gamma\backslash x)(F(\gamma\backslash x) - F(\gamma)) + \int_{\mathbb{R}^d} b(y,\gamma)(F(\gamma \cup y) - F(\gamma))dy$$
$$+ \sum_{x\in\gamma} \int_{\mathbb{R}^d} c(x,y,\gamma\backslash x)(F(\gamma\backslash x \cup y) - F(\gamma))dy. \tag{1}$$

Here $0 \le d(x,\gamma\backslash x)$ is the intensity of death of a cell $x \in \mathbb{R}^d$ depending on all other cells $\gamma\backslash x$, $0 \le b(y,\gamma)$ is the intensity, that a cell is born at position $y \in \mathbb{R}^d$ and $0 \le c(x,y,\gamma\backslash x)$ is the intensity that a cell jumps from position x to the new position $y \in \mathbb{R}^d$. Let us stress, that since we will deal with infinite systems the study of the operator L is extremely hard and was carried out only for a few models, e.g. [16]. In the framework of cell biology typically new cells can only be born due to proliferation and hence we can specify the intensity $b(x,\gamma)$ to be of the form

$$b(x,\gamma) = \sum_{y\in\gamma} b_0(x,y,\gamma\backslash x).$$

This means that each cell at $y \in \gamma$ may split and therefore create a new cell at location $x \in \mathbb{R}^d$. The intensity of such events is given by $b_0(x,y,\gamma\backslash x)$. Put in other words, if $\gamma \in \Gamma_0$ then heuristically

$$\mathbb{P}(X_{t+h}^\gamma = X_t^\gamma \cup \{x\} | X_t^\gamma) = b(x,\gamma)h + o(h)$$

as $h \to 0$. Similar statements hold for $d(x,\gamma\backslash x)$ and $c(x,y,\gamma\backslash x)$. In the case of $|\gamma| = \infty$ such description can be interpreted only heuristically, since in each interval $[t, t+h]$, $h > 0$ infinitely many microscopic events will take place. Hence the notion of first time of a change of a system state if not meaningful, whereas in finite systems an explicit construction of the Markov process deeply uses this fact.

Within this framework we will study distributions $\mathbb{P}(X_t^\gamma)^{-1} =: \mu_t$ called states of the system, instead of the process itself. From cell-biological point of view it is not necessary and realistic to know all positions of cells, but one can observe and model statistics respectively their distribution μ_t, which is probability measure on Γ. One simple example is the poissonian statistics. There the probability of finding n-cells within the volume $\Lambda \subset \mathbb{R}^d$ is given by

$$\mathbb{P}_n(\Lambda) = \frac{1}{n!}\Big(\int\limits_\Lambda \rho(x)dx\Big)^n \exp\Big(-\int\limits_\Lambda \rho(x)dx\Big),$$

where $0 \le \rho$ is locally integrable and describes the cell-density. Let us denote the Poisson measure on Γ by π_ρ and the collection of all probability measures on Γ by $\mathscr{P}$. The Poisson measure plays the role of a chaotic, i.e. free state, where all cells are not correlated. Starting from a state $\mu \in \mathscr{P}$, the description of the microscopic evolution will consist of describing the evolution of states $t \longmapsto \mu_t \in \mathscr{P}$. Compared to the description via a process X_t^μ the evolution of statistics μ_t is connected via the equality

$$\int\limits_\Gamma F(\gamma)\mu_t(d\gamma) = \int\limits_\Omega F(X_t^\mu)d\mathbb{P}, \quad F:\Gamma \longrightarrow \mathbb{R}$$

where Ω is the probability space and $\mathbb{P}$ the probability measure for the process X_t^μ starting with initial distribution $\mu\mathscr{P}$. The study of the evolution μ_t can be done via studying its moments, which are functions of arbitrary large number of variables. The definition of this functions, if they exist, is given as follows, c.f. [13]

$$\int\limits_\Gamma \sum_{\{x_1,\ldots,x_n\}\subset\gamma} f^{(n)}(x_1,\ldots,x_n)d\mu(\gamma)$$
$$= \frac{1}{n!}\int\limits_{(\mathbb{R}^d)^n} f^{(n)}(x_1,\ldots,x_n)k^{(n)}(x_1,\ldots,x_n)dx_1\ldots dx_n \quad (2)$$

for symmetric functions $f^{(n)}$, which are measurable and have compact support. The left-hand side is the mean of the observable $f^{(n)}$, i.e. we sum over all possible n-point configurations $\{x_1,\ldots,x_n\} \subset \gamma$ and afterwards integrate over all possible configurations γ. We assume that this mean can be represented via a density $k^{(n)}$ and the factor $\frac{1}{n!}$ is a combinatorial factor describing the number of all possible choices to order the positions $x_1,\ldots,x_n$. Let us denote the collection of all correlation functions by $(k^{(n)})_{n=0}^\infty = k$, where $0 \le k = k(\eta)$ is a function of finite configurations $\eta \subset \mathbb{R}^d$ $(|\eta| < \infty)$.

Example 1.

- In the case $n = 0$ one has

$$1 = \mu(\Gamma) = \int\limits_\Gamma d\mu(\gamma) = k^{(0)}$$

- For $n = 1$ take a Borel set $A \subset \mathbb{R}^d$ and

$$f^{(1)}(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}.$$

  Then we obtain

$$\int_\Gamma |A \cap \gamma| d\mu(\gamma) = \int_\Gamma \sum_{x \in \gamma} f^{(1)}(x) d\mu(\gamma) = \int_A k^{(1)}(x) dx$$

  and the left-hand side is the expected number of particles within the volume A, whereas the right-hand side is a measure in A. Therefore $k^{(1)}$ is the particle density of the system.

- The same procedure with

$$f^{(2)}(x, y) = \begin{cases} 1, & x, y \in A \\ 0, & \textit{otherwise} \end{cases}$$

  leads to

$$\frac{1}{2} \int_\Gamma |\gamma \cap A|^2 d\mu(\gamma) - \frac{1}{2} \int_\Gamma |\gamma \cap A| d\mu(\gamma)$$
$$= \int_\Gamma \binom{|\gamma \cap A|}{2} d\mu(\gamma) = \frac{1}{2} \int_A \int_A k^{(2)}(x, y) dx dy$$

  and we see that

$$\int_A \int_A k^{(2)}(x, y) dx dy$$

  is the Variance of the cell number operator with kernel $0 \leq k^{(2)}(x, y)$. Similarly $k^{(n)}$ describe higher order moments of the system.

- The correlation functions for the Poisson measure π_ρ are

$$k^{(n)}(x_1, \ldots, x_n) = \rho(x_1) \cdots \rho(x_n).$$

The evolution of states $t \longmapsto \mu_t$ is determined as the solution to the Fokker-Planck equation for measures

$$\frac{\partial}{\partial t} \int_\Gamma F(\gamma) d\mu_t(\gamma) = \int_\Gamma (LF)(\gamma) d\mu_t(\gamma) \tag{3}$$

with initial distribution $\mu_t|_{t=0} = \mu_0$. Let us assume that for each state μ_t the correlation function of arbitrary order $n \in \mathbb{N}$ exists, then the evolution μ_t can be described by the collection of all such correlation functions $k_t = (k_t^{(n)})_{n=0}^\infty$. Similar to equation (3), this collection will satisfy the Fokker–Planck hierarchy

$$\frac{\partial k_t}{\partial t} = L^\Delta k_t \tag{4}$$

or written in components

$$\frac{\partial k_t^{(n)}}{\partial t}(x_1,\dots,x_n) = (L^{\Delta}k_t)^{(n)}(x_1,\dots,x_n), \quad n \in \mathbb{N}_0,$$

where the operator L^{Δ} acts on the whole vector $k_t = (k_t^{(n)})_{n=0}^{\infty}$, i.e. on each correlation function and can be seen as an infinite matrix. Therefore above equation is a vector equation with the matrix operator L^{Δ}. We have therefore transformed the equation for the evolution of states μ_t to an equation for its moments $(k_t^{(n)})_{n=0}^{\infty}$. As a consequence the system of equations for $(k_t^{(n)})_{n=0}^{\infty}$ will be coupled, hence it is not possible to obtain directly a closed equation for $k_t^{(n)}$, where only $k_t^{(n)}$ enters. Attempts to derive from such system a closed equation are known as moment closure techniques. In our approach scaling of the system yields a closed equation for the density of the system. Let us show for a special choice of L how to derive this equation for $k_t^{(1)}$. The general case, will be postponed to the second part of the article.

As a simple example let us look at a free branching process, where each cell has a random exponentially distributed lifetime with parameter $m > 0$ and can proliferate with intensity $\lambda > 0$, i.e the time to create a new cell is also exponentially distributed with parameter $\lambda > 0$. The position of the new cell born from $x \in \gamma$ will be distributed due to the probability distribution $a(x-y)dy$, where $y \in \mathbb{R}^d$ is the position of the new cell. The heuristic Markov generator will have the form

$$\begin{aligned}(LF)(\gamma) = {} & m\sum_{x\in\gamma}(F(\gamma\backslash x) - F(\gamma)) \\ & + \lambda\sum_{x\in\gamma}\int_{\mathbb{R}^d} a(x-y)(F(\gamma\cup y) - F(\gamma))dy. \end{aligned} \tag{5}$$

Let us take in (3) the special choice $F(\gamma) = \sum_{x\in\gamma}\varphi(x)$ with a measurable, bounded function φ with compact support. For this choice the left-hand side of (3) will become, c.f. example 2,

$$\frac{\partial}{\partial t}\int_{\Gamma}\sum_{x\in\gamma}\varphi(x)d\mu_t(\gamma) = \frac{\partial}{\partial t}\int_{\mathbb{R}^d}\varphi(x)k_t^{(1)}(x)dx.$$

A short combinatorial computation shows the equality

$$(LF)(\gamma) = -m\sum_{x\in\gamma}\varphi(x) + \lambda\sum_{x\in\gamma}\int_{\mathbb{R}^d} a(x-y)\varphi(y)dy$$

and thus

$$\begin{aligned}\int_{\Gamma}(LF)(\gamma)d\mu_t(\gamma) &= -m\int_{\Gamma}\sum_{x\in\gamma}\varphi(x)d\mu_t(\gamma) + \lambda\int_{\Gamma}\sum_{x\in\gamma}(a*\varphi)(x)d\mu_t(\gamma) \\ &= -m\int_{\mathbb{R}^d}\varphi(x)k_t^{(1)}(x)dx + \lambda\int_{\mathbb{R}^d}(a*\varphi)(x)k^{(1)}(x)dx.\end{aligned}$$

For the second integral we use Fubini and $a(x-y)=a(y-x)$ to get

$$\int_{\mathbb{R}^d}(a*\varphi)(x)k^{(1)}(x)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x-y)\varphi(y)k^{(1)}(x)dydx$$
$$= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} a(y-x)k^{(1)}(x)dx\Big)\varphi(y)dy.$$

Altogether this gives

$$\int_{\mathbb{R}^d}\varphi(x)\frac{\partial k_t^{(1)}}{\partial t}(x)dx = \int_{\mathbb{R}^d}\varphi(x)\left(-mk_t^{(1)}(x)+\lambda(a*k_t^{(1)})(x)\right)dx$$

for each function φ and thus

$$\frac{\partial k_t^{(1)}}{\partial t}(x) = -mk_t^{(1)}+\lambda(a*k_t)(x).$$

Note, that this is a closed equation, in general it not the case, e.g. the equation for $k_t^{(2)}$

$$\frac{\partial k_t^{(2)}}{\partial t}(x,y) = -2mk_t^{(2)}(x,y)+\lambda\int_{\mathbb{R}^d} a(x-z)k_t^{(2)}(x,z)dz$$
$$+\lambda\int_{\mathbb{R}^d} a(y-z)k_t^{(2)}(z,y)dz + a(x-y)\left(k_t^{(1)}(x)+k_t^{(1)}(y)\right)$$

does depend on the functions of order 1 and 2.

Let us now turn to scaling and outline the general approach. The first step is to scale the intensities of the interaction of the system, usually one dumps the potentials by a factor $\varepsilon>0$, e.g. for (5) this means $a\to\varepsilon a$. In general let us assume we scaled the intensities in a proper way, i.e. have expressions $d_\varepsilon, b_\varepsilon$ and c_ε within expression (1). The exact scaling will be carried out for each model separately. To get a limit, we have also accelerate birth by putting a factor $\frac{1}{\varepsilon}$ in front of it, so in the case of (5) this will mean that L is not changed, which is a direct consequence of the independence of the stochastic evolution of each cell. Finally the resulting generator has the form

$$(L_\varepsilon F)(\gamma) = \sum_{x\in\gamma} d_\varepsilon(x,\gamma\backslash x)(F(\gamma\backslash x)-F(\gamma)) + \frac{1}{\varepsilon}\int_{\mathbb{R}^d} b_\varepsilon(y,\gamma)(F(\gamma\cup y)-F(\gamma))dy$$
$$+\sum_{x\in\gamma}\int_{\mathbb{R}^d} c_\varepsilon(x,y,\gamma\backslash x)(F(\gamma\backslash x\cup y)-F(\gamma))dy.$$

The second step is to increase the density of the system, i.e. we consider initial conditions $k_{0,\varepsilon}^{(n)}$, which satisfy

$$\varepsilon^n k_{0,\varepsilon}^{(n)} \to r_0^{(n)}, \quad \varepsilon\to 0$$

with a symmetric function $r_0^{(n)}$ and $n \in \mathbb{N}_0$. Clearly, this implies that the initial condition $k_{0,\varepsilon}^{(n)}$ has a singularity at $\varepsilon > 0$, which can be interpreted as a growth of the initial densities in $\varepsilon > 0$. The functions $r_0^{(n)}$ are a subject of choice for concrete models. The important case is

$$r_0^{(n)}(x_1, \ldots, x_n) = \rho_0(x_1) \cdots \rho_0(x_n), \tag{6}$$

which represents the poissonian statistics. Denote by L_ε^Δ the operator corresponding to L_ε and by $k_{t,\varepsilon}$ the collection of correlation functions, defined as the solutions of the equation

$$\frac{\partial k_{t,\varepsilon}}{\partial t} = L_\varepsilon^\Delta k_{t,\varepsilon}, \quad k_{t,\varepsilon}|_{t=0} = k_{0,\varepsilon}. \tag{7}$$

In the final step we will seek for a limiting correlation function, describing the scaled system, i.e. we want

$$\varepsilon^n k_{t,\varepsilon}^{(n)} \to r_t, \quad \varepsilon \to 0 \tag{8}$$

for each $n \in \mathbb{N}_0$. Again the collection $r_t = (r_t^{(n)})_{n=0}^\infty$ will satisfy some system of equations similar to (4), i.e.

$$\frac{\partial r_t}{\partial t} = L_V^\Delta r_t, \quad r_t|_{t=0} = r_0.$$

This limiting description is known as the Vlasov hierarchy containing less information as the original model, but is simpler to analyse. Starting from initial function r_0 as he correlation function of the Poisson measure π_{ρ_0}, c.f. (6), one finds that the solution r_t will be of the form

$$r_t^{(n)}(x_1, \ldots, x_n) = \rho_t(x_1) \cdots \rho_t(x_n)$$

and so r_t is again the correlation function of a Poisson measure with the new density ρ_t. This density is determined by the mesoscopic equation, which we will also call kinetic description,

$$\frac{\partial \rho_t}{\partial t}(x) = v(\rho_t)(x), \quad \rho_t|_{t=0} = \rho_0 \tag{9}$$

and this property is known as the Chaos preservation principle. All previous steps can be computed for many models explicitly, which will be realized later for each model. The function ρ_t is the approximate density of this system, i.e. plays the same role as $k_t^{(1)}$, whereas it is determined in general by a non-linear and non-local equation. For the special case (5) the equation for ρ_t is the same as for $k_t^{(1)}$, which again is the consequence of the independence of each cell. Given a microscopic model through its (formal) Markov generator L, we will say that (9) is the kinetic description of the microscopic model. This description is produced by taking the formal limit within (7). Without further investigation it is not known, whether also the corresponding solutions converge, i.e. (8) holds. We will say that the kinetic description (9) corresponds to the microscopic model if (8) holds. In particular this implies that for given initial condition ρ_0 and ρ_t the solution to (9) one has

$$\lim_{\varepsilon\to 0} \varepsilon k_{t,\varepsilon}^{(1)} = \rho_t.$$

The precise notion of the convergence might depend on the particular model and shall be checked for each model separately. In many cases one knows that ρ_t will be bounded and hence the limit is uniformly in all spatial variables.

1.3 Clusterization and pattern formation

From an intuitive point of view many scientists understand under the terminus of clusterization that with increased probability we will observe cells aggregating in some bounded regions. For mathematical analysis such understanding has to be reformulated in terms of mathematical objects. In applications biologists often observe the density of the system, and find regions with peaks and at the same time regions with rather small density. Such phenomena is also often called clusterization. In this work we will call such phenomena pattern formation. One example for pattern formation would be the density, for simplicity one-dimensional,

$$\rho(x) = \begin{cases} a, & x \in [2k, 2k+1) \\ b, & x \in [2k+1, 2(k+1)) \end{cases},$$

where $0 < a < b$ and $k \in \mathbb{Z}$. Such density is periodic, and if b is compared to a much larger we will observe macroscopically in the regions $[2k+1, 2(k+1))$ an aggregation of cells or molecules. One could think about the density for the description of periodic crystal structures, whereas we do not relate such density to any specific model, since we have not it derived from any microscopic model.

The notion of clusters will be used in this work to relate to higher correlations of an interacting cell system, whereas pattern formation is connected only to the first correlation function, i.e. the density of the system. Having in mind that the sequence of correlation functions $k^{(n)}$ describe the densities of the moments of a state of the system, i.e. a probability measure on Γ, we would like to fix in the next step a reference measure, which shall be regarded as completely uncorrelated. In Physics it is know that for a completely uncorrelated system the correlation functions will have product structure, i.e.

$$k^{(n)}(x_1, \ldots, x_n) = \rho(x_1) \cdots \rho(x_n), \quad x_1, \ldots, x_n \in \mathbb{R}^d$$

for each $n \in \mathbb{N}$. Here $0 \le \rho$ is the density. Therefore we regard the Poisson measure π_ρ as the reference measure to measure correlations and clusterization of the system. One important special case is the choice $\rho(x) = z$, where $z > 0$ is constant. In such case the system is distributed uniformly in all $\mathbb{R}^d$ and due to the product structure all cells are independent. Such cases were already analysed in physics for the free gas. We will call a systems non-clustering, if its correlation functions satisfy

$$k^{(n)}(x_1, \ldots, x_n) \le \rho(x_1) \cdots \rho(x_n) C, \quad x_1, \ldots, x_n \in \mathbb{R}^d \tag{10}$$

for all $n \in \mathbb{N}_0$ and some constant $C > 0$. In the case, where

$$k^{(n)}(x_1, \ldots, x_n) \leq \rho(x_1) \cdots \rho(x_n) n!^{\delta}, \quad x_1, \ldots, x_n \in \mathbb{R}^d \tag{11}$$

for all $n \in \mathbb{N}_0$ and some $\delta > 0$, we will say that the system admits clusterization. Note that, this does not mean that the system will be really clustering. In general one should also have a bound from below. We will say the system is clustering, if the following bounds hold for all $n \in \mathbb{N}_0$, $x_1, \ldots, x_n \in \mathbb{R}^d$

$$\rho_0(x_1) \cdots \rho_0(x_n)(n!)^{\delta_0} C_0 \leq k^{(n)}(x_1, \ldots, x_n) \leq \rho_1(x_1) \cdots \rho_1(x_n)(n!)^{\delta_1} C_1,$$

where ρ_0, ρ_1 are non-negative locally integrable functions, $C_0, C_1, \delta_0, \delta_1 > 0$ are constants. The evolution of a system will be clustering if for each fixed time $t > 0$, the correlation functions $k_t^{(n)}$ admits above estimations.

Let us now turn to the interpretation of this conditions. In the case of (10), we observe that the moments of the system are bounded from above by the moments of the Poisson measure. For instance the probability density of finding n cells at positions $x_1, \ldots, x_n$ is given by $k^{(n)}(x_1, \ldots, x_n)$. In the case of (11) this density is fast growing with respect to n and hence it is more likely to find configurations which consist of a higher number of cells. Therefore we see that in contrast to pattern formation, here we incorporate also the microscopic description of the system via its configurations. In the next section we will see, that a free branching process will always be clustering. In order to prevent clusterization it is therefore necessary to introduce microscopic interactions, which will regulate the system. Two examples are given by either increasing the death of cells, in such a way that in dense regions cells will have an increased intensity to kill each other, or dumping down the intensity for the branching of cells, which means that in dense regions cells will have only a small chance to proliferate.

2 Results

2.1 One-component models

Free branching process

The first model we start with is a toy model in the sense that mathematically all corresponding equations can be solved explicitly. This model consists of the two elementary events birth and death of a cell. First of all each cell have an exponentially distributed lifetime with parameter $m > 0$, so the time each cell will survive is given by an exponentially distributed random variable and the mean lifetime is $\frac{1}{m}$. After the death of a cell located at position $x \in \mathbb{R}^d$ the configuration of all cells changes $\gamma \to \gamma \backslash x$. Written in terms of the heuristic Markov generator this part has the form

$$(L_d F)(\gamma) = \sum_{x \in \gamma} m(F(\gamma \backslash x) - F(\gamma)).$$

Moreover, each cell located at position $x \in \gamma$ can divide into two new cells located at the positions y_1, y_2. Thus the configuration changes in the following way $\gamma \to \gamma \backslash x \cup y_1 \cup y_2$. The probability of finding cells in the volume element $\mathrm{d}y_1\mathrm{d}y_2$, is given by

$$a(x - y_1, x - y_2)\mathrm{d}y_1\mathrm{d}y_2,$$

and the intensity of the event of cell-division is given by $\lambda > 0$. We assume that $0 \leq a$ is a probability density, hence is normalized to 1 and assume that this kernel is symmetric in both arguments, so

$$a(x, -y) = a(x, y), \qquad a(-x, y) = a(x, y), \quad x, y \in \mathbb{R}^d.$$

In terms of the heuristic Markov generator this leads to

$$(L_b F)(\gamma) = \lambda \sum_{x \in \gamma} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x - y_1, x - y_2)(F(\gamma \backslash x \cup y_1 \cup y_2) - F(\gamma))\mathrm{d}y_1\mathrm{d}y_2.$$

Incorporating both effects independently of each other in one process, the overall Markov generator will have the form $L = L_d + L_b$. Clearly this description shows, that each cell is independent of all other cells. Thus this description really reflects the effects of free proliferation of cells within some region. If the kernel a is a product of two probability distributions, i.e. $a(x, y) = b(x)c(y)$, then the positions y_1 and y_2 will be distributed independent of each other. In some applications the positions y_1, y_2 are not independent of each other, i.e. choosing position y_1 influences the position of y_2. In the special case, where the position y_2 would be determined completely by the position y_1 one would take e.g. $a(x, y) = b(x)\delta(x + y)$ with a non-negative integrable function b, which is normalized to 1. Therefore the position y_2 is given by $y_2 = x + (x - y_1) = 2x - y_1$, meaning that cells prefer to proliferate in opposite directions, such that the distance $|y_1 - y_2|$ is maximal.

Here we will investigate the general case and analyse some properties of the system. The first observation shows, that if the birth kernel a is such that new cells may appear arbitrary close to the mother cell located at position x, then the dynamics will admit asymptotic clustering.

Theorem 2.1. *Assume that a is continuous such that $a(0) > 0$, then starting from poissonian statistics, i.e. correlation functions $k(\eta) = C^{|\eta|}$, the evolution of correlation functions k_t will satisfy for each η such that all points are sufficiently close to each other*

$$k_t^{(n)}(x_1, \ldots, x_n) \geq \beta^n e^{-(m-\lambda)nt} n!$$

for some constant $\beta > 0$ depending on λ, a and C. Moreover there exists $C(t) > 0$ non-decreasing such that

$$k_t^{(n)}(x_1, \ldots, x_n) \leq C(t)^n n!$$

for all $n \in \mathbb{N}_0$ and $x_1, \ldots, x_n \in \mathbb{R}^d$.

Above estimate implies due to the presence of the factor $n!$, that independent of β, m and λ with high probability many cells can be observed in a small region, which reflects the effect of clustering. The second estimate shows, that factorial growth of correlation functions k_t is the worst case we can observe. This estimate remains true without any conditions on the birth kernel

a. The next Theorem formulates the result concerning the kinetic description of this model.

Theorem 2.2. *For each initial distribution of cells $\rho_0(x) \geq 0$, which is essentially bounded, there exist a unique solution $\rho_t(x) \geq 0$ to the kinetic equation*

$$\frac{\partial \rho_t}{\partial t}(x) = -(m+\lambda)\rho_t(x) + \lambda \int_{\mathbb{R}^d} b(x-y)\rho_t(y)\mathrm{d}y \tag{12}$$

with initial condition $\rho_t|_{t=0} = \rho_0$. Such solution is also essentially bounded and corresponds to the rescaled system, i.e. the Vlasov hierarchy. The function $0 \leq b$ is given by

$$b(x) = \int_{\mathbb{R}^d} a(x,y)\mathrm{d}y + \int_{\mathbb{R}^d} a(y,x)\mathrm{d}y.$$

The absence of non-linearities is due to the absence of interactions of cells. For $m > \lambda$ all cells will die, whereas for $m < \lambda$ the number of cells will grow exponentially. In the critical case $m = \lambda$ the total number of cells is conserved and the equation describes a random walk in continuous time. The general solution to (12) is given by

$$\rho_t(x) = \rho_0(x)e^{-(m+\lambda)t} + e^{-(m+\lambda)t}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}(b^{*n} * \rho_0)(x).$$

The same Mesoscopic equation and the same results about asymptotic clustering of the system can be achieved, if we simplify the birth by setting

$$a(x,y) = \delta(x)b(y),$$

which means that each cell will create a new cell located at position $y \in \mathbb{R}^d$ without disappearing from the system. Mathematically such situation is due to less computational work simpler to analyse. Results concerning invariant states, existence of a Markov process etc. can be found in [15]. In the following we will always restrict ourselves to this case, called Contact model. Its heuristic Markov generator has the form

$$\begin{aligned}(L_{CM}F)(\gamma) = m &\sum_{x\in\gamma}(F(\gamma\backslash x) - F(\gamma)) \\ &+ \lambda\sum_{x\in\gamma}\int_{\mathbb{R}^d} a(x-y)(F(\gamma\cup y) - F(\gamma))\mathrm{d}y.\end{aligned} \tag{13}$$

Spatial logistic model

As already discussed in the Contact model all cells are independent of each other. For a wide class of biologically relevant models such behaviour is not adequate, so one has to introduce additional microscopic mechanisms, which regularize the overall dynamics in such a way that all correlation functions become sub-poissonian. This can be achieved if one includes either additional density dependent mortality or one introduces density dependent birth in such a way, that in regions of high density cells will have a small probability to proliferate. Here we will state some results about the model with additional density dependent mortality.

Let us start with with the usual Contact model, so L_{CM} given in (13) and introduce additional death. Each cell $x \in \gamma$ may cause death of another cell $y \in \gamma\backslash x$ with rate $\lambda^- a^-(x-y)$. The overall rate of death caused by the cell x is simply $\lambda^- \sum\limits_{y\in\gamma^-\backslash x} a^-(x-y)$ and describes some sort of competition of cells for resources within the body. Therefore the complete heuristic Markov generator will have the form

$$(LF)(\gamma) = (L_{CM}F)(\gamma) + \lambda^- \sum_{x\in\gamma} \sum_{y\in\gamma\backslash x} a^-(x-y)(F(\gamma\backslash x) - F(\gamma)). \tag{14}$$

This model was analysed in several articles, c.f. [5, 7]. It is known, c.f. [21] that if $m > 0$ is arbitrary small and there is $\theta > 0$ such that $a^- - \theta a$ is a stable potential, then there exists an evolution of states, such that its correlation functions satisfy $k_t^{(n)} \le C^n$ for some constant $C > 0$. Moreover, it can be shown, that in the regime of high mortality the only invariant state is the one representing the empty configuration. Namely if

$$\lambda a \le \lambda^- a^-$$

then the unique invariant distribution is $\mu_{inv} = \delta_{\{\emptyset\}}$, i.e. $k_{inv}^{(n)} = 0$ for $n \ge 1$ and $k_{inv}^{(0)} = 1$. Here we will only summarize the result for the kinetic description, [6]

Theorem 2.3. *Assume $a, a^- \ge 0$ are symmetric, integrable and normalized to 1. Then for each initial measurable density $\rho_0(x) \le C$ for a.a. $x \in \mathbb{R}^d$ there exists a unique solution ρ_t to the kinetic equation*

$$\frac{\partial \rho_t}{\partial t}(x) = -m\rho_t(x) - \lambda^- \rho_t(x)(a^- * \rho_t)(x) + \lambda(a * \rho_t)(x) \tag{15}$$

with $\rho_t|_{t=0} = \rho_0$. Moreover the function $k_t(\eta) = e_\lambda(\rho_t;\eta)$ is a solution to the Vlasov hierarchy

$$\frac{\partial r_t}{\partial t}(\eta) = L_V^\Delta r_t(\eta), \quad r_t|_{t=0} = r_0$$

with $r_0(\eta) = e_\lambda(\rho_0;\eta)$. This solution ρ_t will again be bounded by the same constant C, i.e. $\rho_t(x) \le C$ for a.a. $x \in \mathbb{R}^d$. Moreover equation (15) is the kinetic description, and if $\lambda^- a^- - \lambda a$ is stable, then also (8) holds.

Clearly there are two stationary solutions to (15) given by 0 and $\dfrac{\lambda - m}{\lambda^-}$. Such solutions are biologically relevant if they are positive, so $m \le \lambda$. Let us now assume, that a^- is strongly localized. Then we can approximate the convolution by a multiplication, which leads to $a^- * \rho_t \approx \rho_t$. In this case the kinetic equation simplifies to

$$\frac{\partial \rho_t}{\partial t}(x) = -m\rho_t(x) - \lambda^- \rho_t(x)^2 + \lambda(a * \rho_t)(x).$$

This equation was analysed in several articles in the one-dimensional case. A function $\rho_t(x)$ is called a traveling wave solution with monotone profile and speed c, if $\rho_t(x) = \psi(x - ct)$ for some monotone function $\phi \in C^1(\mathbb{R})$. E.g. in [20] it was shown that if a is exponentially integrable, i.e. there exist $\alpha > 0$ such that

$$\int_{\mathbb{R}} e^{-\alpha y} a(y) \mathrm{d}y < \infty$$

then there exists $c_* > 0$ such that for each $c \geq c_*$ there exist a traveling wave solution with monotone profile and speed c. A For each $c < c_*$ there exist no periodic traveling wave solution of speed c. The constant c_* is called spreading speed. For the time-inhomogeneous case $\lambda = \lambda(t)$ in [12] a similar result was shown and a formula for c_* has been derived. In contrast if a do not satisfy the exponential integrability condition, then the speed of propagation will be not constant, c.f. [10]. Therefore modelling complex cell systems one has also to distinguish between different classes of kernels a^-, a. For example taking for a a gaussian distribution, one gets a constant spreading speed, whereas taking a as the Cauchy distribution one gets an accelerated spreading speed.

Branching with fecundity

Instead of density dependent mortality here we will summarize the case of density dependent birth. So each cell have again an exponentially distributed lifetime with parameter $m > 0$ and each cell at position $x \in \gamma$ can create a new cell with intensity $e^{-E(x,\gamma \backslash x)}$. The relative energy $E(x, \gamma \backslash x)$ is given by

$$E(x, \gamma \backslash x) = \sum_{y \in \gamma \backslash x} \varphi(x - y) \geq 0.$$

The potential $\varphi \geq 0$ is assumed to be symmetric and integrable. In dense regions around a cell x the energy will be large and thus the exponential $e^{-E(x,\gamma \backslash x)}$ will dump the intensity of creating a new cell at some position. Such kind of self-regulation can be interpreted as a lack of energy, material or resources for the cell at position x. If now x creates a new cell, then again the probability of finding the new cell within the region $\mathrm{d}y$ is given by $a(x - y)\mathrm{d}y$. The generator is given for functions $F : \Gamma \longrightarrow \mathbb{R}$ by

$$\begin{aligned}(LF)(\gamma) &= m \sum_{x \in \gamma} (F(\gamma \backslash x) - F(\gamma)) \\ &\quad + \lambda \sum_{x \in \gamma} e^{-E(x,\gamma \backslash x)} \int_{\mathbb{R}^d} a(x - y)(F(\gamma \cup y) - F(\gamma))\mathrm{d}y. \end{aligned} \tag{16}$$

Such model was discussed in [8] and it was shown, that under some conditions on the potentials a and φ such self-regulation will prevent asymptotic clustering of the system. More precisely, if there is a constant $\theta > 0$ such that for a.a. $x \in \mathbb{R}^d$ the conditions

$$a(x) \leq \theta \varphi(x) e^{-\varphi(x)}$$
$$\lambda \left(1 + \frac{\theta}{eC}\right) < \frac{m}{2} \exp\Big(-C \int_{\mathbb{R}^d} (1 - e^{-\varphi(x)}) \mathrm{d}x\Big)$$

hold, then there is $0 < C' < C$ such that for $k_0(\eta) \leq C'^{|\eta|}$ there exist a unique classical solution k_t to (4) such that $k_t(\eta) \leq C^{|\eta|}$. Here the dispersion kernel a should be dominated by the interaction kernel suppressing the intensity of birth and in the second condition one assumes that the constant mortality is high enough. Pattern formations might still appear, such effects are of mesoscopic nature and thus should by studied within the kinetic description. So let us state the general result for the mesoscopic limit.

Theorem 2.4. *Assume that*

$$a(x) \leq \theta\varphi(x)e^{-\varphi(x)}$$
$$2e^{C\langle\varphi\rangle}\lambda\left(1+\frac{\theta}{eC}\right) < m$$
$$\lambda\left(1+\frac{\theta\langle\varphi\rangle}{e}\right) < m,$$

where $\langle\varphi\rangle = \int_{\mathbb{R}^d}\varphi(x)\mathrm{d}x$ *denotes the mean of the potential* φ*. Then there is* $\alpha_0 \in (0,1)$ *such that for all* $\alpha \in (\alpha_0, 1)$ *and each initial condition* $0 \leq \rho_0 \leq \alpha C$ *there exists a unique solution* $0 \leq \rho_t \leq \alpha C$ *to the kinetic equation*

$$\frac{\partial\rho_t}{\partial t}(x) = -m\rho_t(x) + \lambda\left(a * \rho_t e^{-\varphi*\rho_t}\right)(x) \tag{17}$$

and the function $r_t = e_\lambda(\rho_t)$ *solves the Vlasov-hierarchy.*

The property $\rho_t \leq \alpha C$ means, that the density of the system will be bounded and so no explosions of the cell population may appear. The main difference to the Contact model is the presence of the additional term $e^{-\varphi*\rho_t}$ which suppresses the growth of the density. The first condition states that the interaction should dominate the proliferation The other two conditions require high mortality and are sufficient to prevent the growth of the density of the system. Without these two conditions we expect that the density will grow exponentially, but still will not admit clusterization.

Contact model with motion

Last we would like to draw the attention to another self-regulation mechanism. The usual Contact model described by the heuristic Markov generator L_{CM}, as mentioned before, consists of asymptotic clusters. To avoid this effect, let us assume that each cell have the additional possibility to move within the system. Similar to previous model let us assume that there are two main contributions to the intensity of the motion. On the one hand-side a cell at position $x \in \gamma$ will try to move outside a dense area of cells and on the other hand-side the destination point will be chosen in such way, that it is less dense. All in one cells will try to jump from dense areas to less dense areas. Such heuristic description can be summarized in the following Markov generator

$$\begin{aligned}(LF)(\gamma) &= (L_{CM}F)(\gamma) \\ &\quad + \sum_{x\in\gamma} e^{E_\phi(x,\gamma\backslash x)} \int_{\mathbb{R}^d} e^{-E_\psi(y,\gamma)}c(x-y)(F(\gamma\backslash x \cup y) - F(\gamma))\mathrm{d}y. \end{aligned} \tag{18}$$

As before, the energies have the form $E_\phi(x,\gamma\backslash x) = \sum_{y\in\gamma\backslash x} \phi(x-y)$ and $E_\psi(y,\gamma) = \sum_{z\in\gamma} \psi(y-z)$, where the potentials $\phi, \psi \geq 0$ are symmetric and integrable. The probability of finding the new cell within dy is approximately

$$\frac{1}{N} e^{-E_\psi(y,\gamma)} c(x-y)\mathrm{d}y$$

with a normalization constant $N = N(\gamma)$ and a probability distribution $0 \leq c(x) = c(-x)$. To this time such model was never analysed in this generality and therefore it is not clear how the microscopic behaviour will look like. Nevertheless, simulations suggest that such mechanism can lead to less asymptotic clustering of the evolution, but due to the motion of the system, started from a compactly supported density, will spread out faster then in the Contact model. We also expect that the local density ρ_t within the kinetic description will be growing at most sub-exponential. Within this work we derive the kinetic description for this model. Questions concerned about front wave propagation and bounds on the density should be studied in detail afterwards.

Theorem 2.5. *The kinetic description corresponding to the microscopic description of the Contact model in the presence of density dependent jumps is given by a density $\rho_t \geq 0$, which solves the Mesoscopic equation*

$$\begin{aligned}\frac{\partial \rho_t}{\partial t}(x) &= -m\rho_t(x) + \lambda(a * \rho_t)(x) \\ &\quad + \left(c * \left(\rho_t e^{\phi * \rho_t}\right)\right)(x) e^{-(\psi*\rho_t)(x)} - e^{(\phi*\rho_t)(x)}(c * e^{-\psi*\rho_t})(x)\rho_t(x).\end{aligned}$$

Already here, we can observe how complicated the mesoscopic description might become. Of course one could simplify the situation by only investigating the case, where only one of the potentials ϕ, ψ is non-vanishing. So let us assume $\psi = 0$. Then the equation becomes

$$\frac{\partial \rho_t}{\partial t}(x) = -m\rho_t(x) + \lambda(a^+ * \rho_t)(x) + \left(c * \left(\rho_t e^{\phi*\rho_t}\right)\right)(x) - \langle c\rangle e^{(\phi*\rho_t)(x)}\rho_t(x).$$

The first two terms describe the free proliferation, whereas the last two terms describe the impact of motion on ρ_t. The total number of particles is not affected by this two terms, i.e. $\frac{\partial}{\partial t}\langle\rho_t\rangle = (\lambda - m)\langle\rho_t\rangle$. The local number of cells, $\int_\Lambda \rho_t(x)\mathrm{d}x$, within some volume $\Lambda \subset \mathbb{R}^d$ might have a drastically different behaviour.

2.2 Two-type models

In contrast to previous models here we will present some results about multi-type models. In reality cells have different tasks and hence should be described by different microscopic interactions. In contrast to previous modelling here we will consider two type of configurations $\gamma^+ = \{x_1, \dots, x_n, \dots\}$ and $\gamma^- = \{y_1, \dots, y_n, \dots\}$. Both should be locally finite and distinct, so $\gamma^+ \cap \gamma^- = \emptyset$. The collection of all such configurations will be denoted by Γ^2. Not only the elementary events birth, death and jumping of cells can be treated, we now have the possibility of switching cell-type, i.e. a $+$-cell becomes a $-$-cell and vice versa. More important, the densities for all events might also depend on the cells of other type, so that e.g. $+$-cells are being affected by $-$-cells etc. Within this framework the kinetic description will be a coupled system of two equations, which describe the rescaled density ρ^+ for $+$-cells and the rescaled density ρ^- for $-$-cells.

Let us now outline how to extend previous considerations to this case. A state of the system is a probability distribution, i.e. measure $\mu \in \mathscr{P}(\Gamma^2)$, on the two-component phase space Γ. For the given μ, the corresponding correlation functions $k^{(n,m)}(x_1, \dots, x_n; y_1, \dots, y_m)$, if they exist, are defined via the equation

$$\int_{\Gamma^2} \sum_{\{x_1,\dots,x_n\}\subset\gamma^+} \sum_{\{y_1,\dots,y_m\}\subset\gamma^-} f^{(n,m)}(x_1, \dots, x_n; y_1, \dots y_m) d\mu(\gamma^+, \gamma^-)$$
$$= \frac{1}{n!m!} \int_{(\mathbb{R}^d)^n} \int_{(\mathbb{R}^d)^m} f^{(n,m)}(x_1, \dots, x_n; y_1, \dots, y_m)$$
$$\times k^{(n,m)}(x_1, \dots, x_n; y_1, \dots, y_m) d^n x d^m y$$

for all symmetric functions $f^{(n,m)}$ which are integrable with compact support. Again $k^{(n,m)}$ describe the moments of the state μ of the system and in the special case $n = 1 = m$ the function $k^{(1,1)}$ is the density of the system, whereas $k^{(n,0)}$ and $k^{(0,m)}$ correspond to the boundary distributions.

As before the correlation functional k_t will satisfy the equation

$$\frac{\partial k_t}{\partial t}(\eta^+, \eta^-) = (L^\Delta k_t)(\eta^+, \eta^-), \tag{19}$$

which has to be studied for a rigorous mathematical analysis.

Similar to the one-component case, the kinetic scaling starts with dumping the potentials by multipliying them by a factor $\varepsilon > 0$. Therefore we get a scaled version of the equation (19), i.e. L^Δ_ε instead of L^Δ. Let us assume for the initial conditions $k^{(n,m)}_{0,\varepsilon}$

$$\varepsilon^{n+m} k^{(n,m)}_{0,\varepsilon} \to r^{(n,m)}_0, \quad \varepsilon \to 0$$

with a symmetric function $r^{(n,m)}_0$ and $n, m \in \mathbb{N}_0$. The important case is to take

$$r^{(n,m)}_0(x_1, \dots, x_n; y_1, \dots, y_m) = \rho^+_0(x_1) \cdots \rho^+_0(x_n) \rho^-_0(y_1) \cdots \rho^-_0(y_m). \tag{20}$$

Denote by $k^{(n,m)}_{t,\varepsilon}$ the solutions to equation (19) with L^Δ_ε instead of L^Δ and assume that this solutions preserve the order of singularity, namely

$$\varepsilon^{n+m} k^{(n,m)}_{t,\varepsilon} \to r_t, \quad \varepsilon \to 0 \tag{21}$$

for each $n, m \in \mathbb{N}_0$. This is equivalent to investigate the Cauchy problem for the operators $L^\Delta_{\varepsilon,ren} = R_\varepsilon L^\Delta_\varepsilon R_{\varepsilon^{-1}}$, where

$$(R_\varepsilon k)(\eta^+, \eta^-) = \varepsilon^{|\eta^+|+|\eta^-|} k(\eta^+, \eta^-)$$

and seek for a limit $L^{\Delta}_{\varepsilon,ren} \to L^{\Delta}_{V}$. Using the initial condition r_0 as given in (20), the solution to

$$\frac{\partial r_t}{\partial t} = L^{\Delta}_{V} r_t, \quad r_t|_{t=0} = r_0$$

will again have the form

$$r_t^{(n,m)}(x_1,\ldots,x_n;y_1,\ldots,y_m) = \rho_t^+(x_1)\cdots\rho_t^+(x_n)\rho_t^-(y_1)\cdots\rho_t^-(y_m)$$

and ρ_t^+, ρ_t^- is determined by the kinetic equations

$$\begin{cases} \dfrac{\partial \rho_t^+}{\partial t} = v^+(\rho_t^+, \rho_t^-), \\ \dfrac{\partial \rho_t^-}{\partial t} = v^-(\rho_t^+, \rho_t^-). \end{cases} \tag{22}$$

Then similarly to the one-component case the solutions ρ_t^+ and ρ_t^- to (22) will be called kinetic description of the microscopic model. If in addition (21) holds, then we will say that the kinetic description corresponds to the microscopic model. In such case one has

$$\lim_{\varepsilon\to 0} k_{t,\varepsilon}^{(1,1)}(x,y) = \rho_t^+(x)\rho_t^-(y),$$

where ρ_t^+, ρ_t^- are the solutions to (22) with initial condition ρ_0^+ and ρ_0^-. Let us explain the details and state the results for several important models in the last part of this section. Since these models were not investigated mathematically, we will give only some simple preliminary results and state the kinetic description. Its analysis and properties of the description should be analyzed for each model separately.

Necrosis model

Looking at a cell system, with free branching and constant mortality $m > 0$, i.e the Contact model, one possible extension to more realistic situations is to modify the death of cells. After the death of a cell, it triggers several biological mechanisms which effect surrounding cells. If the number of deaths will exceed some critical value, then the surrounding cells will have an increased intensity of death. Such effects will cause cascades of dying cells infecting neighbouring cells. To model this effect we will introduce to types of cells. The +-cells will be the usual cells with constant mortality and free proliferation, i.e. the generator is similar to the generator L_{CM} from the Contact model. The −-cells will represent the dead cells, which exceeded the critical value and therefore will cause death of +-cells. These dead cells will disappear due to some exponentially distributed time with parameter $m_1 > 0$. The affect of −-cells on +-cells will be described similar to the spatial logistic model, c.f. (14). To summarize this explanation we will write down the form of the heuristic Markov generator, i.e.

$$(LF)(\gamma^+,\gamma^-) = (AF)(\gamma^+,\gamma^-) + (BF)(\gamma^+,\gamma^-) + (VF)(\gamma^+,\gamma^-). \tag{23}$$

The first operator A is similar to the Contact model for the normal cells and has the form

$$(AF)(\gamma^+,\gamma^-) = m_0 \sum_{x\in\gamma^+} (F(\gamma^+\backslash x, \gamma^- \cup x) - F(\gamma^+,\gamma^-)) \\ + \lambda^+ \sum_{x\in\gamma^+} \int_{\mathbb{R}^d} a^+(x-y)(F(\gamma^+ \cup y, \gamma^-) - F(\gamma^+,\gamma^-))\mathrm{d}y.$$

The operator B describes the evolution of $-$ cells, which can only disappear from our system, so it is simply

$$(BF)(\gamma^+,\gamma^-) = m_1 \sum_{x\in\gamma^-} (F(\gamma^+,\gamma^-\backslash x) - F(\gamma^+,\gamma^-)).$$

The last part describes the interaction of both types and is assumed to be of the form

$$(VF)(\gamma^+,\gamma^-) = \lambda^- \sum_{x\in\gamma^+} \Big(\sum_{y\in\gamma^-} \varphi(x-y) \Big) (F(\gamma^+\backslash x, \gamma^- \cup x) - F(\gamma^+,\gamma^-)).$$

The potentials $a^+, \varphi \geq 0$ are assumed to be symmetric, integrable, normalized to 1, and the constants $m_0, m_1, \lambda^+, \lambda^-$ are strictly positive. Ignoring the effects caused by changing the types $+$ to $-$ and vice versa, the overall evolution should be similar to the dynamics of the spatial logistic model with constant mortality $m_0 + m_1$, dispersion $\lambda^+ a^+$ and competition kernel $a^- = \varphi$. Effects caused by changing the type may cause waves of dying cells and by this regulate the local density, which will prevent the explosion of the local number of cells. A rigorous mathematical analysis and simulations are the first steps for a better understanding of this system.

Finally let us give the kinetic description of this model.

Theorem 2.6. *Let $\rho_0 \geq 0$ be essentially bounded and ρ_t a non-negative solution to the system of mesoscopic equations*

$$\frac{\partial \rho_t^+}{\partial t}(x) = -(m_0 + \lambda^-(\varphi * \rho_t^-)(x))\rho_t^+(x) + \lambda^+(a^+ * \rho_t^+)(x) \tag{24}$$

$$\frac{\partial \rho_t^-}{\partial t}(x) = -m_1\rho_t^-(x) + \lambda^-\rho_t^+(x)(\varphi * \rho_t^-)(x) + m_0\rho_t^+(x). \tag{25}$$

Then $r_t(\eta^+,\eta^-) = e_\lambda(\rho_t^+;\eta^+)e_\lambda(\rho_t^-;\eta^-)$ is the solution with initial condition $r_0(\eta^+,\eta^-) = e_\lambda(\rho_0^+;\eta^+)e_\lambda(\rho_0^-;\eta^-)$ corresponding to the scaled Vlasov hierarchy.

Go-and-grow models

Here we will assume that tumour cells have two possible states. On the one side the cells can be in a proliferating state, which we call $-$-state. This state is responsible for the growth of the tumour. In the second state, called $+$-state, a cell will be moving and so contribute to additional spreading of the tumour, where the length of the distance should be large compared with the spreading size of the proliferation. We have the freedom to take several different types of interactions and intensities for proliferation, movement, and changing the type of state. Let us first summarize briefly all common effects and afterwards give an extended description for each choice of intensities.

In principal all proliferating cells have their own development and will spread within the system due to either the Contact model or the Contact model with fecundity. Moreover, they will have the possibility to change their type to moving cells by random. Such switching can be either spontaneously or triggered by surrounding cells in dense areas. This moving cell will start to randomly hop inside the tumour, essentially with high probability this jumps will be far compared to the distance of proliferation. After a certain time this moving cell will reach a substantially less dense region and will start to proliferate again. Such microscopic dynamics may cause the creation of new tumour patters where the distance to the old pattern is large compared to proliferation length.

A medical difficulty is to observe such moving cells, therefore a treatment of a tumour is essentially restricted to the treatment of proliferating cells. One goal is to determine the front wave propagation, derive reasonable extremal statistics, and consequently predict the size and possible locations of a significantly wider amount of tumour cells. We expect that this kind of insights will lead to a better understanding of the microscopic structure of the tumours and hence to new therapeutical treatments of tumour and cancer.

In the following we will give 4 examples with concrete types of intensities and derive their kinetic description. The moving cells will always evolve as a free jumping process, meaning each moving cell will independent of all other cells randomly hop within the system. In addition each moving cell will have a density independent death of parameter $d \geq 0$. The heuristic Markov generator for the moving cells is simply

$$(L_{hop}F)(\gamma^+,\gamma^-) = d\sum_{x\in\gamma^+}(F(\gamma^+\backslash x,\gamma^-) - F(\gamma^+,\gamma^-)$$
$$+\sum_{x\in\gamma^+}\int_{\mathbb{R}^d} c(x-y)(F(\gamma^+\backslash x\cup y,\gamma^-) - F(\gamma^+,\gamma^-))\mathrm{d}y.$$

Each model will either have different rates at which the cells will change their state or the type of proliferation is varying.

First model

Let us assume that the proliferating cells will be described by the Contact model, c.f. (13) and that within dense areas the proliferating cells have an increased intensity to change their state to moving cells. For simplicity we assume first, that cells within the moving state will stay an exponential distributed time with parameter $q > 0$ in this state and afterwards start to proliferate again.

Changing the state is the described by the heuristic Markov generator

$$(VF)(\gamma^+,\gamma^-) = q\sum_{x\in\gamma^+}(F(\gamma^+\backslash x,\gamma^-\cup y) - F(\gamma^+,\gamma^-)) \\ + \sum_{x\in\gamma^-}\Big(p + \sum_{y\in\gamma^-\backslash x}\varphi(x-y)\Big)(F(\gamma^+\cup x,\gamma^-\backslash x) - F(\gamma^+,\gamma^-)).$$

Here $p,q>0$ are the intensities to change the type independent of all other cells and $0\le\varphi\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ is a symmetric potential. The overall dynamics is a superposition of all three type of dynamics and has the form $L = L_{CM} + L_{hop} + V$.

The kinetic description for this model is given by:

$$\frac{\partial\rho_t^+}{\partial t}(x) = -(\langle c\rangle + d + q)\rho_t^+(x) + (c*\rho_t^+)(x) + p\rho_t^-(x) + \rho_t^-(x)(\varphi*\rho_t^-)(x)$$
$$\frac{\partial\rho_t^-}{\partial t}(x) = -(m+p)\rho_t^-(x) - \rho_t^-(x)(\varphi*\rho_t^-)(x) + \lambda(a*\rho_t^-)(x) + q\rho_t^+(x).$$

Despite the presence of motion, this example is similar to previous one. Here the spreading speed should be due to the motion increased, whereas in the previous model the spreading speed is constant for exponentially integrable dispersion kernels. The local cell number may be dumped by the motion, but the overall particle number will still grow asymptotically as $e^{(\lambda-m-d)t}\rho_0$, with ρ_0 the initial distribution of cells.

Second model

Let us include density dependent changes from moving to proliferating cells, so the moving cell will have a small probability to change its type if it is still in a dense area of proliferating cells. Such changes could be achieved by the following change of the operator V

$$(VF)(\gamma^+,\gamma^-) = q\sum_{x\in\gamma^+}\exp\Big(-\sum_{y\in\gamma^-}\psi(x-y)\Big)(F(\gamma^+\backslash x,\gamma^-\cup y) - F(\gamma^+,\gamma^-)) \\ + \sum_{x\in\gamma^-}\Big(p + \sum_{y\in\gamma^-\backslash x}\varphi(x-y)\Big)(F(\gamma^+\cup x,\gamma^-\backslash x) - F(\gamma^+,\gamma^-)).$$

Here p,q,φ are the same as before and $0\le\psi\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ is a symmetric, non-negative potential. This model will lead to the following pair of equations describing the local densities ρ_t^+,ρ_t^-

$$\frac{\partial\rho_t^+}{\partial t}(x) = -\,(\langle c\rangle + d + qe^{-(\psi*\rho_t^-)(x)})\rho_t^+(x) + (c*\rho_t^+)(x) \\ + p\rho_t^-(x) + \rho_t^-(x)(\varphi*\rho_t^-)(x)$$
$$\frac{\partial\rho_t^-}{\partial t}(x) = -\,(m+p)\rho_t^-(x) - \rho_t^-(x)(\varphi*\rho_t^-)(x) + \lambda(a*\rho_t^-)(x) \\ + q\rho_t^+(x)e^{-(\psi*\rho_t^-)(x)}.$$

Here the cells will stay a not exponentially distributed lifetime in the moving state. With high probability they will move until they reach an area with less proliferating cells and start to proliferate again. Thus we expect, that the motion outside of a pattern is higher and therefore the speed of growth of the boundary of the tumour is increased compared to previous model.

Third model

Let us assume constant intensities $p, q > 0$ for changing from proliferation to motion and vice versa, i.e. $\varphi = \psi = 0$ from the previous model, so

$$\begin{aligned}(VF)(\gamma^+,\gamma^-) = q \sum_{x\in\gamma^+} &(F(\gamma^+\backslash x, \gamma^- \cup x) - F(\gamma^+,\gamma^-)) \\ + p \sum_{x\in\gamma^-} &(F(\gamma^+ \cup x, \gamma^-\backslash x) - F(\gamma^+,\gamma^-)).\end{aligned} \tag{26}$$

Instead, we introduce additional density dependent death of proliferating cells, so they are self-regulating themselves, c.f. spatial logistic model. The generator for the $-$ cells is given in such case by

$$\begin{aligned}(L_-F)(\gamma) = &\sum_{x\in\gamma_-} m(F(\gamma^+,\gamma^-\backslash x) - F(\gamma)) \\ &+ \sum_{x\in\gamma^-}\sum_{y\in\gamma^-\backslash x} a^-(x-y)(F(\gamma^+,\gamma^-\backslash x) - F(\gamma)) \\ &+ \sum_{x\in\gamma^+}\int_{\mathbb{R}^d} a^+(x-y)(F(\gamma^+,\gamma^-\cup y) - F(\gamma))\mathrm{d}y\end{aligned}$$

and the overall generator by $L = L_- + L_{hop} + V$. A proliferating cell will have an increased rate for death and will start moving according to an exponentially distributed time with parameter $p > 0$. This cell will continue to move for an another exponentially distributed time with parameter $q > 0$ and afterwards start to proliferate again. Such behaviour will cause a diffusion like movement of the cells where the speed of growth of the patterns should be less then in the previous models. Instead, here the local regulation mechanism will bound the local density in time. Altogether this will lead to the following kinetic description for the local densities ρ_t^+, ρ_t^-

$$\begin{aligned}\frac{\partial\rho_t^+}{\partial t}(x) &= -(\langle c\rangle + q + d)\rho_t^+(x) + (c * \rho_t^+)(x) + p\rho_t^-(x) \\ \frac{\partial\rho_t^-}{\partial t}(x) &= -(m+p)\rho_t^-(x) - \rho_t^-(x)(a^- * \rho_t^-)(x) + \lambda(a * \rho_t^-)(x) + q\rho_t^+(x).\end{aligned}$$

Fourth model

Instead of looking at density dependent mortality for self-regulation of the proliferating cells, we could also take density dependent birth, i.e. branching with fecundity, c.f. [8]. Here the generator is given by $L = L_- + L_{hop} + V$, where L_- is given in (16) and V in (26). Using the same notations we will get the kinetic description

$$\frac{\partial \rho_t^+}{\partial t}(x) = -(\langle c \rangle + q + d)\rho_t^+(x) + (c * \rho_t^+)(x) + p\rho_t^-(x) \tag{27}$$

$$\frac{\partial \rho_t^-}{\partial t}(x) = -(m+p)\rho_t^-(x) + \lambda\left(a * \rho_t^- e^{-\varphi * \rho_t^-}\right)(x) + q\rho_t^+(x). \tag{28}$$

3 General Markov evolutions on configuration spaces

In this section we are going to summarize all necessary definitions and results, so that we can prove the given statements of previous section. First we briefly outline our approach for one-component systems and afterwards point out the steps for a natural extension to two-component systems. The last part deals with the mesoscopic scaling, here all machinery needed to derive the kinetic description for a wide class of models is introduced.

3.1 One-component models

The phase space of the evolutions is described by locally finite configurations $\gamma \in \Gamma$, i.e.

$$\Gamma = \{\gamma \subset \mathbb{R}^d : \ |\gamma \cap K| < \infty \ \ \forall K \subset \mathbb{R}^d \text{ bounded}\}.$$

The topology on Γ is defined as the smallest, such that all maps

$$\Gamma \ni \gamma \longmapsto \langle f, \gamma \rangle = \sum_{x \in \gamma} f(x), \quad f \in C_c(\mathbb{R}^d)$$

are continuous and Γ equipped with this topology has the structure of a polish space, c.f. [14], [1]. Here $C_c(\mathbb{R}^d)$ is the space of all continuous functions f on $\mathbb{R}^d$ with compact support. Denote by $\mathscr{B}(\Gamma)$ the corresponding Borel $\sigma-$algebra and remember that the space of all probability measurses on Γ, i.e. states of the system, is denoted by $\mathscr{P}$. The Poisson measure $\pi_\rho \in \mathscr{P}$ on Γ is defined via the Laplace transform, c.f. [1]

$$\widehat{\pi}(f) = \exp\Bigg(\int_{\mathbb{R}^d} (e^{f(x)} - 1)\rho(x)\mathrm{d}x\Bigg), \quad f \in C_c(\mathbb{R}^d),$$

where $0 \le \rho \in L^1_{loc}(\mathbb{R}^d)$. It is also possible to construct π_ρ directly using the projective structure of Γ. Since we are not going to use this construction, we will refer to [1]. The space of finite configurations $\eta \in \Gamma_0$ is

$$\Gamma_0 = \{\eta \subset \mathbb{R}^d : \ |\eta| < \infty\} = \bigsqcup_{n \in \mathbb{N}} \Gamma_0^{(n)} \tag{29}$$

with $\Gamma_0^{(n)} = \{\eta \subset \mathbb{R}^d : \ |\eta| = n\}$. Also this space can be equipped with a natural topology and the Borel $\sigma-$algebra is denoted by $\mathscr{B}(\Gamma_0)$, c.f. [13]. Denote the bijective symmetrization map by

$$\mathrm{sym}_n : \widetilde{(\mathbb{R}^d)^n} \to \Gamma_0^{(n)}, \quad (x_1, \ldots, x_n) \longmapsto \{x_1, \ldots, x_n\}$$

with $(x_1, \ldots, x_n) \in \widetilde{(\mathbb{R}^d)^n}$ if and only if $x_j \neq x_j$ for all $j \neq k$. The Lebesgue-poisson measure on Γ_0 is defined by

$$\lambda = \delta_{\{\emptyset\}} + \sum_{n=1}^{\infty} \frac{\mathrm{d}x^{(n)}}{n!},$$

where $\mathrm{d}x^{(n)}$ is the image measure of the Lebesgue measure $\mathrm{d}x^{\otimes n}$ on $(\mathbb{R}^d)^n$ under the symmetrization map sym_n. Functions on Γ_0, will be written by $G, k : \Gamma_0 \to \mathbb{R}$, whereas functions on Γ are denoted by $F : \Gamma \to \mathbb{R}$. From (29) we conclude that each function k respectively $G : \Gamma_0 \longrightarrow \mathbb{R}$ has a decomposition to a sequence of symmetric functions $k = (k^{(n)})_{n=0}^{\infty}$ respectively $G = (G^{(n)})_{n=0}^{\infty}$. There is a combinatorial operator similar to Fourier transform translating functions $G : \Gamma_0 \longrightarrow \mathbb{R}$ to functions $F : \Gamma \longrightarrow \mathbb{R}$. This Transformation is called K-transform, see [13], and is defined by

$$(KG)(\gamma) = \sum_{\eta \Subset \gamma} G(\eta). \tag{30}$$

Here the symbol $\Subset$ means, that the summation is taken only about all finite configurations $\eta \subset \gamma$. The inverse map K^{-1} has the form

$$(K^{-1}G)(\eta) = \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} G(\xi).$$

Expression (30) is well-defined for instance for bounded functions G having bounded support, i.e. there is $\Lambda \subset \mathbb{R}^d$ compact, $N \in \mathbb{N}$ and $C > 0$ such that $|G(\eta)| \leq C$, and for all $\eta \in \Gamma_0$ with $|\eta| > N$ or $\eta \not\subset \Lambda$ one has $G(\eta) = 0$. In such case KG is a cylindrical function on Γ, for details see [13].

Next introduce a convolution for measurable functions $G, H : \Gamma_0 \longrightarrow \mathbb{R}$ via

$$(G \star H)(\eta) = \sum_{\xi \subset \eta} \sum_{\zeta \subset \xi} G(\xi) H(\eta \setminus \xi \cup \zeta). \tag{31}$$

This convolution will satisfy a similar property to the Fourier transform of functions, namely

$$(KG)(KH) = K(G \star H), \tag{32}$$

provided $G, H \in L^1(\Gamma_0, d\lambda)$. This transformation allows us to associate to each probability measure $\mu \in \mathscr{P}(\Gamma)$ with finite local moments, i.e.

$$\int_{\Gamma} |\gamma \cap \Lambda|^n \mu(\mathrm{d}\gamma) < \infty$$

for all compacts $\Lambda \subset \mathbb{R}^d$, a locally finite measure ρ_μ on Γ_0 via an extension of the relation

$$\rho_\mu(A) = \int_{\Gamma} (K1_A)(\gamma) \mu(\mathrm{d}\gamma), \quad A \in \mathscr{B}(\Gamma_0).$$

Let us assume, that ρ_μ is absolutely continuous with respect to the Lebesgue-Poisson measure λ. then the Radon-Nikodym derivative $k_\mu = \dfrac{\mathrm{d}\rho_\mu}{\mathrm{d}\lambda}$ is the correlation function defined in (2) (corresponding to the measure μ). Conversely given a function $k : \Gamma_0 \longrightarrow \mathbb{R}$ the following inverse statement for the construction of a measure $\mu \in \mathscr{P}$ from k holds. The proof can be found in [13].

Theorem 3.1. *Assume that k is positive definite in the sense that*

$$\int_{\Gamma_0} G(\eta)k(\eta)\mathrm{d}\lambda(\eta) \geq 0 \tag{33}$$

for all G bounded with bounded support, such that $KG \geq 0$. Then there exists a probability measure μ on Γ with correlation function k.

The Lebesgue-Poisson exponential $e_\lambda(f;\eta) := \prod_{x\in\eta} f(x)$ satisfy the combinatorial formula $Ke_\lambda(f) = e_\lambda(f+1)$, i.e.

$$\sum_{\xi\subset\eta} e_\lambda(\rho;\xi) = e_\lambda(\rho+1;\eta).$$

The following equality will be useful for several computations

$$\int_{\Gamma_0} e_\lambda(\rho;\eta)\mathrm{d}\lambda(\eta) = \exp\Big(\int_{\mathbb{R}^d} \rho(x)\mathrm{d}x\Big).$$

Let us take $f \in C_c(\mathbb{R}^d)$ and compute on the one-hand-side

$$\int_{\Gamma} e^{\langle f,\gamma\rangle}\mathrm{d}\pi_\rho(\gamma) = \exp\Big(\int_{\mathbb{R}^d} (e^{f(x)}-1)\rho(x)\mathrm{d}x\Big) = \int_{\Gamma_0} e_\lambda(e^f-1)e_\lambda(\rho)\mathrm{d}\lambda$$
$$= \int_{\Gamma_0} Ke_\lambda(e^f)e_\lambda(\rho)\mathrm{d}\lambda$$

thus

$$\int_{\Gamma} e^{\langle f,\gamma\rangle}\mathrm{d}\pi_\rho(\gamma) = \int_{\Gamma_0} Ke_\lambda(e^f)e_\lambda(\rho)\mathrm{d}\lambda,$$

which shows that the correlation measure for π_ρ is given by $e_\lambda(\rho)\mathrm{d}\lambda$. Finally we will explain the approach to describe statistical dynamics on this spaces, i.e. the approach to analyse the evolution $t \longmapsto \mu_t$. So let us start with a heuristic Markov generator L, e.g. (1) or (13). In the general framework of Markov processes one would study the evolution of observables, i.e. solutions to the equation

$$\frac{\partial F_t}{\partial t} = LF_t, \quad F_t|_{t=0} = F_0.$$

Its solution can give the possibility to construct under certain conditions a Markov process $(X_t^\gamma)_{t\geq 0}$ such that

$$F_t(\gamma) = \mathbb{E}^\gamma(F_0(X_t)).$$

Alternatively we can try to investigate the equation for measures μ_t, c.f. (3). But since we are dealing with infinite configurations, both approaches are very difficult and it was possible to realize them only in a few examples, c.f. [16]. Instead one tries to rewrite the equation using the $K-$transform to an equation for functions on Γ_0 and investigate this equation. This approach should be interpreted as a change of variables, so we define the operator $\widehat{L} = K^{-1}LK$, which acts now on functions $G : \Gamma_0 \longrightarrow \mathbb{R}$ and try to solve the Cauchy problem

$$\frac{\partial G_t}{\partial t} = \widehat{L}G_t, \quad G_t|_{t=0} = G_0.$$

In this article we will investigate this equation in one of the following Banach spaces

$$\mathbb{B}_\alpha = L^1(\Gamma_0, e^{\alpha|\cdot|}\mathrm{d}\lambda)$$

with $\alpha \in \mathbb{R}$ and the norm given by

$$\|G\|_\alpha = \int\limits_{\Gamma_0} |G(\eta)|e^{\alpha|\eta|}\mathrm{d}\lambda(\eta) = \sum_{n=0}^{\infty} \frac{e^{\alpha n}}{n!} \int\limits_{(\mathbb{R}^d)^n} |G^{(n)}(x_1, \dots, x_n)|\mathrm{d}x_1 \dots \mathrm{d}x_n.$$

An evolution $t \longmapsto G_t \in \mathbb{B}_\alpha$ determines a dual evolution $t \longmapsto k_t^D$ by

$$\int\limits_{\Gamma_0} G_t(\eta)k_0(\eta)\mathrm{d}\lambda(\eta) = \int\limits_{\Gamma_0} G_0(\eta)k_t^D(\eta)\mathrm{d}\lambda(\eta)$$

and since $G_t \in \mathbb{B}_\alpha$ this dual evolution will obey the Ruelle bound

$$|k_t^D(\eta)| \leq Ce^{\alpha|\eta|}, \quad \eta \in \Gamma_0$$

and hence be sub-poissonian. As already mentioned such an evolution describes a system, which is not asymptotically clustering, but still could include some pattern formation. It is also possible to study the equation for k_t directly, therefore using duality it is possible to compute the expression for $L^\triangle$ directly via

$$\int\limits_{\Gamma_0} (\widehat{L}G)(\eta)k(\eta)\mathrm{d}\lambda(\eta) = \int\limits_{\Gamma_0} G(\eta)(L^\triangle k)(\eta)\mathrm{d}\lambda(\eta)$$

for each function G bounded with bounded support and k locally integrable. One special case was computed already for the first and second correlation functions. Finally one would seek for a solution to

$$\frac{\partial k_t}{\partial t} = L^\triangle k_t, \quad k_t|_{t=0} = k_0$$

and construct if possible the evolution of states $t \longmapsto \mu_t$. This sketch has to be realized for each model separately, like all operators $L, \widehat{L}, L^\triangle$ have to be defined on a proper set of functions, which is large enough to determine the evolution of states. Note that even if we have solved the equation (4) it is not clear, whether the evolution $t \longmapsto k_t$ really determines an evolution of states $t \longmapsto \mu_t$ and therefore is of biological interest. Such task has to be carefully proved and was realized for several important models, c.f. [15]. The main problem is that the evolution has to be positive definite, c.f. (33).

3.2 Two-component models

Let us now outline the major differences of two-component models. Afterwards it will be clear how to extend all considerations to models with any number of components $n \in \mathbb{N}$. First of all let us denote by $+$ respectively $-$ the types of cells and by γ^+ and γ^- their (locally finite) configurations. Since no cells of different type can be located at the same position we will assume $\gamma^+ \cap \gamma^- = \emptyset$, therefore

$$\Gamma^2 = \{(\gamma^+, \gamma^-) : \quad \gamma^+, \gamma^- \in \Gamma, \quad \gamma^+ \cap \gamma^- = \emptyset\}.$$

Similarly the space of finite configurations Γ_0^2 and the topologies on these spaces are defined. Since for each $\xi \in \Gamma_0$ the set

$$\{\eta \in \Gamma_0 : \quad \eta \cap \xi \neq \emptyset\}$$

is a set of measure zero with respect to λ we can define the Lebesgue Poisson measure λ^2 on Γ_0^2 as the product measure $\lambda \otimes \lambda$ and calculate as in the one-component case. Similarly the Poisson measure will be the product measure of two copies of π. The K-transform is a composition of two K-transforms for each type of cells, i.e.

$$(KG)(\gamma^+, \gamma^-) = \sum_{\eta^+ \Subset \gamma^+} \sum_{\eta^- \Subset \gamma^-} G(\eta^+, \eta^-)$$

and K^{-1} is just

$$(K^{-1}F)(\eta^+, \eta^-) = \sum_{\xi^+ \subset \eta^+} \sum_{\xi^- \subset \eta^-} (-1)^{|\eta^+ \setminus \xi^+|} (-1)^{|\eta^- \setminus \xi^-|} F(\xi^+, \xi^-).$$

The Lebesgue-Poisson exponential will be the product of the Lebesgue-Poisson exponentials for each type of cells and the correlation functions become a double indexed vector, i.e. $k^{(n,m)}(x_1, \ldots, x_n; y_1, \ldots y_m)$. The heuristic Markov generator L now acts on functions $F : \Gamma^2 \longrightarrow \mathbb{R}$ and L^Δ on collections of correlation functions $k = (k^{(n,m)})_{n,m=0}^\infty$.

3.3 Mesoscopic scaling

As before the approach to derive the kinetic description, respectively the mesoscopic equation, can be described within three steps. In the first step one rescales the potentials, and thus the generator L. The outcome is a new system with smaller interactions, with generator denoted by L_ε. In the second step we will choose some admissible class of initial states $k_{0,\varepsilon} = (k_{0,\varepsilon}^{(n)})_{n=0}^\infty$ such that

$$\varepsilon^{|\eta|} k_{0,\varepsilon}(\eta) \to r_0(\eta), \quad \varepsilon \to 0$$

for each $\eta \in \Gamma_0$. Finally let $k_{t,\varepsilon}$ be the solution of

$$\frac{\partial k_{t,\varepsilon}}{\partial t}(\eta) = L_\varepsilon^\Delta k_{t,\varepsilon}(\eta), \tag{34}$$

where L_ε^Δ is the adjoint operator to $\widehat{L}_\varepsilon = K^{-1} L_\varepsilon K$. We will seek for a limit

$$\varepsilon^{|\eta|} k_{t,\varepsilon}(\eta) \to r_t(\eta), \quad \varepsilon \to 0$$

for each $\eta \in \Gamma_0$ and t. Such limit implies that

$$\varepsilon^n k_{t,\varepsilon}^{(n)}(x_1, \ldots, x_n) \to \rho_t(x_1) \cdots \rho_t(x_n), \quad \varepsilon \to 0 \tag{35}$$

if $r_0^{(n)}(x_1, \ldots, x_n) = \rho_0(x_1) \cdots \rho_n(x_n)$ for all $n \in \mathbb{N}$. This is equivalent to solve the equations

$$\frac{\partial k_{t,\varepsilon}^{ren}}{\partial t} = L_{\varepsilon,ren}^{\Delta} k_{t,\varepsilon}^{ren}, \quad k_{t,\varepsilon}^{ren} = R_\varepsilon k_{0,\varepsilon}$$

and seek for the limits

$$\lim_{\varepsilon \to 0} k_{t,\varepsilon}^{ren} = r_t, \tag{36}$$

where r_t solves the equation

$$\frac{\partial r_t}{\partial t} = L_V^{\Delta} r_t, \quad r_t|_{t=0} = r_0. \tag{37}$$

Here

$$L_{\varepsilon,ren}^{\Delta} = R_\varepsilon L_\varepsilon^{\Delta} R_\varepsilon^{-1} \to L_V^{\Delta} \tag{38}$$

and $(R_\varepsilon k)(\eta) = \varepsilon^{|\eta|} k(\eta)$. Summarizing this approach, we first rescale the system and arrive at an expression for the operator $L_{\varepsilon,ren}^{\Delta}$. Form this one computes the expression for L_V^{Δ}. Finally putting $r_0 = e_\lambda(\rho_0)$ in equation (37) one deduces the kinetic description

$$\frac{\partial \rho_t}{\partial t} = v(\rho_t), \quad \rho_t|_{t=0} = \rho_0. \tag{39}$$

The analysis of (36) is quite hard and needs several technical tool and such problem should be solved for each model separately. Nevertheless it is important, since it relates the mesoscopic evolution as the limiting evolution of the microscopic evolution. This means for instance, that starting with $r_0(x_1, \ldots, x_n) = \rho_0(x_1) \cdots \rho_0(x_n)$, and denoting by ρ_t the solution to (39), we get for all $n \in \mathbb{N}$ (35). The precise notion of convergence should be chosen adequately to the model. In this work we will focus on (38) and compute equations (37) and (39) for several models. However, convergence of equations (38) does not imply convergence of solutions, i.e. (36), thus it is important to determine conditions which imply $k_{t,\varepsilon}^{ren} \to r_t$. If such convergence happens to be false in some case, then we know that this kinetic description, also if it is well analysed, will not describe the original model and hence has no biological significance.

4 One-component systems

Within this section we will prove the results stated in the previous section and derive for many possible individual based interactions their related operators on quasi-observables, correlation functions and the kinetic description. The main technical tools introduced in the last section will be applied for each model directly.

We will work in scales of Banach spaces defined for $\alpha \in \mathbb{R}$ by $\mathbb{B}_\alpha = L^1(\Gamma_0, e^{\alpha|\cdot|}\mathrm{d}\lambda)$, i.e. equivalence classes of measurable functions $G : \Gamma_0 \longrightarrow \mathbb{R}$ satisfying

$$\begin{aligned}\|G\|_\alpha &= \int_{\Gamma_0} |G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta) \\ &= \sum_{n=0}^{\infty} \frac{e^{\alpha n}}{n!} \int_{(\mathbb{R}^d)^n} |G^{(n)}(x_1, \dots, x_n)| \mathrm{d}x_1 \dots \mathrm{d}x_n < \infty. \end{aligned} \tag{40}$$

The dual space is given by $\mathbb{B}_\alpha^* = L^\infty(\Gamma_0, e^{-\alpha|\cdot|}\mathrm{d}\lambda)$, so measurable functions $k : \Gamma_0 \longrightarrow \mathbb{R}$ such that

$$\|k\|_\alpha = \operatorname*{ess\,sup}_{\eta \in \Gamma_0} |k(\eta)| e^{-\alpha|\eta|} < \infty. \tag{41}$$

The duality pairing is simply

$$\langle G, k \rangle = \int_{\Gamma_0} G(\eta) k(\eta) \mathrm{d}\lambda(\eta) \tag{42}$$

and satisfies $|\langle G, k\rangle| \leq \|G\|_\alpha \|k\|_\alpha$. Let $\widehat{L} \in L(\mathbb{B}_\alpha, \mathbb{B}_{\alpha'})$ for each $\alpha' < \alpha$ and L^Δ the dual operator with respect to (42). Then

$$\|\widehat{L}\|_{\alpha\alpha'} = \|L^\Delta\|_{\alpha'\alpha} \tag{43}$$

where the norms are determined by (40) and (41). Consequently for several aspects it is enough to analyse only the operator $\widehat{L}$. It is possible to assign to each $\widehat{L}$ a measurable function $M_\alpha : \Gamma_0 \longrightarrow \mathbb{R}_+$ such that

$$\|\widehat{L}G\|_\alpha = \int_{\Gamma_0} |\widehat{L}G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta) \leq \int_{\Gamma_0} |G(\eta)| M_\alpha(\eta) e^{\alpha|\eta|} \mathrm{d}\lambda(\eta) = \|M_\alpha G\|_\alpha.$$

The operator $(\widehat{L}, D(M_\alpha))$ is well-defined on

$$D(M_\alpha) = \{G \in \mathbb{B}_\alpha : \ M_\alpha \cdot G \in \mathbb{B}_\alpha\} \tag{44}$$

and if $M_\alpha(\eta) \leq P_\alpha(|\eta|) e^{\delta|\eta|}$ with some polynomial P_α and $\delta > 0$, then the estimate

$$|\eta|^k e^{-\delta|\eta|} \leq \left(\frac{k}{e\delta}\right)^k \tag{45}$$

implies $\widehat{L} \in L(\mathbb{B}_\alpha, \mathbb{B}_{\alpha'})$ for all $\alpha - \alpha' > \delta$. From (43) the same estimate is valid for L^Δ. Practically we have only to determine the expression for M_α and analyse its growth. Concerning the construction of microscopic dynamics via semigroups on the scale of Banach spaces $\mathbb{B}_\alpha$ one would compute the function $D_\alpha : \Gamma_0 \longrightarrow \mathbb{R}_+$ given by

$$\int_{\Gamma_0} (\widehat{L}G)(\eta) e^{\alpha|\eta|} \mathrm{d}\lambda(\eta) = \int_{\Gamma_0} G(\eta) D_\alpha(\eta) e^{\alpha|\eta|} \mathrm{d}\lambda(\eta).$$

By (42) it means $(L^{\Delta}e_{\lambda}(e^{\alpha}))(\eta) = D_{\alpha}(\eta)$, and analyse its properties. In many cases both functions M_{α} and D_{α} have a simple relation, but M_{α} is not unique. Similarly define $D_{\Delta}(M_{\alpha}) \subset \mathbb{B}^*_{\alpha}$ as the set of all $k \in \mathbb{B}^*_{\alpha}$ such that $M_{\alpha}k \in \mathbb{B}^*_{\alpha}$. Then L^{Δ} is well-defined on $D_{\Delta}(M_{\alpha})$.

Within the mesoscopic scaling we will consider the rescaled operators $\widehat{L}_{\varepsilon,ren}$ and $L^{\Delta}_{\varepsilon,ren}$. Denote by N_{α} the function determined by

$$\int_{\Gamma_0} |\widehat{L}_{\varepsilon,ren}G(\eta)|e^{\alpha|\eta|}d\lambda(\eta) \leq \int_{\Gamma_0} N_{\alpha}(\eta)|G(\eta)|e^{\alpha|\eta|}d\lambda(\eta).$$

Note, that in general such function does not need to exist, it will be necessary to show that for all reasonable models under some general assumptions we can find the function N_{α}. As before the operators $(\widehat{L}_{\varepsilon,ren}, D(N_{\alpha}))$ and $(L^{\Delta}_{\varepsilon,ren}, D_{\Delta}(N_{\alpha}))$ are well-defined. The limiting operator $\widehat{L}_V$ given by $\widehat{L}_{\varepsilon,ren} \to \widehat{L}_V$ as $\varepsilon \to 0$ determines another function N^V_{α} via

$$\int_{\Gamma_0} |\widehat{L}_V G(\eta)|e^{\alpha|\eta|}\mathrm{d}\lambda(\eta) \leq \int_{\Gamma_0} N^V_{\alpha}(\eta)|G(\eta)|e^{\alpha|\eta|}\mathrm{d}\lambda(\eta).$$

So we define $(\widehat{L}_V, D(N^V_{\alpha}))$ and $(L^{\Delta}_V, D_{\Delta}(N^V_{\alpha}))$.

Theorem 4.1. *For all subsequent interactions, the following holds.*

1. *For any $G \in D(N_{\alpha}) \cap D(N^V_{\alpha})$ the convergence*

$$\widehat{L}_{\varepsilon,ren}G \to \widehat{L}_V G, \quad \varepsilon \to 0$$

holds. If in addition there is a polynomial P and $\delta > 0$ such that $N_{\alpha}(\eta), N^V_{\alpha}(\eta) \leq P(|\eta|)e^{\delta|\eta|}$, then $\widehat{L}, \widehat{L}_{\varepsilon,ren}$ and $\widehat{L}_V$ act as bounded linear operators in $L(\mathbb{B}_{\alpha}, \mathbb{B}_{\alpha'})$ for any $\alpha - \alpha' > \delta$ and

$$\|\widehat{L}_{\varepsilon,ren} - \widehat{L}_V\|\alpha\alpha' \to 0, \quad \varepsilon \to 0.$$

2. *For any $k \in D_{\Delta}(N_{\alpha}) \cap D_{\Delta}(N^V_{\alpha})$ the convergence*

$$L^{\Delta}_{\varepsilon,ren}k \to L^{\Delta}_V k, \quad \varepsilon \to 0$$

holds. And if in addition there is a polynomial P and $\delta > 0$ such that $N_{\alpha}(\eta), N^V_{\alpha}(\eta) \leq P(|\eta|)e^{\delta|\eta|}$, then $L^{\Delta}, L^{\Delta}_{\varepsilon,ren}$ and L^{Δ}_V act as bounded linear operators in $L(\mathbb{B}^*_{\alpha'}, \mathbb{B}^*_{\alpha})$ for any $\alpha - \alpha' > \delta$ and

$$\|L^{\Delta}_{\varepsilon,ren} - L^{\Delta}_V\|\alpha'\alpha \to 0, \quad \varepsilon \to 0.$$

3. *If ρ_t is a solution to the corresponding kinetic description determined by L^{Δ}_V, c.f.* (39)*, then $e_{\lambda}(\rho_t)$ is a solution to the Cauchy problem associated to L^{Δ}_V, i.e. solves the Cauchy problem* (37)*.*

In the following denote by $E(x, \gamma\backslash x) = \sum_{y\in\gamma\backslash x} \phi(x-y)$ the relative energy of the cell x with respect to the rest of the configuration $\gamma\backslash x$. Here $0 \leq \phi \in L^1(\mathbb{R}^d)$ is assumed to be symmetric. For infinite configurations such sum will be infinite in general, but e.g. for the Poisson measure it is possible to define $E(x, \gamma\backslash x)$ for almost all $\gamma \in \Gamma$, such that this sum is convergent.

We will use also the following well-known result.

Lemma 4.2. *Let $H : \mathbb{R}^d \times \Gamma_0 \longrightarrow \mathbb{R}$ and $G : \Gamma_0 \times \Gamma_0 \times \Gamma_0 \longrightarrow \mathbb{R}$ be measurable, then the following formulas hold, provided one side of the corresponding equality exists*

$$\int_{\Gamma_0} \sum_{x\in\eta} H(x,\eta) d\lambda(\eta) = \int_{\Gamma_0} \int_{\mathbb{R}^d} H(x, \eta \cup x) dx d\lambda(\eta)$$

and

$$\int_{\Gamma_0} \sum_{\xi\subset\eta} H(\xi, \eta\backslash\xi, \eta) d\lambda(\eta) = \int_{\Gamma_0} \int_{\Gamma_0} H(\xi, \eta, \eta \cup \xi) d\lambda(\xi) d\lambda(\eta).$$

4.1 Death dynamics

Let us investigate here the dynamics of the microscopic event death.

Example 2 (constant mortality)**.** The Markov generator has here the form

$$(LF)(\gamma) = \sum_{x\in\gamma} m(x)(F(\gamma\backslash x) - F(\gamma)),$$

where $0 \leq m \in L^\infty_{loc}(\mathbb{R}^d)$. Each cell located in position $x \in \mathbb{R}^d$ has an exponential distributed liftetime with parameter $m(x)$. In the case when $m(x) = 0$, the cell will not die due to this mechanism. The operator $\widehat{L}$ on quasi-observables has the form

$$(\widehat{L}G)(\eta) = -\sum_{x\in\eta} m(x)G(\eta)$$

and likewise L^Δ is given by the same expression. Moreover we see that it is possible to take $M_\alpha(\eta) = N_\alpha(\eta) = N_\alpha^V(\eta) = \sum\limits_{x\in\eta} m(x)$. Since here no scaling is necessary we obtain $\widehat{L}_V = \widehat{L}$ and $L_V^\Delta = L_V$. Consequently the kinetic description is simply

$$\frac{\partial\rho_t}{\partial t}(x) = -m(x)\rho_t(x).$$

Example 3 (quadratic mortality)**.** The Markov generator is given by

$$(LF)(\gamma) = \sum_{x\in\gamma} E(x, \gamma\backslash x)(F(\gamma\backslash x) - F(\gamma)).$$

Here each cell located at position $x \in \mathbb{R}^d$ may die, where the intensity of death is given by the intensity $\sum\limits_{y\in\gamma\backslash x} \phi(x - y)$, i.e. the death of the cell is caused by interaction with another cell located at position $y \in \gamma\backslash x$. The case where $y = x$ is already included in the constant mortality $m = m(x)$. The operator for quasi-observables is now given by

$$(\widehat{L}G)(\eta) = -\sum_{x\in\eta} E(x, \eta\backslash x)G(\eta) - \sum_{x\in\eta} E(x, \eta\backslash x)G(\eta\backslash x)$$

and the operator on correlation functions by

$$(L^{\triangle}k)(\eta) = -\sum_{x\in\eta} E(x,\eta\backslash x)k(\eta) - \sum_{x\in\eta}\int_{\mathbb{R}^d} \phi(x-y)k(\eta\cup y)dy.$$

Similarly we can choose $M_\alpha(\eta) = N_\alpha(\eta) = \sum_{x\in\eta} E(x,\eta\backslash x) + \langle a\rangle e^\alpha|\eta|$. Within the scaling and after renormalization we arrive at new operators, where only the multiplicative part will be multiplied by $\varepsilon > 0$. Hence after limit transition $\varepsilon\to 0$ we obtain the operators for the Vlasov hierarchy given by

$$(\widehat{L}_V G)(\eta) = -\sum_{x\in\eta}\sum_{y\in\eta\backslash x} a^-(x-y)G(\eta\backslash x)$$

and likewise

$$(L_V^{\triangle}k)(\eta) = -\sum_{x\in\eta}\int_{\mathbb{R}^d} a^-(x-y)k(\eta\cup y)\mathrm{d}y$$

so that $N_\alpha^V(\eta) = e^\alpha\langle a^-\rangle|\eta|$. Finally the kinetic description is given by

$$\frac{\partial\rho_t}{\partial t}(x) = -\rho_t(x)(a^- * \rho_t)(x).$$

Example 4. Let us look at the stronger death intensity described by the Markov generator

$$(LF)(\gamma) = \sum_{x\in\gamma} e^{E(x,\gamma\backslash x)}(F(\gamma\backslash x) - F(\gamma)),$$

Here each particle located at position $x\in\mathbb{R}^d$ may die, whereas the intensity of such microscopic event is given by $e^{E(x,\gamma\backslash x)}$, in the case of $E(x,\gamma\backslash x) = \infty$ one can think of immediate death. The corresponding operator on quasi-observables is given by

$$(\widehat{L}G)(\eta) = -\sum_{\xi\subset\eta} G(\xi)\sum_{x\in\xi} e^{E(x,\xi\backslash x)} e_\lambda(e^{\phi(x-\cdot)} - 1;\eta\backslash\xi)$$

and on correlation functions by

$$(L^{\triangle}k)(\eta) = -\sum_{x\in\eta} e^{E(x,\eta\backslash x)}\int_{\Gamma_0} e_\lambda(e^{\phi(x-\cdot)} - 1;\xi)k(\eta\cup\xi)\mathrm{d}\lambda(\xi).$$

We can choose the function $M_\alpha(\eta) = \beta_1(\alpha)\sum_{x\in\eta} e^{E(x,\eta\backslash x)}$, where

$$\beta_1(\alpha) = \exp\Big(e^\alpha\int_{\mathbb{R}^d}(e^{\phi(x)} - 1)\mathrm{d}x\Big).$$

For the mesoscopic scaling let us rescale the potential as $\phi\to\varepsilon\phi$ and after renormalization we arrive at

$$(\widehat{L}_{\varepsilon,ren}G)(\eta) = -\sum_{\xi\subset\eta} G(\xi)\sum_{x\in\xi} e^{\varepsilon E(x,\xi\backslash x)} e_\lambda\left(\frac{e^{\varepsilon\phi(x-\cdot)} - 1}{\varepsilon};\eta\backslash\xi\right)$$

and

$$(L^{\Delta}_{\varepsilon,ren}k)(\eta) = -\sum_{x\in\eta} e^{\varepsilon E(x,\eta\setminus x)} \int_{\Gamma_0} e_\lambda\left(\frac{e^{\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\xi\right) k(\eta\cup\xi)\mathrm{d}\lambda(\xi).$$

After limit transition $\varepsilon \to 0$ we arrive at

$$(\widehat{L}_V G)(\eta) = -\sum_{\xi\subset\eta} G(\xi) \sum_{x\in\xi} e_\lambda(\phi(x-\cdot);\eta\setminus\xi)$$

and

$$(L^{\Delta}_V k)(\eta) = -\sum_{x\in\eta}\int_{\Gamma_0} e_\lambda(\phi(x-\cdot);\xi)k(\eta\cup\xi)\mathrm{d}\lambda(\xi).$$

Here we can take $N_\alpha(\eta) = \beta(\alpha)\sum\limits_{x\in\eta} e^{E(x,\eta\setminus x)}$, where we have to assume that

$$\beta(\alpha) = \sup_{\varepsilon\in(0,1]} \exp\left(\frac{e^\alpha}{\varepsilon}\int_{\mathbb{R}^d} |e^{\varepsilon\phi(x)}-1|\mathrm{d}x\right) < \infty.$$

Finally, $N^V_\alpha(\eta) = \exp(e^\alpha\langle\phi\rangle)|\eta|$ and for the kinetic description

$$\frac{\partial\rho_t}{\partial t}(x) = -\rho_t(x)e^{(\phi*\rho_t)(x)}.$$

4.2 Birth dynamics

Here we will describe the microscopic event responsible for the appearance of new cells.

Example 5 (Sourgailis birth)**.** The most simple form of birth, is where in each region $\Lambda\subset\mathbb{R}^d$ the intensity that a new cell appear in Λ is given by $\int\limits_\Lambda z(x)\mathrm{d}x$, where $0\le z\in L^1_{loc}(\mathbb{R}^d)$ is the intensity. Each such event is independent of the other and describes thus free growth of the system. In this case the Markov generator is given by

$$(LF)(\gamma) = \int_{\mathbb{R}^d} z(x)(F(\gamma\cup x)-F(\gamma))\mathrm{d}x$$

and on quasi-observables by

$$(\widehat{L}G)(\eta) = \int_{\mathbb{R}^d} z(x)G(\eta\cup x)\mathrm{d}x.$$

For correlation functions the adjoint operator is given by

$$(L^{\Delta}k)(\eta) = \sum_{x\in\eta} z(x)k(\eta\setminus x).$$

Take $M_\alpha(\eta) = N_\alpha(\eta) = N^V_\alpha(\eta) = e^{-\alpha}\sum\limits_{x\in\eta} z(x)$. Since scaling will not affect this operators, we immediately arrive at the kinetic description given by

$$\frac{\partial\rho_t}{\partial t}(x) = z(x).$$

Example 6 (Gibbs-type birth)**.** Let us assume that L is of the form

$$(LF)(\gamma) = z \int_{\mathbb{R}^d} e^{-E(x,\gamma)}(F(\gamma \cup x) - F(\gamma))\mathrm{d}x,$$

where $z > 0$. The creation of cells in some volume $\Lambda \subset \mathbb{R}^d$ is given by the intensity $\int_\Lambda ze^{-E(x,\gamma)}dx \leq z|\Lambda|$, where $|\Lambda|$ denotes the Lebesgue volume of Λ. The operator for quasi-observables is given by

$$(\widehat{L}G)(\eta) = z \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} e_\lambda(e^{-\phi(x-\cdot)} - 1; \eta \backslash \xi) e^{-E(x,\xi)} G(\xi \cup x)\mathrm{d}x$$

and for correlation functions by

$$(L^\Delta k)(\eta) = z \sum_{x \in \eta} e^{-E(x,\eta\backslash x)} \int_{\Gamma_0} e_\lambda(e^{-\phi(x-\cdot)} - 1; \xi) k(\eta \backslash x \cup \xi) d\lambda(\xi).$$

Here we can take $M_\alpha(\eta) = \beta(\alpha) \sum_{x \in \eta} e^{-E(x,\eta\backslash x)}$, where

$$\beta(\alpha) = \exp\Big(e^\alpha \int_{\mathbb{R}^d} |1 - e^{-\phi(x)}|\mathrm{d}x\Big)$$

and $N_\alpha(\eta) = \exp(e^\alpha\langle\phi\rangle) \sum_{x \in \eta} e^{-E(x,\eta\backslash x)}$. After scaling and renormalization we will arrive at

$$(\widehat{L}_\varepsilon G)(\eta) = z \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} e_\lambda \left(\frac{e^{-\varepsilon\phi(x-\cdot)} - 1}{\varepsilon}; \eta \backslash \xi \right) e^{-\varepsilon E(x,\xi)} G(\xi \cup x)\mathrm{d}x$$

which tends in the limit $\varepsilon \to 0$ to

$$(L_V G)(\eta) = z \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} e_\lambda \left(\phi(x - \cdot); \eta \backslash \xi\right) G(\xi \cup x)\mathrm{d}x.$$

In the same way we obtain

$$(L_\varepsilon^\Delta k)(\eta) = z \sum_{x \in \eta} e^{-\varepsilon E(x,\eta\backslash x)} \int_{\Gamma_0} e_\lambda \left(\frac{e^{-\varepsilon\phi(x-\cdot)} - 1}{\varepsilon}; \xi \right) k(\eta \backslash x \cup \xi)\mathrm{d}\lambda(\xi)$$

and hence when $\varepsilon \to 0$

$$(L_V^\Delta k)(\eta) = z \sum_{x \in \eta} \int_{\Gamma_0} e_\lambda(\phi(x - \cdot); \xi) k(\eta \backslash x \cup \xi)\mathrm{d}\lambda(\xi).$$

The function N_α^V can be chosen as $N_\alpha^V(\eta) = z \exp(e^\alpha\langle\phi\rangle)e^{-\alpha}|\eta|$. Therefore the kinetic description is given by

$$\frac{\partial \rho_t}{\partial t}(x) = ze^{(\phi * \rho_t)(x)}.$$

Example 7 (free branching)**.** In the simplest way free branching is described by

$$(LF)(\gamma) = \sum_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} a^+(x-y)(F(\eta \cup y) - F(\eta))\mathrm{d}y,$$

where $0 \le a^+ \in L^1(\mathbb{R}^d)$ is symmetric. Here each cell located at position $x \in \gamma$ may create a new cell located at position $y \in \mathbb{R}^d$. The intensity of such event is given by $\langle a^+ \rangle = \int_{\mathbb{R}^d} a^+(z)\mathrm{d}z$ and the new particle is distributed according to the probability measure $\frac{1}{\langle a^+ \rangle} a^+(x-y)\mathrm{d}y$. On the level of quasi-observables this effect is described via

$$(\widehat{L}G)(\eta) = \sum_{x \in \eta} \int_{\mathbb{R}^d} a^+(x-y)G(\eta \backslash x \cup y)\mathrm{d}y + \sum_{x \in \eta} \int_{\mathbb{R}^d} a^+(x-y)G(\eta \cup y)\mathrm{d}y.$$

Likewise on correlation functions it is given by

$$(L^{\Delta}k)(\eta) = \sum_{x \in \eta} \int_{\mathbb{R}^d} a^+(x-y)k(\eta \backslash x \cup y)\mathrm{d}y + \sum_{x \in \eta} \sum_{y \in \eta \backslash x} a^+(x-y)k(\eta \backslash x).$$

It is sufficient to take $M_\alpha(\eta) = N_\alpha(\eta) = e^{-\alpha} \sum_{x \in \eta} \sum_{y \in \eta \backslash x} a^+(x-y) + \langle a^+ \rangle |\eta|$. Here we can explicitly compute the correlation functions, which will be done later on. After scaling and renormalization we observe that only the second summands will be multiplied by $\varepsilon > 0$. Hence in the limit we arrive at

$$(\widehat{L}_V G)(\eta) = \sum_{x \in \eta} \int_{\mathbb{R}^d} a^+(x-y)G(\eta \backslash x \cup y)\mathrm{d}y$$

and likewise for correlation functions $L^{\Delta}k$ is given by the same expression, namely we can chose $N_\alpha^V(\eta) = \langle a^+ \rangle |\eta|$. For the kinetic description we obtain

$$\frac{\partial \rho_t}{\partial t}(x) = (a^+ * \rho_t)(x).$$

Similarly we can also consider the case, where each cell at $x \in \gamma$ may split into two new cells at positions y_1, y_2. The intensity of such transition would be, for simplicity, again constant $\langle a^+ \rangle$. The probability distribution is given by $\frac{1}{\langle a^+ \rangle} a^+(x-y_1, x-y_2)\mathrm{d}y_1\mathrm{d}y_2$, where $0 \le a^+ \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$ is symmetric in both variables. The Markov generator is of the form

$$(LF)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a^+(x-y_1, x-y_2)(F(\gamma \backslash x \cup y_1 \cup y_2) - F(\gamma))\mathrm{d}y_1\mathrm{d}y_2.$$

For quasi-observables this yields

$$\begin{aligned}(\widehat{L}G)(\eta) = &\sum_{x \in \eta} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a^+(x-y_1, x-y_2)G(\eta \backslash x \cup y_1 \cup y_2)\mathrm{d}y_1\mathrm{d}y_2 \\ &+ \sum_{x \in \eta} \int_{\mathbb{R}^d} b(x-y)(G(\eta \backslash x \cup y) - G(\eta))\mathrm{d}y + \langle a^+ \rangle |\eta| G(\eta),\end{aligned}$$

where $b(x) = \int_{\mathbb{R}^d} a(x,y)\mathrm{d}y + \int_{\mathbb{R}^d} a(y,x)\mathrm{d}y$. Likewise we can compute the adjoint operator, which is given by

$$(L^{\triangle}k)(\eta) = \sum_{y_1\in\eta}\sum_{y_2\in\eta\setminus y_1}\int_{\mathbb{R}^d} a^+(x-y_1,x-y_2)k(\eta\cup x\setminus y_1\setminus y_2)dx$$
$$+\sum_{x\in\eta}\int_{\mathbb{R}^d} b(x-y)(k(\eta\setminus x\cup y)-k(\eta))dy + \langle a^+\rangle k(\eta).$$

Similarly we can choose

$$M_\alpha(\eta) = N_\alpha(\eta) = e^{-\alpha}\sum_{y_1\in\eta}\sum_{y_2\in\eta\setminus y_1}\int_{\mathbb{R}^d} a(x-y_1,x-y_2)\mathrm{d}x + 3\langle a^+\rangle|\eta|.$$

Within the scaling we have to multiply a^+ by ε and afterwards rescale the operators. Effectively it will consist only of multiplying the first term by ε, and after limit transition we arrive at

$$(\widehat{L}_V G)(\eta) = \sum_{x\in\eta}\int_{\mathbb{R}^d} b(x-y)(G(\eta\setminus x\cup y)-G(\eta))\mathrm{d}y + \langle a^+\rangle|\eta|G(\eta)$$

and

$$(L_V^{\triangle}k)(\eta) = \sum_{x\in\eta}\int_{\mathbb{R}^d} b(x-y)(k(\eta\setminus x\cup y)-k(\eta))\mathrm{d}y + \langle a^+\rangle|\eta|k(\eta),$$

so that $N_\alpha^V(\eta) = 3\langle a^+\rangle|\eta|$. Therefore the kinetic description is simply given by

$$\frac{\partial\rho_t}{\partial t}(x) = -\langle a^+\rangle\rho_t(x) + (b*\rho_t)(x) = \big((b*\rho_t)(x) - \langle b^+\rangle\rho_t(x)\big) + \langle a^+\rangle\rho_t(x).$$

Note that the solution is increasing, e.g. if ρ_0 is integrable, then the solution ρ_t will be integrable as well and satisfy

$$\frac{\partial}{\partial t}\langle\rho_t\rangle = \langle a^+\rangle\langle\rho_t\rangle,$$

which yields $\langle\rho_t\rangle = e^{\langle a^+\rangle t}\langle\rho_0\rangle$.

Example 8 (establishment)**.** Let us take a look on the birth dynamics with establishment. Here each cell located at position $x\in\gamma$ will have a dumped probability to produce a new cell at position $y\in\mathbb{R}^d$, if there are many cells around y. The Markov generator is given by

$$(LF)(\gamma) = \sum_{x\in\gamma}\int_{\mathbb{R}^d} e^{-E(y,\gamma)}a^+(x-y)(F(\gamma\cup y)-F(\gamma))dy,$$

where $0\le a^+\in L^1(\mathbb{R}^d)$ is symmetric. Each cell at position $x\in\gamma$ will create a new cell at position y, but the intensity of this effect is dumped by the relative energy in the exponential. Calculations yield the following form for the generator on quasi-observables

$$
\begin{aligned}
&(\widehat{L}G)(\eta)\\
&=\sum_{\xi\subset\eta}\sum_{x\in\xi}\int_{\mathbb{R}^d} G(\xi\cup y)e_\lambda(e^{-\phi(y-\cdot)}-1;\eta\backslash\xi)e^{-E(y,\xi)}a^+(x-y)\mathrm{d}y\\
&\quad+\sum_{\xi\subset\eta}\sum_{x\in\eta\backslash\xi}\int_{\mathbb{R}^d} G(\xi\cup y)e^{-E(y,\xi)}e_\lambda(e^{-\phi(y-\cdot)}-1;\eta\backslash\xi\backslash x)a^+(x-y)e^{-\phi(x-y)}\mathrm{d}y.
\end{aligned}
$$

Likewise we obtain for L^Δ on correlation functions

$$
\begin{aligned}
&(L^\Delta k)(\eta)\\
&=\sum_{x\in\eta}\sum_{y\in\eta\backslash x} e^{-E(x,\eta\backslash x)}a^+(x-y)\int_{\Gamma_0} e_\lambda(e^{-\phi(x-\cdot)}-1;\xi)k(\eta\cup\xi\backslash x)\mathrm{d}\lambda(\xi)\\
&\quad+\sum_{x\in\eta} e^{-E(x,\eta\backslash x)}\int_{\Gamma_0}\int_{\mathbb{R}^d} e_\lambda(e^{-\phi(x-\cdot)}-1;\xi)a^+(x-y)e^{-\phi(x-y)}k(\eta\cup\xi\backslash x\cup y)\mathrm{d}y\mathrm{d}\lambda(\xi).
\end{aligned}
$$

Hence M_α is given by

$$
\begin{aligned}
M_\alpha(\eta)=e^{-\alpha}\beta(\alpha)&\sum_{x\in\eta}\sum_{y\in\eta\backslash x} a^+(x-y)e^{-E(x,\eta\backslash x)}\\
&+\beta(\alpha)\langle a^+e^{-\phi}\rangle\sum_{x\in\eta}e^{-E(x,\eta\backslash x)}
\end{aligned}
$$

with $\beta(\alpha)=\exp\Big(e^\alpha\int\limits_{\mathbb{R}^d}|1-e^{-\phi(x)}|\mathrm{d}x\Big)$. Rescaling the interactions, i.e. $a^+\to\varepsilon a^+$, $\phi\to\varepsilon\phi$, putting $L\to\dfrac{1}{\varepsilon}L_\varepsilon$ and rescaling both operators we arrive at

$$
\begin{aligned}
&(\widehat{L}_{\varepsilon,ren}G)(\eta)\\
&=\varepsilon\sum_{\xi\subset\eta}\sum_{x\in\xi}\int_{\mathbb{R}^d} G(\xi\cup y)e_\lambda\left(\frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon};\eta\backslash\xi\right)e^{-\varepsilon E(y,\xi)}a^+(x-y)dy\\
&\quad+\sum_{\xi\subset\eta}\sum_{x\in\eta\backslash\xi}\int_{\mathbb{R}^d} G(\xi\cup y)e^{-\varepsilon E(y,\xi)}e_\lambda\left(\frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon};\eta\backslash\xi\backslash x\right)a^+(x-y)e^{-\varepsilon\phi(x-y)}dy.
\end{aligned}
$$

and

$$
\begin{aligned}
&(L^\Delta_{\varepsilon,ren}k)(\eta)\\
&=\varepsilon\sum_{x\in\eta}\sum_{y\in\eta\backslash x} e^{-\varepsilon E(x,\eta\backslash x)}a^+(x-y)\int_{\Gamma_0} e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\xi\right)k(\eta\cup\xi\backslash x)\mathrm{d}\lambda(\xi)\\
&\quad+\sum_{x\in\eta} e^{-\varepsilon E(x,\eta\backslash x)}\int_{\Gamma_0}\int_{\mathbb{R}^d} e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\xi\right)a^+(x-y)e^{-\varepsilon\phi(x-y)}\\
&\qquad\qquad\times k(\eta\cup\xi\backslash x\cup y)dy\mathrm{d}\lambda(\xi),
\end{aligned}
$$

which shows

$$N_\alpha(\eta) = e^{-\alpha}\exp(e^\alpha\langle\phi\rangle)\sum_{x\in\eta}\sum_{y\in\eta\backslash x} a^+(x-y) + \exp(e^\alpha\langle\phi\rangle)\langle a^+\rangle|\eta|.$$

The limiting operators as $\varepsilon \to 0$ are given by

$$(\widehat{L}_V G)(\eta) = \sum_{\xi\subset\eta}\sum_{x\in\eta\backslash\xi}\int_{\mathbb{R}^d} G(\xi\cup y)e_\lambda(-\phi(y-\cdot);\eta\backslash\xi\backslash x)a^+(x-y)\mathrm{d}y.$$

and

$$(L_V^\Delta k)(\eta) = \sum_{x\in\eta}\int_{\Gamma_0}\int_{\mathbb{R}^d} e_\lambda(-\phi(x-\cdot);\xi)a^+(x-y)k(\eta\cup\xi\backslash x\cup y)\mathrm{d}y\mathrm{d}\lambda(\xi)$$

and so $N_\alpha^V(\eta) = \langle a^+\rangle\exp(e^\alpha\langle\phi\rangle)|\eta|$. Therefore the kinetic description is given by

$$\frac{\partial\rho_t}{\partial t}(x) = (a^+ * \rho_t)(x)e^{-(\phi*\rho_t)(x)}.$$

Example 9. (fecundity) Let us take a look on the birth dynamics with fecundity. Here each cell at $x \in \gamma$ will produce new cells according to the distribution $a^+(x-y)dy$, whereas the intensity is dumped by a factor $e^{-E(x,\gamma\backslash x)}$. The Markov generator is given by

$$(LF)(\gamma) = \sum_{x\in\gamma} e^{-E(x,\gamma\backslash x)}\int_{\mathbb{R}^d} a^+(x-y)(F(\gamma\cup y)-F(\gamma))dy,$$

where $0 \le a^+ \in L^1(\mathbb{R}^d)$ is symmetric. Calculations yield the following form for the generator on quasi-observables

$$\begin{aligned}(\widehat{L}G)(\eta) = &\sum_{\xi\subset\eta}\sum_{x\in\eta\backslash\xi} e^{-E(x,\xi)}e_\lambda(e^{-\phi(x-\cdot)}-1;\eta\backslash\xi\backslash x)\int_{\mathbb{R}^d} G(\xi\cup y)a^+(x-y)\mathrm{d}y\\ &+\sum_{\xi\subset\eta}\sum_{x\in\xi} e^{-E(x,\xi\backslash x)}e_\lambda(e^{-\phi(x-\cdot)}-1;\eta\backslash\xi)\int_{\mathbb{R}^d} G(\xi\cup y)a^+(x-y)\mathrm{d}y.\end{aligned}$$

Likewise we obtain for L^Δ on correlation functions

$$\begin{aligned}&(L^\Delta k)(\eta)\\ &= \sum_{x\in\eta}\int_{\Gamma_0}\int_{\mathbb{R}^d} e^{-E(y,\xi\backslash x)}a^+(x-y)e_\lambda(e^{-\phi(y-\cdot)}-1;\xi)k(\eta\cup\xi\backslash x\cup y)\mathrm{d}y\mathrm{d}\lambda(\xi)\\ &\quad+\sum_{x\in\eta}\sum_{y\in\eta\backslash x} e^{-E(x,\eta\backslash x)}e^{\phi(x-y)}a^+(x-y)\int_{\Gamma_0} e_\lambda(e^{-\phi(x-\cdot)}-1;\eta)k(\eta\cup\xi\backslash y)\mathrm{d}\lambda(\xi)\end{aligned}$$

and hence

$$M_\alpha(\eta) = e^{-\alpha}\beta(\alpha)\sum_{x\in\eta}\sum_{y\in\eta\backslash x} a^+(x-y)e^{\phi(x-y)}e^{-E(x,\eta\backslash x)} + \beta(\alpha)\langle a^+e^\phi\rangle|\eta|.$$

Rescaling the interactions, i.e. $a^+ \to \varepsilon a^+$, $\phi \to \varepsilon\phi$, putting $L \to \frac{1}{\varepsilon} L_\varepsilon$ and rescaling both operators we arrive at

$$\begin{aligned}
&(\widehat{L}_{\varepsilon,ren} G)(\eta) \\
&= \sum_{\xi\subset\eta} \sum_{x\in\eta\setminus\xi} e^{-\varepsilon E(x,\xi)} e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon}; \eta\setminus\xi\setminus x\right) \int_{\mathbb{R}^d} G(\xi\cup y) a^+(x-y)\mathrm{d}y \\
&\quad + \varepsilon \sum_{\xi\subset\eta} \sum_{x\in\xi} e^{-\varepsilon E(x,\xi\setminus x)} e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon}; \eta\setminus\xi\right) \int_{\mathbb{R}^d} G(\xi\cup y) a^+(x-y)\mathrm{d}y.
\end{aligned}$$

Likewise we obtain for L^Δ on correlation functions

$$\begin{aligned}
&(L^\Delta_{\varepsilon,ren} k)(\eta) \\
&= \sum_{x\in\eta} \int_{\Gamma_0} \int_{\mathbb{R}^d} e^{-\varepsilon E(y,\xi\setminus x)} a^+(x-y) e_\lambda\left(\frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon}; \xi\right) k(\eta\cup\xi\setminus x\cup y)\mathrm{d}y\mathrm{d}\lambda(\xi) \\
&\quad + \varepsilon \sum_{x\in\eta} \sum_{y\in\eta\setminus x} e^{-\varepsilon E(x,\eta\setminus x)} e^{\varepsilon\phi(x-y)} a^+(x-y) \\
&\qquad\qquad \times \int_{\Gamma_0} e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon}; \eta\right) k(\eta\cup\xi\setminus y)\mathrm{d}\lambda(\xi)
\end{aligned}$$

so

$$N_\alpha(\eta) = e^{-\alpha}\exp(e^\alpha\langle\phi\rangle) \sum_{x\in\eta} \sum_{y\in\eta\setminus x} a^+(x-y) e^{\phi(x-y)} + \langle a^+\rangle \exp(e^\alpha\langle\phi\rangle)|\eta|.$$

The limiting operators as $\varepsilon \to 0$ are given by

$$(\widehat{L}_V G)(\eta) = \sum_{\xi\subset\eta} \sum_{x\in\eta\setminus\xi} e_\lambda(-\phi(x-\cdot); \eta\setminus\xi\setminus x) \int_{\mathbb{R}^d} G(\xi\cup y) a^+(x-y)\mathrm{d}y$$

and

$$(L^\Delta_V k)(\eta) = \sum_{x\in\eta} \int_{\Gamma_0} \int_{\mathbb{R}^d} a^+(x-y) e_\lambda(-\phi(y-\cdot); \xi) k(\eta\cup\xi\setminus x\cup y)\mathrm{d}y\mathrm{d}\lambda(\xi)$$

Therefore the kinetic description is given by

$$\frac{\partial\rho_t}{\partial t}(x) = (a^+ * (e^{-\phi*\rho_t}\rho_t))(x).$$

4.3 Moving cells

Here let us describe possible microscopic events, which lead to a motion of cells. The first model describes the most simple possibility.

Example 10 (free jumps)**.** The Markov generator of a system of free jumping cells is given by

$$(LF)(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} c(x-y)(F(\gamma\setminus x\cup y) - F(\gamma))\mathrm{d}y,$$

where $0 \leq c \in L^1(\mathbb{R}^d)$ is symmetric. Here each cell jumps independently of the others according to a jump rate $\langle c \rangle$ and a probability distribution $\frac{1}{\langle c \rangle} c(x-y)\mathrm{d}y$, where $x \in \gamma$. The mechanism can be described on quasi-observables via

$$(\widehat{L}G)(\eta) = \sum_{x\in\eta} \int_{\mathbb{R}^d} c(x-y)(G(\eta\backslash x \cup y) - G(\eta))\mathrm{d}y$$

and L^Δ is given by the same formula. Since after scaling nothing is changed we obtain immediately for the kinetic description

$$\frac{\partial \rho_t}{\partial t}(x) = (c * \rho_t)(x) - \rho_t(x) = \int_{\mathbb{R}^d} c(x-y)(\rho_t(y) - \rho_t(x))\mathrm{d}x,$$

which is the equation describing a random walk in continuous time. Then functions M_α, N_α and N_α^V can be chosen as $\langle c \rangle |\eta|$.

Another possibility of describing the free motion of particles is given by the next example.

Example 11 (free diffusion)**.** Let the Markov generator be given by

$$(LF)(\gamma) = \sum_{x\in\Gamma} (\Delta_x F)(\gamma).$$

Here each cell undergoes a free diffusion independent of all other cells. Rewriting this operators to quasi-observables we arrive at

$$(\widehat{L}G)(\eta) = \sum_{x\in\eta} (\Delta_x G)(\eta)$$

and likewise the expression for L^Δ is given by the same formula. Since scaling will not change the operators we arrive at the kinetic description

$$\frac{\partial \rho_t}{\partial t}(x) = \Delta \rho_t(x).$$

Example 12 (jumps with additive intensity)**.** The Markov generator for jumping cells, with density dependent intensity is given by

$$(LF)(\gamma) = \sum_{x\in\gamma} \sum_{y\in\gamma\backslash x} b(x-y) \int_{\mathbb{R}^d} c(x-z)(F(\gamma\backslash x \cup z) - F(\gamma))\mathrm{d}z,$$

where $0 \leq c, b \in L^1(\mathbb{R}^d)$ are symmetric. Each cell at $x \in \gamma$ will jump with intensity $\langle c \rangle \sum_{y\in\gamma\backslash x} b(x-y)$ and the position is determined by the distribution $\frac{1}{\langle c \rangle} c(x-z)\mathrm{d}z$. The description via quasi-observables will give

$$\begin{aligned}(\widehat{L}G)(\eta) = &\sum_{x\in\eta} \sum_{y\in\eta\backslash x} b(x-y) \int_{\mathbb{R}^d} c(x-z)(G(\eta\backslash x\backslash y \cup z) - G(\eta\backslash y))\mathrm{d}z \\ &+ \sum_{x\in\eta} \sum_{y\in\eta\backslash x} b(x-y) \int_{\mathbb{R}^d} c(x-z)(G(\eta\backslash x \cup z) - G(\eta))\mathrm{d}z\end{aligned}$$

and similarly for correlation functions

$$(L^{\Delta}k)(\eta) = \sum_{x\in\eta}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} c(x-y)b(y-z)(k(\eta\backslash x\cup y\cup z)-k(\eta\cup z))\mathrm{d}y\mathrm{d}z$$
$$+\sum_{x\in\eta}\sum_{y\in\eta\backslash x} b(x-y)\int_{\mathbb{R}^d} a(x-z)(k(\eta\backslash x\cup z)-k(\eta))\mathrm{d}z.$$

The rigorous derivation of the kinetic description was already done in [4]. Scaling the potential as $b\to\varepsilon b$ and rescaling, we see that only the last terms in $\widehat{L}$ and L^{Δ} will be multiplied by ε. Hence after limit transition $\varepsilon\to 0$ we arrive at

$$(\widehat{L}_V G)(\eta) = \sum_{x\in\eta}\sum_{y\in\eta\backslash x} b(x-y)\int_{\mathbb{R}^d} c(x-z)(G(\eta\backslash x\backslash y\cup z)-G(\eta\backslash y))\mathrm{d}z$$

and

$$(L^{\Delta}_V k)(\eta) = \sum_{x\in\eta}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} c(x-y)b(y-z)(k(\eta\backslash x\cup y\cup z)-k(\eta\cup z))\mathrm{d}y\mathrm{d}z,$$

which yields the kinetic description

$$\frac{\partial\rho_t}{\partial t}(x) = (c*((b*\rho_t)\cdot\rho_t))(x)-(c*(b*\rho_t))(x)\rho_t(x).$$

Example 13 (density dependent jumps)**.** Let $0\le\phi,\psi,c\in L^1(\mathbb{R}^d)$ symmetric with $\phi\in L^\infty(\mathbb{R}^d)$, set $E_\phi(x,\gamma\backslash x) = \sum\limits_{y\in\gamma\backslash x}\phi(x-y)$ and likewise $E_\psi(y,\gamma) = \sum\limits_{x\in\gamma}\psi(x-y)\ge 0$. Define the formal Markov generator

$$(LF)(\gamma) = \sum_{x\in\gamma} e^{E_\phi(x,\gamma\backslash x)}\int_{\mathbb{R}^d} e^{-E_\psi(y,\gamma)}c(x-y)(F(\gamma\backslash x\cup y)-F(\gamma))\mathrm{d}y.$$

Here each cell located at $x\in\gamma$ will have a high intensity to jump, if there are many other cells around and due to $e^{-E_\psi(y,\gamma)}$ it will prefer to jump in regions, which have a small density of cells. Let us compute the the operator $\widehat{L}$. For $G\in B_{bs}(\Gamma_0)$, $x\in\gamma$ and $y\notin\gamma$ we obtain

$$\begin{aligned}
&(KG)(\gamma\backslash x\cup y)-(KG)(\gamma)\\
&=\sum_{\eta\Subset\gamma\backslash x\cup y} G(\eta)-\sum_{\eta\Subset\gamma} G(\eta)\\
&=\sum_{\eta\Subset\gamma\backslash x} G(\eta)+\sum_{\eta\Subset\gamma\backslash x} G(\eta\cup y)-\sum_{\eta\Subset\gamma\backslash x} G(\eta)-\sum_{\eta\Subset\gamma\backslash x} G(\eta\cup x)\\
&=\sum_{\eta\Subset\gamma\backslash x}(G(\eta\cup y)-G(\eta\cup x)).
\end{aligned}$$

Therefore using (31) and (32) we get

$$\sum_{x\in\gamma} e^{E_\phi(x,\gamma\setminus x)} \int_{\mathbb{R}^d} e^{-E_\psi(y,\gamma\setminus x)} e^{-\psi(x-y)} c(x-y) \sum_{\eta\Subset\gamma\setminus x} (G(\eta\cup y) - G(\eta\cup x))$$

$$= \sum_{x\in\gamma} \int_{\mathbb{R}^d} e^{-\psi(x-y)} c(x-y) K e_\lambda(f(x,y))(\gamma\setminus x) \sum_{\eta\Subset\gamma\setminus x} (G(\eta\cup y) - G(\eta\cup x)) \mathrm{d}y$$

$$= \sum_{x\in\gamma} \sum_{\eta\Subset\gamma\setminus x} \int_{\mathbb{R}^d} e_\lambda\left(f(x,y)\right) \star (G(\cdot\cup y) - G(\cdot\cup x))(\eta) e^{-\psi(x-y)} c(x-y) \mathrm{d}y$$

$$= \sum_{\eta\Subset\gamma} \sum_{x\in\eta} \int_{\mathbb{R}^d} e_\lambda\left(f(x,y)\right) \star (G(\cdot\cup y) - G(\cdot\cup x))(\eta\setminus x) e^{-\psi(x-y)} c(x-y) \mathrm{d}y$$

with $f(x,y;w) = e^{\phi(x-w)-\psi(y-w)} - 1$. Again using the definition (31) we get

$$(\widehat{L}G)(\eta)$$

$$= \sum_{x\in\eta} \int_{\mathbb{R}^d} e_\lambda\left(f(x,y)\right) \star (G(\cdot\cup y) - G(\cdot\cup x))(\eta\setminus x) e^{-\psi(x-y)} c(x-y) \mathrm{d}y$$

$$= \sum_{\xi\subset\eta} \sum_{x\in\eta} e^{E_\phi(x,\xi\setminus x)} \int_{\mathbb{R}^d} e^{-E_\psi(y,\xi)} c(x-y) e_\lambda\left(f(x,y);\eta\setminus\xi\right) \left(G(\xi\cup y) - G(\xi\cup x)\right) \mathrm{d}y.$$

This yields the following formula

$$\sum_{\xi\subset\eta} \sum_{x\in\xi} e^{E_\phi(x,\xi\setminus x)} \int_{\mathbb{R}^d} e^{-E_\psi(y,\xi)} c(x-y) e_\lambda\left(f(x,y);\eta\setminus\xi\right) \left(G(\xi\setminus x\cup y) - G(\xi)\right) \mathrm{d}y.$$

Let us show that the function M_α can be chosen by

$$M_\alpha(\eta) = e^{e^\alpha\kappa} \sum_{x\in\eta} e^{-E_\psi(x,\eta\setminus x)} \int_{\mathbb{R}^d} c(x-y) e^{-\phi(x-y)-\psi(x-y)} e^{E_\phi(y,\eta)} \mathrm{d}y$$
$$+ e^{e^\alpha\kappa} \sum_{x\in\eta} e^{E_\phi(x,\eta\setminus x)} \int_{\mathbb{R}^d} c(x-y) e^{-E_\psi(y,\eta)} \mathrm{d}y,$$

where $\kappa = e^{\|\phi\|_{L^\infty}} \langle\phi\rangle$. So let $G \in D(M_\alpha)$ and note that

$$\int_{\mathbb{R}^d} f(x,y;w) \mathrm{d}w = \int_{\mathbb{R}^d} \left(e^{\phi(x-w)-\psi(y-w)} - 1\right) \mathrm{d}w \le e^{\|\phi\|_{L^\infty}} \langle\phi\rangle = \kappa.$$

Now using the formulas from Lemma 4.2 we arrive at

$$e^\alpha \int_{\Gamma_0^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{E_\phi(x,\xi)} e^{-E_\psi(y,\xi\cup x)} c(x-y) e_\lambda\left(|f(x,y)|;\eta\right)$$
$$\times |G(\xi\cup y)| e^{\alpha|\xi|} e^{\alpha|\eta|} \mathrm{d}y \mathrm{d}x \mathrm{d}\lambda(\eta,\xi)$$
$$\le e^\alpha e^{e^\alpha\kappa} \int_{\Gamma_0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{\alpha|\xi|} e^{-E_\psi(y,\xi\cup x)} e^{E_\phi(x,\xi)} c(x-y) |G(\xi\cup y)| \mathrm{d}x \mathrm{d}y \mathrm{d}\lambda(\xi)$$

$$= e^{e^{\alpha}\kappa} \int_{\Gamma_0} |G(\xi)| \sum_{y\in\xi} e^{-E_{\psi}(y,\xi\backslash y)} \int_{\mathbb{R}^d} c(x-y) e^{-\psi(x-y)-\phi(x-y)} e^{E_{\phi}(x,\xi)} e^{\alpha|\xi|} \mathrm{d}x \mathrm{d}\lambda(\xi)$$

and for the second part of $\widehat{L}$ at

$$\int_{\Gamma_0^2} \mathrm{d}\lambda^2(\eta,\xi) e^{\alpha|\eta|} e^{\alpha|\xi|} \sum_{x\in\eta} \int_{\mathbb{R}^d} dy c(x-y) e^{E_{\phi}(x,\xi\backslash x)} e^{-E_{\psi}(y,\xi)} e_{\lambda}\left(|f(x,y)|;\eta\right) |G(\xi)|$$

$$\leq e^{e^{\alpha}\kappa} \int_{\Gamma_0} \Big(\sum_{x\in\xi} e^{E_{\phi}(x,\xi\backslash x)} \int_{\mathbb{R}^d} c(x-y) e^{-E_{\psi}(y,\xi)} dy \Big) |G(\xi)| e^{\alpha|\xi|} \mathrm{d}\lambda(\xi).$$

Since $M_{\alpha}(\eta) \leq 2\langle c\rangle e^{e^{\alpha}\kappa} |\eta| e^{\|\phi\|_{L^{\infty}}|\eta|}$ we get for $\alpha - \alpha' > \|\phi\|_{L^{\infty}}$ that $\widehat{L} \in L(\mathbb{B}_{\alpha}, \mathbb{B}_{\alpha'})$ and

$$\|\widehat{L}\|_{\alpha\alpha'} \leq \frac{2\langle c\rangle e^{e^{\alpha'}\kappa}}{e(\alpha - \alpha' - \|\phi\|_{L^{\infty}})}.$$

Turning now to correlation functions, the action of the operator L^{Δ} is given by

$$\sum_{y\in\eta} \int_{\Gamma_0} \mathrm{d}\lambda(\xi) \int_{\mathbb{R}^d} \mathrm{d}x c(x-y) e^{E_{\phi}(x,\eta\backslash y)} e^{-E_{\psi}(y,\eta\cup x\backslash y)} e_{\lambda}\left(f(x,y);\xi\right) k(\eta\cup\xi\backslash y\cup x)$$
$$- \sum_{y\in\eta} \int_{\Gamma_0} \mathrm{d}\lambda(\xi) \int_{\mathbb{R}^d} \mathrm{d}y c(x-y) e^{E_{\phi}(y,\eta\backslash y)} e^{-E_{\psi}(y,\eta)} e_{\lambda}\left(f(x,y);\eta\right) k(\eta\cup\xi)$$

and similarly L^{Δ} is a bounded linear operator in $L(\mathbb{B}_{\alpha'}^*, \mathbb{B}_{\alpha}^*)$ for each $\alpha - \alpha' > \|\phi\|_{L^{\infty}}$. In order to see this let $G \in B_{bs}(\Gamma_0)$ and $k \in \mathbb{B}_{\alpha}$ for some $\alpha \geq 0$, then for the first term we get

$$\int_{\Gamma_0} k(\eta) \sum_{\xi\subset\eta} \sum_{x\in\xi} \int_{\mathbb{R}^d} e^{E_{\phi}(x,\xi\backslash x)} e^{-E_{\psi}(y,\xi)} c(x-y) e_{\lambda}\left(f(x,y);\eta\right) G(\xi\backslash x\cup y) \mathrm{d}y \mathrm{d}\lambda(\eta)$$
$$= \int_{\Gamma_0^2} k(\eta\cup\xi) \sum_{x\in\xi} \int_{\mathbb{R}^d} c(x-y) e^{E_{\phi}(x,\xi\backslash x)} e^{-E_{\psi}(y,\xi)} e_{\lambda}\left(f(x,y);\eta\right) G(\xi\backslash x\cup y) \mathrm{d}y \mathrm{d}\lambda(\eta,\xi)$$
$$= \int_{\Gamma_0^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k(\eta\cup\xi\cup x) G(\xi\cup y) c(x-y) e^{E_{\phi}(x,\xi)} e^{-E_{\psi}(y,\xi\cup x\backslash y)} e_{\lambda}\left(f(x,y);\eta\right) \mathrm{d}x \mathrm{d}y \mathrm{d}\lambda(\eta,\xi)$$
$$= \int_{\Gamma_0^2} \sum_{y\in\xi} \int_{\mathbb{R}^d} k(\eta\cup\xi\cup x\backslash y) G(\xi) c(x-y) e^{E_{\phi}(x,\xi\backslash y)} e^{-E_{\psi}(y,\xi\cup x\backslash y)} e_{\lambda}\left(f(x,y);\eta\right) \mathrm{d}x \mathrm{d}\lambda(\eta,\xi).$$

For the second term we have

$$\int_{\Gamma_0} k(\eta) \sum_{\xi\subset\eta} \sum_{x\in\xi} \int_{\mathbb{R}^d} c(x-y) e^{E_{\phi}(x,\xi\backslash x)} e^{-E_{\psi}(y,\xi)} e_{\lambda}\left(f(x,y);\eta\backslash\xi\right) G(\xi) \mathrm{d}y \mathrm{d}\lambda(\eta)$$
$$= \int_{\Gamma_0^2} k(\eta\cup\xi) \sum_{x\in\xi} \int_{\mathbb{R}^d} c(x-y) e^{E_{\phi}(x,\xi\backslash x)} e^{-E_{\psi}(y,\xi)} e_{\lambda}\left(f(x,y);\eta\right) G(\xi) \mathrm{d}y \mathrm{d}\lambda(\eta,\xi).$$

Here we have to rescale the potentials $\phi \to \varepsilon\phi$ and $\psi \to \varepsilon\psi$. Since we are interested in the limit $\varepsilon \to 0$, we will restrict the range of ε to $(0,1]$. The rescaled operator will have the

$$\sum_{\xi\subset\eta}\sum_{x\in\xi} e^{\varepsilon E_\phi(x,\xi\backslash x)} \int_{\mathbb{R}^d} e^{-\varepsilon E_\psi(y,\xi)} c(x-y) \\ \times e_\lambda\left(e^{\varepsilon\phi(x-\cdot)-\varepsilon\psi(y-\cdot)}-1;\eta\backslash\xi\right)(G(\xi\backslash x\cup y)-G(\xi))\mathrm{d}y.$$

In order to get the normalized expression we have again to consider the composition $\widehat{L}_\varepsilon^{ren} = R_{\varepsilon^{-1}}\widehat{L}_\varepsilon R_\varepsilon$, this leads to the following expression for $\widehat{L}_\varepsilon^{ren}$

$$\sum_{\xi\subset\eta}\sum_{x\in\xi} e^{\varepsilon E_\phi(x,\xi\backslash x)} \int_{\mathbb{R}^d} e^{-\varepsilon E_\psi(y,\xi)} c(x-y) e_\lambda\left(f_\varepsilon(x,y);\eta\backslash\xi\right)(G(\xi\backslash x\cup y)-G(\xi))dy$$

with $f_\varepsilon(x,y;w) = \dfrac{e^{\varepsilon\phi(x-w)-\varepsilon\psi(y-w)}-1}{\varepsilon}$. For each fixed $\eta \in \Gamma_0$ this expression converges to

$$(\widehat{L}_V G)(\eta) = \sum_{\xi\subset\eta}\sum_{x\in\xi}\int_{\mathbb{R}^d} c(x-y)e_\lambda\left(\phi(x-\cdot)-\psi(y-\cdot);\eta\backslash\xi\right)(G(\xi\backslash x\cup y)-G(\xi))\mathrm{d}y.$$

Lemma 4.3. *For $\widehat{L}_\varepsilon^{ren}$ the corresponding function N_α is given by*

$$N_\alpha(\eta) = e^{e^\alpha\kappa}\sum_{x\in\eta}\int_{\mathbb{R}^d} c(x-y)e^{E_\phi(y,\eta)}\mathrm{d}y + \langle c\rangle e^{e^\alpha\kappa}\sum_{x\in\eta} e^{E_\phi(x,\eta\backslash x)}$$

and $N_\alpha^V(\eta) = 2\langle c\rangle \exp(e^\alpha(\langle\phi\rangle + \langle\psi\rangle))|\eta|$. Moreover for each $\alpha - \alpha' > \|\phi\|_{L^\infty}$ the estimate holds:

$$\|\widehat{L}_{\varepsilon,ren}\|_{\alpha\alpha'} \le \frac{2\langle c\rangle e^{e^{\alpha'}\kappa}}{e(\alpha-\alpha'-\varepsilon\|\phi\|_{L^\infty})} \le \frac{2\langle c\rangle e^{e^{\alpha'}\kappa}}{e(\alpha-\alpha'-\|\phi\|_{L^\infty})},$$
$$\|\widehat{L}_V\|_{\alpha\alpha'} \le \frac{2\langle c\rangle \exp(e^{\alpha'}(\langle\phi\rangle+\langle\psi\rangle))}{e(\alpha-\alpha')}$$

for all $\alpha' < \alpha$.

Proof. For this purpose we have first to estimate $f_\varepsilon(x,y)$ by

$$|f_\varepsilon(x,y;w)| \le e^{\|\phi\|_{L^\infty}}\phi(x-w)$$

for almost all $w \in \mathbb{R}^d$ and afterwards to use

$$N_\alpha(\eta) \le 2\langle ce^{e^\alpha\kappa}\rangle|\eta|e^{\|\phi\|_{L^\infty}|\eta|}.$$ □

Clearly we have $D(M_\alpha) \subset D(N_\alpha) \subset D(N_\alpha^V)$.

Theorem 4.4. *Let $G \in D(N_\alpha)$ such that $|\eta|^2 e^{\|\phi\|_{L^\infty}|\eta|}G \in \mathbb{B}_\alpha$, then*

$$\widehat{L}_\varepsilon^{ren}G \to \widehat{L}_V G, \quad \varepsilon \to 0$$

in $\mathbb{B}_\alpha$. In addition the renormalized operator for quasi-observables converges in the uniform operator topology of $L(\mathbb{B}_\alpha, \mathbb{B}_{\alpha'})$, i.e. the following holds

$$\|\widehat{L}_V - \widehat{L}_\varepsilon^{ren}\|_{\alpha\alpha'} \to 0, \quad \varepsilon \to 0.$$

Proof. Fix $\alpha' < \alpha$ and $G \in D(M_{\alpha'})$ such that $|\eta|^2 e^{\|\phi\|_{L^\infty}|\eta|} G \in \mathbb{B}_{\alpha'}$. Let us divide $\widehat{L}_V$ and $\widehat{L}_\varepsilon^{ren}$ in two parts according to $G(\xi\backslash x \cup y)$ and $G(\xi)$ and investigate their differences separately. Starting with the term containing $G(\xi\backslash x \cup y)$ we obtain for the difference

$$\begin{aligned} & e^{\alpha'} \int_{\Gamma_0^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{d}\lambda(\eta,\xi)\mathrm{d}x\mathrm{d}y e^{\alpha'|\eta|}e^{\alpha'|\xi|}c(x-y)|G(\xi\cup y)| \\ & \quad \times \left| e^{\varepsilon E_\phi(x,\xi)} e^{-\varepsilon E_\psi(y,\xi\cup x)} e_\lambda\left(f_\varepsilon(x,y);\eta\right) - e_\lambda(\phi(x-\cdot)-\psi(y-\cdot);\eta)\right| \\ & = \int_{\Gamma_0^2} \mathrm{d}\lambda(\eta,\xi) e^{\alpha'|\eta|}e^{\alpha'|\xi|}|G(\xi)| \sum_{y\in\xi}\int_{\mathbb{R}^d} \mathrm{d}x c(x-y) \\ & \quad \times \left| e^{\varepsilon E_\phi(x,\xi\backslash y)} e^{-\varepsilon E_\psi(y,\xi\backslash y\cup x)} e_\lambda\left(f_\varepsilon(x,y);\eta\right) - e_\lambda(\phi(x-\cdot)-\psi(y-\cdot);\eta)\right|. \end{aligned}$$

Using $|f_\varepsilon(x,y;w)| \le e^{\|\phi\|_{L^\infty}}\phi(x-w)$,

$$|f_\varepsilon(x,y;w) - (\phi(x-w)-\psi(y-w))| \le e^r|\phi(x-w)-\psi(y-w)|$$

where $r = \|\phi\|_{L^\infty} + \|\psi\|_{L^\infty}$ and

$$|e^{\varepsilon E_\phi(x,\xi\backslash y)} - 1| \le \varepsilon E_\phi(x,\xi\backslash y) e^{\|\phi\|_{L^\infty}|\xi\backslash y|} \le \varepsilon \|\phi\|_{L^\infty}|\xi| e^{\|\phi\|_{L^\infty}|\xi|}$$

the modulus in the integral can be estimated by

$$\begin{aligned} & \left| e^{\varepsilon E_\phi(x,\xi\backslash y)} e^{-\varepsilon E_\psi(y,\xi\backslash y\cup x)} e_\lambda\left(f_\varepsilon(x,y);\eta\right) - e_\lambda(\phi(x-\cdot)-\psi(y-\cdot);\eta)\right| \\ & \le |e^{\varepsilon E_\phi(x,\xi\backslash y)} - 1| \left| e^{-\varepsilon E_\psi(y,\xi\backslash y\cup x)} e_\lambda\left(f_\varepsilon(x,y);\eta\right)\right| \\ & \quad + |e^{-\varepsilon E_\psi(y,\xi\backslash y\cup x)} - 1|\, |e_\lambda\left(|f_\varepsilon(x,y)|;\eta\right)| \\ & \quad + |e_\lambda\left(|f_\varepsilon(x,y)|;\eta\right) - e_\lambda(\phi(x-\cdot)-\psi(y-\cdot);\eta)| \\ & \le \varepsilon\|\phi\|_{L^\infty}|\xi| e^{\varepsilon\|\phi\|_{L^\infty}|\xi|} e_\lambda\left(e^{\|\phi\|_{L^\infty}}\phi(x-\cdot)|;\eta\right) \\ & \quad + \varepsilon(E_\psi(y,\xi\backslash y) + \psi(x-y)) e_\lambda\left(e^{\|\phi\|_{L^\infty}}\phi(x-\cdot)|;\eta\right) \\ & \quad + \varepsilon e^r \sum_{w\in\eta} |\phi(x-w)-\psi(y-w)| e_\lambda(e^r|\phi(x-\cdot)-\psi(y-\cdot)|;\eta\backslash w). \end{aligned}$$

Invoking this in previous estimations one obtains with some generic constant $C > 0$ independent of α', α, and ε

$$\begin{aligned} & \le \varepsilon C e^{e^{\alpha'+r}\kappa} \int_{\Gamma_0} e^{\alpha'|\xi|}|G(\xi)| \sum_{y\in\xi}\int_{\mathbb{R}^d} c(x-y)\mathrm{d}x |\xi| e^{\varepsilon\|\phi\|_{L^\infty}|\xi|}\mathrm{d}\lambda(\xi) \\ & \quad + \varepsilon C e^{e^{\alpha'+r}\kappa} \int_{\Gamma_0} e^{\alpha'|\xi|}|G(\xi)| \sum_{y\in\xi}(\langle c\rangle + \langle c\psi\rangle) E_\psi(y,\xi\backslash y) \\ & \quad + \varepsilon C e^{\alpha'} e^{e^{\alpha'+r}\kappa} \int_{\Gamma_0} e^{\alpha'|\xi|}|G(\xi)||\xi|\mathrm{d}\lambda(\xi) \\ & \le \varepsilon C e^{e^{\alpha'+r}\kappa} \int_{\Gamma_0} |G(\xi)| \left(|\xi|^2 e^{\varepsilon\|\phi\|_{L^\infty}|\xi|} e^{\alpha'} + 1 + |\xi| + |\xi|^2\right) e^{\alpha'|\xi|}\mathrm{d}\lambda(\xi) \end{aligned}$$

This shows the first part. For the second part take $0 < \varepsilon < \dfrac{\alpha - \alpha'}{\|\phi\|_{L^\infty}}$, then $\mathbb{B}_\alpha \subset D(N_{\alpha'})$ and for $G \in \mathbb{B}_\alpha$ above integral is bounded by

$$\leq \varepsilon C \left(\frac{4}{e^2(\alpha - \alpha' - \|\phi\|_{L^\infty})^2} + \frac{4}{e^2(\alpha - \alpha')^2} + \frac{1}{e(\alpha - \alpha')} + 1 \right) \|G\|_\alpha e^{e^{\alpha' + r}\kappa}$$

which shows the assertion for the terms containing $G(\xi \cup y \backslash x)$. Similarly the differences including $G(\xi)$ can be estimated. $\square$

For correlation functions the rescaled operator has the form

$$\begin{aligned}
&(L_\varepsilon^\Delta k)(\eta) \\
&= \sum_{y\in\eta} \int_{\Gamma_0} \mathrm{d}\lambda(\xi) \int_{\mathbb{R}^d} \mathrm{d}x c(x-y) e^{\varepsilon E_\phi(x,\eta\backslash y)} e^{-\varepsilon E_\psi(y,\eta\cup x\backslash y)} \\
&\qquad\qquad \times e_\lambda\left(e^{\varepsilon\phi(x-\cdot)-\varepsilon\psi(y-\cdot)} - 1; \xi\right) k(\eta \cup \xi \cup x \backslash y) \\
&\quad - \sum_{y\in\eta} \int_{\Gamma_0} \mathrm{d}\lambda(\xi) \int_{\mathbb{R}^d} \mathrm{d}x c(x-y) e^{\varepsilon E_\phi(y,\eta\backslash y)} e^{-\varepsilon E_\psi(x,\eta)} \\
&\qquad\qquad \times e_\lambda\left(e^{\varepsilon\phi(y-\cdot)-\varepsilon\psi(x-\cdot)} - 1; \xi\right) k(\eta \cup \xi).
\end{aligned}$$

Again computing the renormalized operator one gets similarly to the case for quasi-observables

$$\begin{aligned}
&(L_\varepsilon^{\Delta,ren} k)(\eta) \\
&= \sum_{y\in\eta} \int_{\Gamma_0} \mathrm{d}\lambda(\xi) \int_{\mathbb{R}^d} dx c(x-y) e^{\varepsilon E_\phi(x,\eta\backslash y)} e^{-\varepsilon E_\psi(y,\eta\cup x\backslash y)} e_\lambda\left(f_\varepsilon(x,y);\xi\right) k(\eta\cup\xi\cup x\backslash y) \\
&\quad - \sum_{y\in\eta} \int_{\Gamma_0} \mathrm{d}\lambda(\xi) \int_{\mathbb{R}^d} dx c(x-y) e^{\varepsilon E_\phi(y,\eta\backslash y)} e^{-\varepsilon E_\psi(x,\eta)} e_\lambda\left(f_\varepsilon(x,y);\xi\right) k(\eta\cup\xi)
\end{aligned}$$

and for each fixed $\eta \in \Gamma_0$ this operator converges to

$$\begin{aligned}
(L_V^\Delta k)(\eta) &= \sum_{y\in\eta} \int_{\Gamma_0} \mathrm{d}\lambda(\xi) \int_{\mathbb{R}^d} dx c(x-y) e_\lambda\left(\phi(x-\cdot) - \psi(y-\cdot);\xi\right) k(\eta\cup\xi\backslash y\cup x) \\
&\quad - \sum_{y\in\eta} \int_{\Gamma_0} \mathrm{d}\lambda(\xi) \int_{\mathbb{R}^d} dx c(x-y) e_\lambda\left(\phi(y-\cdot) - \psi(x-\cdot);\xi\right) k(\eta\cup\xi).
\end{aligned}$$

Lemma 4.5. *The renormalized operator on correlation functions converges in the uniform operator topology of* $L(\mathbb{B}_{\alpha'}^*, \mathbb{B}_\alpha^*)$ *to* L_V^Δ*, i.e. for all* $\alpha' < \alpha$

$$\|L_V^\Delta - L_\varepsilon^{\Delta,ren}\|_{\alpha'\alpha} \to 0, \quad \varepsilon \to 0.$$

Since $L_\varepsilon^{\Delta,ren}$ is dual to $\widehat{L}_\varepsilon^{ren}$ with respect to (42) the assertion follows from (45). Finally let us derive the kinetic description for this model. Therefore we have to compute $L_V^\Delta e_\lambda(\rho)$ for a function $0 \leq \rho \in L^\infty(\mathbb{R}^d)$. This expression is given by

$$
\begin{aligned}
&(L_V^{\Delta} e_\lambda)(\eta) \\
&= \sum_{y\in\eta} e_\lambda(\rho;\eta\backslash y) \int_{\Gamma_0} \mathrm{d}\lambda(\xi) \int_{\mathbb{R}^d} \mathrm{d}x c(x-y)\rho(x) e_\lambda(\phi(x-\cdot)-\psi(y-\cdot);\xi) e_\lambda(\rho;\xi) \\
&\quad - \sum_{y\in\eta} e_\lambda(\rho;\eta\backslash y)\rho(y) \int_{\Gamma_0} \mathrm{d}\lambda(\xi) \int_{\mathbb{R}^d} \mathrm{d}x c(x-y) e_\lambda(\phi(y-\cdot)-\psi(x-\cdot);\xi) e_\lambda(\rho;\xi) \\
&= \sum_{y\in\eta} e_\lambda(\rho;\eta\backslash y) \int_{\mathbb{R}^d} \mathrm{d}x c(x-y)\rho(x) \exp\Big(\int_{\mathbb{R}^d} \mathrm{d}w \left(\phi(x-w)-\psi(y-w)\right)\rho(w) \Big) \\
&\quad - \sum_{y\in\eta} e_\lambda(\rho;\eta\backslash y)\rho(y) \int_{\mathbb{R}^d} \mathrm{d}x c(x-y) \exp\Big(\int_{\mathbb{R}^d} \mathrm{d}w \left(\phi(y-w)-\psi(x-w)\right)\rho(w) \Big) \\
&= \sum_{y\in\eta} e_\lambda(\rho;\eta\backslash y) \left(e^{-(\psi*\rho)(y)} (c*\rho\cdot e^{\phi*\rho})(y) - \rho(y) e^{(\phi*\rho)(y)} (c*e^{-(\psi*\rho)})(y) \right)
\end{aligned}
$$

and since

$$\frac{\partial}{\partial t} e_\lambda(\rho_t;\eta) = \sum_{x\in\eta} e_\lambda(\rho_t;\eta\backslash y) \frac{\partial\rho_t}{\partial t}(y)$$

we obtain for the contribution of the jumps to the mesoscopic equation the terms

$$\left(c*\left(\rho_t e^{\phi*\rho_t}\right)\right)(y) e^{-(\psi*\rho_t)(y)} - e^{(\phi*\rho_t)(y)} (c*e^{-\psi*\rho_t})(y)\rho_t(y).$$

4.4 Free branching process

Let us recap shortly the description of the free branching process. Here the heuristic Markov generator is given by

$$
\begin{aligned}
(LF)(\gamma) &= m \sum_{x\in\gamma} (F(\gamma\backslash x) - F(\gamma)) \\
&\quad + \lambda \sum_{x\in\gamma} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x-y_1, x-y_2)(F(\gamma\backslash x\cup y_1\cup y_2) - F(\gamma))\mathrm{d}y_1\mathrm{d}y_2
\end{aligned}
$$

with constant mortality $m>0$ and intensity of cell-division $\lambda>0$. The potential $0\le a\in L^1(\mathbb{R}^d\times\mathbb{R}^d)$ is assumed to by symmetric in both coordinates and the total mass is normalized to 1. This model describes a cell population, where each cell will die with exponential distributed lifetime of parameter $m>0$ and will divide into two cells after another exponential distributed time of parameter $\lambda>0$. The position of the new cells is determined by the probability distribution

$$a(x-y_1, x-y_2)\mathrm{d}y_1\mathrm{d}y_2,$$

where $x\in\gamma$ is the position of the mother cell. The generator L is well defined for all functions $F=KG$, where $G\in B_{bs}(\Gamma_0)$, i.e. is bounded and has bounded support, i.e. there exist a compact $\Lambda\subset\mathbb{R}^d$ and $N\in\mathbb{N}$ such that G is bounded and for any $\eta \in \Gamma_0$ with $|\eta| > N$ or $\eta \not\subset \Lambda$ on has $G(\eta) = 0$. Following the general approach of section 3, we are first going to calculate the operators $\widehat{L}$ for quasi-observables G and L^{Δ} for correlation functions k.

Theorem 4.6. *For $G \in B_{bs}(\Gamma_0)$ the operator $\widehat{L} = \widehat{L}_V + \widehat{B}$ is given by*

$$(\widehat{L}_V G)(\eta) = -(m+\lambda)|\eta| G(\eta) + \lambda \sum_{x\in\eta} \int_{\mathbb{R}^d} b(x-y) G(\eta\backslash x \cup y) \mathrm{d}y \tag{46}$$

with $\widehat{B}$ given by

$$(\widehat{B}G)(\eta) = \lambda \sum_{x\in\eta} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x-y_1, x-y_2) G(\eta\backslash x \cup y_1 \cup y_2) \mathrm{d}y. \tag{47}$$

Here $0 \le b$ describes the effective proliferation and is given by

$$b(x) = \int_{\mathbb{R}^d} a(x,y)\mathrm{d}y + \int_{\mathbb{R}^d} a(y,x)\mathrm{d}y.$$

The function $M_\alpha = M_\alpha^V + M_\alpha^B$ is given by $M_\alpha^V(\eta) = (m+3\lambda)|\eta|$ and

$$M_\alpha^B(\eta) = \lambda e^{-\alpha} \sum_{y_1\in\eta} \sum_{y_2\in\eta\backslash y_1} \int_{\mathbb{R}^d} a(x-y_1, x-y_2)\mathrm{d}x$$

If in addition the expression

$$\theta = \min\left\{ \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} a(x-y,x)dx, \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} a(x,x-y)dx \right\} \tag{48}$$

is finite, then $\widehat{L}$ acts as a bounded operator from $\mathbb{B}_\alpha$ to $\mathbb{B}_{\alpha'}$ for each $\alpha' < \alpha$. In this case the estimate

$$\|\widehat{L}\|_{\alpha\alpha'} \le \|\widehat{L}_V\|_{\alpha\alpha'} + \|\widehat{B}\|_{\alpha\alpha'} \le \frac{m+3\lambda}{e(\alpha-\alpha')} + \frac{4\lambda\theta e^{-\alpha'}}{e^2(\alpha-\alpha')^2}. \tag{49}$$

holds.

Proof. Using the $K-$transform we obtain for $x \in \gamma$

$$(KG)(\gamma\backslash x) - (KG)(\gamma) = -\sum_{\eta \Subset \gamma\backslash x} G(\eta \cup x)$$

and therefore for the first part

$$\begin{aligned} m\sum_{x\in\gamma}((KG)(\gamma\backslash x) - (KG)(\gamma)) &= -m\sum_{x\in\gamma}\sum_{\eta\Subset\gamma\backslash x} G(\eta\cup x) \\ &= -m\sum_{\eta\Subset\gamma}\sum_{x\in\eta} G(\eta) = -mK(|\cdot|G)(\gamma). \end{aligned}$$

Applying the inverse $K-$transform we arrive at the expression $-m|\eta|G(\eta)$ reflecting the natural death of each cell. For the cell-division we first note that for $x \in \gamma$ and $y_1, y_1 \notin \gamma$

$$\begin{aligned}
&(KG)(\gamma \backslash x \cup y_1 \cup y_1) - (KG)(\gamma) \\
&= \sum_{\eta \Subset \gamma \backslash x} \left(G(\eta \cup y_1) + G(\eta \cup y_2) + G(\eta \cup y_1 \cup y_2) - G(\eta \cup x) \right).
\end{aligned}$$

Therefore the birth-part is given by

$$\begin{aligned}
&\sum_{x \in \eta} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x - y_1, x - y_2) \\
&\qquad \times \left(G(\eta \backslash x \cup y_1) + G(\eta \backslash x \cup y_2) + G(\eta \backslash x \cup y_1 \cup y_2) - G(\eta) \right) \mathrm{d}y_1 \mathrm{d}y_2.
\end{aligned}$$

In the first two terms of the second part the integration over y_1 respectively y_2 can be carried out, which gives together with the substitution $y_1, y_2 \to y$

$$\begin{aligned}
&\lambda \sum_{x \in \eta} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x - y_1, x - y_2) \left(G(\eta \backslash x \cup y_1) + G(\eta \backslash x \cup y_2) \right) \mathrm{d}y_1 \mathrm{d}y_2 \\
&= \lambda \sum_{x \in \eta} \int_{\mathbb{R}^d} b(x - y) G(\eta \backslash x \cup y) \mathrm{d}y.
\end{aligned}$$

Altogether we obtain formulas (46) and (47). Let us now compute M_α, so let $G \in D(M_\alpha)$ defined in (44), then

$$\int_{\Gamma_0} |\widehat{L}G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta) \leq \int_{\Gamma_0} |\widehat{L}_V G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta) + \int_{\Gamma_0} |\widehat{B}G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta)$$

and for the $\widehat{L}_V$ we get

$$\begin{aligned}
&\int_{\Gamma_0} |\widehat{L}_V G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta) \\
&\leq \int_{\Gamma_0} (m + \lambda)|\eta||G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta) + \lambda \int_{\Gamma_0} \sum_{x \in \eta} \int_{\mathbb{R}^d} b(x - y)|G(\eta \backslash x \cup y)| e^{\alpha|\eta|} \mathrm{d}y \mathrm{d}\lambda(\eta) \\
&= \int_{\Gamma_0} (m + \lambda)|\eta||G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta) + \lambda e^{\alpha} \int_{\Gamma_0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b(x - y)|G(\eta \cup y)| e^{\alpha|\eta|} \mathrm{d}y \mathrm{d}x \mathrm{d}\lambda(\eta) \\
&= \int_{\Gamma_0} (m + \lambda)|\eta||G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta) + \lambda \int_{\Gamma_0} \sum_{y \in \eta} \int_{\mathbb{R}^d} b(x - y)|G(\eta)| e^{\alpha|\eta|} \mathrm{d}x \mathrm{d}\lambda(\eta) \\
&= \int_{\Gamma_0} (m + \lambda)|\eta||G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta) + 2\lambda \int_{\Gamma_0} |\eta||G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta)
\end{aligned}$$

and hence $M_\alpha^V(\eta) = (m+3\lambda)|\eta|$. For the second part we get

$$\begin{aligned}
&\int_{\Gamma_0} |\widehat{B}G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta) \\
&\leq \lambda \int_{\Gamma_0} \sum_{x\in\eta} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x-y_1, x-y_2)|G(\eta\backslash x \cup y_1 \cup y_2)| e^{\alpha|\eta|} \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}\lambda(\eta) \\
&= e^{-\alpha}\lambda \int_{\Gamma_0} \sum_{y_1\in\eta} \sum_{y_2\in\eta\backslash y_1} \int_{\mathbb{R}^d} a(x-y_1, x-y_2)|G(\eta)| e^{\alpha|\eta|} \mathrm{d}x \mathrm{d}\lambda(\eta).
\end{aligned}$$

If (48) holds, then $M_\alpha^B(\eta) \leq \lambda e^{-\alpha}\theta|\eta|^2$, which shows the estimate for the norm of $\|\widehat{L}\|_{\alpha\alpha'}$. □

Let us take a closer look at $\widehat{L}$. This operator is a sum of a particle number preserving part $\widehat{L}_V$ and a upper diagonal part $\widehat{B}$. Rewrite this number preserving part $\widehat{L}_V$ in the form

$$(\widehat{L}_V G)(\eta) = -(m-\lambda)|\eta|G(\eta) + \lambda \sum_{x\in\eta} \int_{\mathbb{R}^d} b(x-y) \left(G(\eta\backslash x \cup y) - G(\eta)\right) \mathrm{d}y.$$

By previous proof we know, that $(\widehat{L}_V, D(\widehat{L}_V))$ is a well-defined linear operator satisfying

$$\|\widehat{L}_V\|_{\alpha\alpha'} \leq \frac{m+3\lambda}{e(\alpha-\alpha')}.$$

Let $G = (G^{(n)})_{n=0}^\infty$ be the decomposition of a measurable function $G : \Gamma_0 \longrightarrow \mathbb{R}$ to its components and set for $n \in \mathbb{N}$

$$\begin{aligned}
&(D_n G^{(n)})(x_1, \ldots, x_n) \\
&= -(m-\lambda)nG^{(n)}(x_1, \ldots, x_n) \\
&\quad + \lambda \sum_{k=1}^n \int_{\mathbb{R}^d} b(x_k - y) \Big(G^{(n)}(x_1, \ldots, \hat{x}_k, y, \ldots, x_n) - G^{(n)}(x_1, \ldots, x_n)\Big) \mathrm{d}y \\
&= -(m-\lambda)nG^{(n)}(x_1, \ldots, x_n) + (A_n G)^{(n)}(x_1, \ldots, x_n),
\end{aligned}$$

where $\hat{x}_k$ means that integration over the variable x_k should be omitted. For each $n \in \mathbb{N}_0$ the operator $\widehat{L}_V$ is diagonal, i.e. it acts only on $G^{(n)}$. The equation

$$\frac{\partial G_t^{(n)}}{\partial t} = D_n G_t^{(n)}, \quad G_t^{(n)}|_{t=0} = G_0^{(n)}$$

has a solution $G_t^{(n)} = e^{-(m-\lambda)nt} H_t^{(n)}$, were $H_t^{(n)}$ solves

$$\frac{\partial H_t^{(n)}}{\partial t} = A_n H_t^{(n)}, \quad H_t^{(n)}|_{t=0} = G_0^{(n)}.$$

Therefore let us try to understand the meaning of A_n. This part describes a Random walk in continuous time of each cell with intensity 2λ and the probability of a cell located at $x \in \mathbb{R}^d$ to jump in the region $\mathrm{d}y$ is given by

$$\frac{1}{2} b(x-y) \mathrm{d}y.$$

Lemma 4.7. *D_n is a bounded linear operator on $L^1((\mathbb{R}^d)^n)$ and $L^\infty((\mathbb{R}^d)^n)$ and the corresponding semigroup is a positive contraction semigroup. Moreover, if $\lambda \leq m$, then $(\widehat{L}_V, D(\widehat{L}_V))$ has an extension to a of a sub-stochastic semigroup on $\mathbb{B}_\alpha$ for each α.*

Proof. The first assertion is a consequence of the Beurling-Deny-Criterion, c.f. [17]. Assume $\lambda \leq m$ and consider

$$(\widehat{L}_V G)(\eta) = -(m+\lambda)|\eta|G(\eta) + \sum_{x\in\eta} \int_{\mathbb{R}^d} b(x-y)G(\eta\backslash x \cup y)\mathrm{d}y,$$

the second summand is positive and defined on the same domain as the negative multiplication operator $-(m+\lambda)|\eta|$. Now an application of [19] shows the assertion, provided

$$\int_{\Gamma_0} (\widehat{L}_V G)(\eta) e^{-\alpha|\eta|}\mathrm{d}\lambda(\eta) \leq 0$$

for $0 \leq G \in D(\widehat{L}_V)$. But this is true, since $\lambda \leq m$. □

Note that also for $m < \lambda$ an evolution $t \longmapsto G_t$ can be constructed. Let $G_0 = (G_0^{(n)})_{n\in\mathbb{N}}$ be measurable such that each component $G_0^{(n)}$ is integrable. Then $e^{-(m-\lambda)nt}e^{tA_n}G_0^{(n)} = e^{tD_n}G_0^{(n)}$ is well-defined and the vector $G_t = (e^{tD_n}G_0^{(n)})_{n=0}^\infty$ is the unique component-wise solution to

$$\begin{cases} \dfrac{\partial G_t}{\partial t} = \widehat{L}_V G_t \\ G_t|_{t=0} = G_0 \end{cases}.$$

This solution, if $G_0 \in \mathbb{B}_\alpha$, evolves in the scale of Banach spaces $\mathbb{B}_\alpha$ with $\alpha(t) = \alpha + (m-\lambda)t$, i.e. $G_t \in \mathbb{B}_{\alpha(t)}$, which follows from

$$\begin{aligned} \|G_t\|_{\alpha(t)} &= \sum_{n=0}^\infty \frac{e^{-(m-\lambda)nt}e^{\alpha(t)n}}{n!} \int_{(\mathbb{R}^d)^n} |e^{tA_n}G_0^{(n)}(x_1,\ldots,x_n)|\mathrm{d}x_1\ldots\mathrm{d}x_n \\ &\leq \sum_{n=0}^\infty \frac{e^{\alpha n}}{n!} \int_{(\mathbb{R}^d)^n} |G_0^{(n)}(x_1,\ldots,x_n)|\mathrm{d}x_1\ldots\mathrm{d}x_n = \|G_0\|_\alpha. \end{aligned}$$

The presence of the perturbation $\widehat{B}$ implies that the solution cannot satisfy $G_t \in \mathbb{B}_{\alpha(t)}$ for $t \geq 0$ and any $\alpha(t)$. Since $\widehat{B}$ sends functions of $n+1$ variables to functions of n variables it is not helpful to discuss a direct solution formula, though it is possible. More precise results will be investigated in terms of correlation functions.

Lemma 4.8. *For $k : \Gamma_0 \longrightarrow \mathbb{R}$ such that $|k(\eta)| \leq |\eta|!C^{|\eta|}$ for some constant $C > 0$ the operator L^Δ is given by*

$$L^\Delta = L_V^\Delta + B^\Delta,$$

where L_V^{Δ} *is given by the same expression as* $\widehat{L}_V$ *and* B^{Δ} *by*

$$(B^{\Delta}k)(\eta) = \lambda \sum_{y_1 \in \eta} \sum_{y_2 \in \eta \setminus y_1} \int_{\mathbb{R}^d} a(x-y_1, x-y_2) k(\eta \cup x \setminus y_1 \setminus y_2) dx. \tag{50}$$

Moreover $L_V^{\Delta} \in L(\mathbb{B}_{\alpha'}^*, \mathbb{B}_{\alpha}^*)$ *and if* (48) *holds, then* $B^{\Delta} \in L(\mathbb{B}_{\alpha'}^*, \mathbb{B}_{\alpha}^*)$ *with* $\|L_V^{\Delta}\|_{\alpha'\alpha} = \|\widehat{L}_V\|_{\alpha\alpha'}$ *and* $\|B^{\Delta}\|_{\alpha'\alpha} = \|\widehat{B}\|_{\alpha\alpha'}$.

Proof. For $G \in B_{bs}(\Gamma_0)$ and k as described above, the operator L^{Δ} is uniquely determined by the pairing

$$\int_{\Gamma_0} (\widehat{L}G)(\eta) k(\eta) \mathrm{d}\lambda(\eta) = \int_{\Gamma_0} G(\eta)(L^{\Delta}k)(\eta) \mathrm{d}\lambda(\eta).$$

The negative multiplication part will therefore not change and for the second part we get by the formula from Lemma 4.2

$$\begin{aligned}
&\lambda \int_{\Gamma_0} \sum_{x \in \eta} \int_{\mathbb{R}^d} b(x-y) G(\eta \setminus x \cup y) \mathrm{d}y k(\eta) \mathrm{d}\lambda(\eta) \\
&= \lambda \int_{\Gamma_0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b(x-y) G(\eta \cup y) k(\eta \cup x) \mathrm{d}y \mathrm{d}x \mathrm{d}\lambda(\eta) \\
&= \lambda \int_{\Gamma_0} \sum_{y \in \eta} \int_{\mathbb{R}^d} b(x-y) k(\eta \cup x \setminus y) \mathrm{d}x G(\eta) \mathrm{d}\lambda(\eta).
\end{aligned}$$

Finally

$$\begin{aligned}
&\int_{\Gamma_0} (\widehat{B}G)(\eta) k(\eta) \mathrm{d}\lambda(\eta) \\
&= \lambda \int_{\Gamma_0} \sum_{x \in \eta} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x-y_1, x-y_2) G(\eta \setminus x \cup y_1 \cup y_2) dy_1 dy_2 k(\eta) \mathrm{d}\lambda(\eta) \\
&= \lambda \int_{\Gamma_0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x-y_1, x-y_2) G(\eta \cup y_1 \cup y_2) k(\eta \cup x) \mathrm{d}x \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}\lambda(\eta) \\
&= \lambda \int_{\Gamma_0} \sum_{y_1 \in \eta} \sum_{y_2 \in \eta \setminus y_1} \int_{\mathbb{R}^d} a(x-y_1, x-y_2) k(\eta \cup x \setminus y_1 \setminus y_2) dx G(\eta) \mathrm{d}\lambda(\eta),
\end{aligned}$$

proves the assertion. The second part follows from (43). □

Again, the equation for L_V^{Δ} can be solved explicitly and since B^{Δ} has now lower diagonal structure the equation

$$\frac{\partial k_t}{\partial t} = L^{\Delta} k_t$$

has a unique solution given by a recursive formula. More precisely let $k_0 = (k_0^{(n)})_{n=0}^{\infty}$ be non-negative and measurable such that $k_0^{(n)} \in L^{\infty}((\mathbb{R}^d)^n)$. Denote by B_n^{Δ} the operator given by (50) taking functions from $n-1$ variables to functions with n variables. The solution to (4) is given by

$$k_t^{(n+1)} = e^{-(m-\lambda)(n+1)t} e^{tA_{n+1}} k_0^{(n+1)} + \int_0^t e^{-(m-\lambda)(n+1)(t-s)} e^{(t-s)A_{n+1}} B_{n+1}^{\Delta} k_s^{(n)} ds. \tag{51}$$

Theorem 4.9. *For each $k_0 \geq 0$ measurable, such that $k_0^{(n)} \in L^\infty((\mathbb{R}^d)^n)$, there exist a unique solution $k_t \geq 0$, given recursively by formula (51). If θ is finite, then for each initial conditions satisfying $k_0(\eta) \leq |\eta|! C^{|\eta|}$ for some constant $C > 0$, this solution obeys the bound*

$$k_t(\eta) \leq |\eta|!(C+t)^{|\eta|}(1+\theta)^{|\eta|}\kappa(t)^{|\eta|} e^{-(m-\lambda)|\eta| t}$$

with $\kappa(t) = \max\{1, \lambda, \lambda e^{(m-\lambda)t}\}$. If, in addition there is $\delta > 0$ such that $a(x,y) \geq \alpha > 0$ for some $\alpha > 0$ and all $|x|, |y| \leq \delta$, then for each $k_0(\eta) = C^{|\eta|}$ the solution k_t cannot be sub-poissonian, i.e. for any $\eta \in \Gamma_0$ with:

$$\forall x, y \in \eta,\ x \neq y: \ |x-y| < \delta$$

the estimate

$$k_t(\eta) \geq \beta^{|\eta|} e^{-(m-\lambda)|\eta| t} |\eta|! \quad t \geq 1$$

holds, where $\beta = \min\{C, \lambda\alpha, \delta, |B_\delta|\}$ with $\delta = \begin{cases} \frac{1}{\lambda - m}, & \lambda > m \\ 1, & \lambda \leq m \end{cases}$ and $|B_\delta|$ is the Lebesgue volume of the Ball B_δ of radius δ.

Proof. For the bound from above we will proceed by induction on the number of cells $|\eta|$. The first correlation function is given by

$$k_t^{(1)} = e^{-(m-\lambda)t} e^{tA_1} k_0^{(1)}$$

and hence by positivity of $(e^{tA_1})_{t \geq 0}$ and $e^{tA_1} C = C$

$$k_t^{(1)} \leq e^{-(m-\lambda)t} C \leq (C+t)(1+\theta)\kappa(t) e^{-(m-\lambda)t}.$$

For $n \to n+1$ we get with $|\eta| = n+1$

$$\begin{aligned} k_t^{(n+1)} &\leq e^{-(m-\lambda)(n+1)t}(n+1)! C^{n+1} + \int_0^t e^{-(m-\lambda)(n+1)(t-s)} e^{(t-s)A_{n+1}} B_{n+1}^{\Delta} k_s^{(n)} ds \\ &\leq e^{-(m-\lambda)(n+1)t}(n+1)! C^{n+1} \\ &\quad + (n+1)!\kappa(t)^{n+1}(1+\theta)^{n+1}\left((C+t)^{n+1} - C^{n+1}\right) e^{-(m-\lambda)(n+1)t} \\ &\leq (n+1)!(C+t)^{n+1}(1+\theta)^{n+1}\kappa(t)^{n+1} e^{-(m-\lambda)(n+1)t}. \end{aligned}$$

Here we used the fact that for $s \leq t$ we have $\kappa(s) \leq \kappa(t)$. For the second part let $k_0^{(n)} = C^n$, then $e^{tA_n} k_0 = C^n$ and therefore $k_t^{(1)} = e^{-(m-\lambda)t} C \geq \beta e^{-(m-\lambda)t}$.

For $n \to n+1$ and $t \geq 1$ we obtain

$$\begin{aligned} k_t^{(n+1)} &\geq e^{-(m-\lambda)(n+1)t} C^{n+1} \\ &\qquad + \lambda \int_0^t e^{-(m-\lambda)(n+1)(t-s)} (n+1) n\alpha e^{-(m-\lambda)ns} \beta^n n! \mathrm{d}s |B_\delta| \\ &\geq e^{-(m-\lambda)(n+1)t} \int_0^t e^{(m-\lambda)s} ds \beta^n (n+1)! \alpha\lambda \\ &\geq e^{-(m-\lambda)(n+1)t} \beta^{n+1} (n+1)! \end{aligned}$$

□

This Theorem shows that if the probability distribution for the new cells, has no hard core, i.e. $a(0) > 0$ for continuous distributions, than the system will consist of clusters. Appearance of such clusters are due to the operator B^Δ. The part L_V^Δ contains information about asymptotic behaviour, speed of propagation etc., whereas B^Δ contains information about correlations of the system. Assume for simplicity, that in the cell-division the position of the new cells are independent of each other, then we may write $a(x,y) = c(x)c(y)$ for some symmetric function $0 \leq c \in L^1(\mathbb{R}^d)$ normalized to 1. If for example c is continuous and non-vanishing, then previous assumptions are satisfied and we get the bound

$$\beta^n n! e^{-(m-\lambda)nt} \leq k_t^{(n)}$$

on $\mathbb{R}^d$. Hence the system will be always clustering. The same results were shown for the case $a(x,y) = c(x)\delta(y)$, where each cell creates a new cell and its location is described by the kernel c. In contrast to this model, the old cell will not die. Clearly such models should have the same properties. Previous Theorem justifies the assumption, that it is enough to work with the usual Contact Model $a(x,y) = c(x)\delta(y)$.

Scaling

Following the general scheme of mesoscopic scaling described in previous chapter, we have to scale potentials like $a \longmapsto \varepsilon a$ and accelerate birth by a factor $\frac{1}{\varepsilon}$. Clearly, since the birth only consists of the a-part, this will not change the operator itself, i.e. $L_\varepsilon = L$. First we will look at Quasi-observables. In this case the renormalized operator is given by $\widehat{L}_{\varepsilon,ren} = R_{\varepsilon^{-1}} \widehat{L} R_\varepsilon$, where $R_\varepsilon G(\eta) = \varepsilon^{|\eta|} G(\eta)$. Applying this to this model, one gets $\widehat{L}_{\varepsilon,ren} = \widehat{L}_V + \varepsilon \widehat{B}$. Hence we can realize $\widehat{L}_{\varepsilon,ren}$ on the same domain as $\widehat{L}$.

Lemma 4.10. *For each $G \in D(N_\alpha)$ one has*

$$\widehat{L}_{\varepsilon,ren} G \longrightarrow \widehat{L}_V G, \quad \varepsilon \to 0$$

in the norm $\|\cdot\|_\alpha$ for each $\alpha \in \mathbb{R}$. Moreover if (48) *holds, the operator $\widehat{L}_{\varepsilon,ren}$ converges to $\widehat{L}_V$ in the operator norm of $L(\mathbb{B}_\alpha, \mathbb{B}_{\alpha'})$ for each $\alpha' < \alpha$.*

Proof. Let $G \in D(M_\alpha)$, then $\widehat{L}_{\varepsilon,ren} G - \widehat{L}_V G = \varepsilon \widehat{B} G \in \mathbb{B}_\alpha$, which shows the first assertion. For the second part we know $\widehat{B} \in L(\mathbb{B}_\alpha, \mathbb{B}_{\alpha'})$ and thus, for $G \in \mathbb{B}_\alpha$,

$$\|\widehat{L}_{\varepsilon,ren}G - \widehat{L}_V G\|_{\alpha'} = \varepsilon\|\widehat{B}G\|_\alpha \le \varepsilon\|\widehat{B}\|_{\alpha\alpha'}\|G\|_\alpha. \qquad \square$$

Similarly we get.

Lemma 4.11. *Assume* (48) *holds, then for each* $\alpha' < \alpha$ *the operator* $L^\Delta_{\varepsilon,ren}$ *converges in the operatornorm of* $L(\mathbb{B}^*_{\alpha'}, \mathbb{B}^*_\alpha)$ *to the operator* L^Δ_V.

Proof. Let $k \in \mathbb{B}^*_{\alpha'}$, then

$$\|L^\Delta_{\varepsilon,ren}k - L^\Delta_V k\|_\alpha = \varepsilon\|B^\Delta k\|_\alpha \le \varepsilon\|B^\Delta\|_{\alpha'\alpha}\|k\|_{\alpha'}$$

implies the assertion. $\square$

Hence mesoscopic scaling suppresses the microscopic effects like cell-correlations etc. The resulting model has less information but is simpler to analyse. As already shown $\widehat{L}_V$ or L^Δ_V will lead to evolutions $t \longmapsto G_t$ or $t \longmapsto k_t$, which can preserve the spaces $\mathbb{B}_\alpha$ respectively $\mathbb{B}^*_\alpha$. Finally we will show the chaos preservation property and derive the equations for the local densities of the kinetic description.

Theorem 4.12. *Let* $k_0(\eta) = \prod\limits_{x\in\eta} \rho_0(x)$ *with* $0 \le \rho_0 \in L^\infty(\mathbb{R}^d)$. *Then the unique solution to*

$$\begin{cases} \dfrac{\partial k_t}{\partial t} = L^\Delta_V k_t \\ k_t|_{t=0} = e_\lambda(\rho_0) \end{cases} \tag{52}$$

is given by $k_t(\eta) = \prod\limits_{x\in\eta} \rho_t(x)$, *where* $\rho_t \ge 0$ *is a classical solution to the mesoscopic equation*

$$\begin{cases} \dfrac{\partial \rho_t}{\partial t} = -(m+\lambda)\rho_t + b * \rho_t \\ \rho_t|_{t=0} = \rho_0. \end{cases}$$

Proof. Since for each $k_0 = (k_0^{(n)})_{n=0}^\infty$ such that all $k_0^{(n)}$ are essentially bounded there exists a unique solution, we have only to check that also $k_t(\eta) = \prod\limits_{x\in\eta} \rho_t(x)$ solves (52). Note, that for the given function ρ_0 a unique classical solution for the mesoscopic equation exists on $\mathbb{R}_+$. Computing

$$\frac{\partial k_t}{\partial t}(\eta) = \sum_{x\in\eta} \frac{\partial \rho_t}{\partial t}(x) e_\lambda(\rho_t; \eta\backslash x)$$

and

$$(L^\Delta_V e_\lambda(\rho_t))(\eta) = \sum_{x\in\eta} e_\lambda(\rho_t;\eta\backslash x)\left(-(m+\lambda)\rho_t(x) + \int\limits_{\mathbb{R}^d} b(x-y)\rho_t(y)dy\right)$$

we conclude that k_t given by the formula is a solution. $\square$

In this model all cells are independent of each other, which implies that the equation in the kinetic description will be linear. Non-linearities enter through interactions of cells. So in more realistic models the typical equation will consist of convolutions and powers of ρ_t.

5 Two-component models

The extension to two-component models is straightforward. The Banach spaces $\mathbb{B}_\alpha$ of functions $G : \Gamma_0^2 \longrightarrow \mathbb{R}$ becomes $\mathbb{B}_\alpha = L^1(\Gamma_0^2, e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}d\lambda)$ with $\alpha = (\alpha^+, \alpha^-)$ equipped with the norm

$$\|G\|_\alpha = \int_{\Gamma_0^2} |G(\eta^+, \eta^-)| e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}\lambda(\eta^+, \eta^-)$$

and the dual space of correlation functions $k \in \mathbb{B}_\alpha^* = L^\infty(\Gamma_0^2, e^{-\alpha^+|\eta^+|}e^{-\alpha^-|\eta^-|}\mathrm{d}\lambda)$ with the norm

$$\|k\|_\alpha = \operatorname*{ess\,sup}_{(\eta^+,\eta^-)\in\Gamma_0^2} |k(\eta^+, \eta^-)| e^{-\alpha^+|\eta^+|} e^{-\alpha^-|\eta^-|}.$$

The dual pairing for these spaces is given by

$$\langle G, k\rangle = \int_{\Gamma_0^2} G(\eta^+, \eta^-) k(\eta^+, \eta^-) \mathrm{d}\lambda(\eta^+, \eta^-)$$

and satisfies $|\langle G, k\rangle| \leq \|G\|_\alpha \|k\|_\alpha$. For pairs $\alpha' = (\alpha'^+, \alpha'^-)$ and $\alpha = (\alpha^+, \alpha^-)$ we will write $\alpha' < \alpha$ if $\alpha'^+ < \alpha^+$ and $\alpha'^- < \alpha^-$ holds. In such case for an operator $\widehat{L} \in L(\mathbb{B}_\alpha, \mathbb{B}_{\alpha'})$ for all $\alpha' < \alpha$ and its dual operator $L^\Delta \in L(\mathbb{B}_{\alpha'}^*, \mathbb{B}_\alpha^*)$

$$\|\widehat{L}\|_{\alpha\alpha'} = \|L^\Delta\|_{\alpha'\alpha}$$

holds. Also there exists a measurable function $M_\alpha : \Gamma_0^2 \longrightarrow \mathbb{R}_+$ such that

$$\int_{\Gamma_0^2} |\widehat{L}G(\eta^+, \eta^-)| e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}\lambda(\eta^+, \eta^-) \leq \int_{\Gamma_0^2} M_\alpha(\eta) |G(\eta^+, \eta^-)| e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}\lambda(\eta^+, \eta^-),$$

so all previous methods can be applied in this extended two-component setting.

In this section we will derive, similarly as for the one-component case, all necessary formulas to derive the kinetic description. Such list of interactions will be not complete, but should cover most of the interesting models in cell biology. Here we will restrict in many cases to interactions on $+$-cells. The case of $-$-cells in the presence of interactions with $+$-cells can be derived in the same way, simply exchanging all $+$ with $-$ and vice versa.

Define the relative energies $E(x, \gamma^\pm) = \sum_{y\in\gamma^\pm} a(x-y)$ and E_ϕ, E_ψ in the same way with a replaced by ϕ respectively ψ. We will assume that $0 \leq a, \phi, \psi \in L^1(\mathbb{R}^d)$ are symmetric.

Example 14. Let us consider first consider the Markov generator

$$(LF)(\gamma) = \sum_{x\in\gamma^+} E(x, \gamma^-)(F(\gamma^+ \backslash x, \gamma^-) - F(\gamma)).$$

Each cell at position $x \in \gamma^+$ can die due to the interaction $\sum\limits_{y\in\gamma^-} a(x-y)$ with cells from different type. The operator on quasi-observables is given by

$$(\widehat{L}G)(\eta) = -\sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)G(\eta) - \sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)G(\eta^+,\eta^-\backslash y).$$

The functions M_α and N_α are in such case given by

$$N_\alpha(\eta) = M_\alpha(\eta) = \sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y) + e^{\alpha^-}\langle a\rangle|\eta^+|.$$

After scaling, i.e. $a \to \varepsilon a$ and renormalization, we arrive in the limit to the operator

$$(\widehat{L}_V G)(\eta) = -\sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)G(\eta^+,\eta^-\backslash y),$$

which is defined on $D(N_\alpha^V)$ with $N_\alpha^V(\eta) = e^{-\alpha^-}\langle a\rangle|\eta^+|$. The convergence holds for each $G \in D(N_\alpha)$ in $\mathbb{B}_\alpha$, since only the multiplicative part is multiplied by ε. On the level of correlation functions L^Δ is given by

$$(L^\Delta k)(\eta) = -\sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)k(\eta) - \sum_{x\in\eta^+}\int_{\mathbb{R}^d} a(x-y)k(\eta^+,\eta^-\cup y)\mathrm{d}y$$

and

$$(L_V^\Delta k)(\eta) = -\sum_{x\in\eta^+}\int_{\mathbb{R}^d} a(x-y)k(\eta^+,\eta^-\cup y)\mathrm{d}y.$$

Therefore the kinetic description is given by

$$\frac{\partial\rho_t^+}{\partial t}(x) = -\rho_t^+(x)(a*\rho_t^-)(x), \quad \frac{\partial\rho_t^-}{\partial t}(x) = 0.$$

Example 15. Let us consider here the case, where the interaction is not quadratic in the number of particles, but exponential instead. In such case the Markov generator is given by

$$(LF)(\gamma) = \sum_{x\in\gamma^+} e^{E_\phi(x,\gamma^-)}e^{E_\psi(x,\gamma^+\backslash x)}(F(\gamma^+\backslash x,\gamma^-) - F(\gamma)).$$

The operator on quasi-observables is given by

$$\begin{aligned}(\widehat{L}G)(\eta) = &-\sum_{\xi\subset\eta}\sum_{x\in\xi^+} e^{E_\psi(x,\xi^+\backslash x)}e^{E_\phi(x,\xi^-)}\\ &\times e_\lambda(e^{\psi(x-\cdot)}-1;\eta^+\backslash\xi^+)e_\lambda(e^{\phi(x-\cdot)}-1;\eta^-\backslash\xi^-)G(\xi)\end{aligned}$$

and on correlation functions by

$$\begin{aligned}(L^\Delta k)(\eta) = &-\sum_{x\in\eta^+} e^{E_\psi(x,\eta^+\backslash x)}e^{E_\phi(x,\eta^-)}\\ &\times\int_{\Gamma_0^2} e_\lambda(e^{\psi(x-\cdot)}-1;\xi^+)e_\lambda(e^{\phi(x-\cdot)}-1;\xi^-)k(\eta\cup\xi)\mathrm{d}\lambda^2(\xi),\end{aligned}$$

where

$$\beta_\psi(\alpha^+) = \exp\Big(e^{\alpha^+}\int_{\mathbb{R}^d}(e^{\psi(x)}-1)\mathrm{d}x\Big), \quad \beta_\phi(\alpha^-) = \exp\Big(e^{\alpha^-}\int_{\mathbb{R}^d}(e^{\phi(x)}-1)\mathrm{d}x\Big),$$

and

$$M_\alpha(\eta) = \beta_\psi(\alpha^+)\beta_\phi(\alpha^-)\sum_{x\in\eta^+} e^{E_\psi(x,\eta^+\setminus x)}e^{E_\phi(x,\eta^-)}.$$

The rescaled operators $\widehat{L}_{\varepsilon,ren}G$ have for $\eta\in\Gamma_0^2$ the form

$$-\sum_{\xi\subset\eta}\sum_{x\in\xi^+} e^{\varepsilon E_\psi(x,\xi^+\setminus x)}e^{\varepsilon E_\phi(x,\xi^-)} \times e_\lambda\left(\frac{e^{\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\eta^+\setminus\xi^+\right)e_\lambda\left(\frac{e^{\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\eta^-\setminus\xi^-\right)G(\xi)$$

and on correlation functions $L^\Delta_{\varepsilon,ren}k$ is given by

$$-\sum_{x\in\eta^+} e^{\varepsilon E_\psi(x,\eta^+\setminus x)}e^{\varepsilon E_\phi(x,\eta^-)} \times\int_{\Gamma_0^2} e_\lambda\left(\frac{e^{\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\xi^+\right)e_\lambda\left(\frac{e^{\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\xi^-\right)k(\eta\cup\xi)\mathrm{d}\lambda^2(\xi),$$

so

$$\beta^*_\psi(\alpha^+) = \sup_{\varepsilon\in(0,1]}\exp\Big(\frac{e^{\alpha^+}}{\varepsilon}\int_{\mathbb{R}^d}(e^{\varepsilon\psi(x)}-1)\mathrm{d}x\Big),$$

$$\beta^*_\phi(\alpha^+) = \sup_{\varepsilon\in(0,1]}\exp\left(\frac{e^{\alpha^+}}{\varepsilon}\int_{\mathbb{R}^d}(e^{\varepsilon\phi(x)}-1)\mathrm{d}x\right),$$

and

$$N_\alpha(\eta) = \beta^*_\psi(\alpha^+)\beta^*_\phi(\alpha^-)\sum_{x\in\eta^+} e^{E_\psi(x,\eta^+\setminus x)}e^{E_\phi(x,\eta^-)}.$$

Taking the limit $\varepsilon\to 0$ we obtain

$$(\widehat{L}_V G)(\eta) = -\sum_{\xi\subset\eta}\sum_{x\in\xi^-} e_\lambda(\psi(x-\cdot);\eta^+\setminus\xi^+)e_\lambda(\phi(x-\cdot);\eta^-\setminus\xi^-)G(\xi)$$

and

$$(L^\Delta_V k)(\eta) = -\sum_{x\in\eta^+}\int_{\Gamma_0^2} e_\lambda(\psi(x-\cdot);\xi^+)e_\lambda(\phi(x-\cdot);\xi^-)k(\eta\cup\xi)\mathrm{d}\lambda^2(\xi),$$

so $N^V_\alpha(\eta) = \exp(e^{\alpha^+}\langle\psi\rangle + e^{\alpha^-}\langle\phi\rangle)|\eta^+|$. Finally we see that the kinetic description is given by

$$\frac{\partial\rho_t^+}{\partial t}(x) = -\rho_t^+(x)e^{(\psi*\rho_t^+)(x)}e^{(\phi*\rho_t^-)(x)}, \quad \frac{\partial\rho_t^-}{\partial t}(x) = 0.$$

Example 16. Let us look at the model with fecundity including interactions with both types of cells. The Markov generator is given by

$$(LF)(\gamma) = \sum_{x\in\gamma^+} e^{-E_\phi(x,\gamma^-)} e^{-E_\psi(x,\gamma^+\setminus x)} \int_{\mathbb{R}^d} a^+(x-y)(F(\gamma^+\cup y,\gamma^-) - F(\gamma))\mathrm{d}y,$$

where E_ϕ, E_ψ are given by the same expressions as in the previous example. In such case the operator on quasi-observables is given by

$$\begin{aligned}(\widehat{L}G)(\eta) = &\sum_{\xi\subset\eta}\sum_{x\in\xi^+} e^{-E_\phi(x,\xi^-)} e^{-E_\psi(x,\xi^+\setminus x)} \\ &\times \int_{\mathbb{R}^d} a(x-y) e_\lambda(e^{-\phi(x-\cdot)}-1;\eta^-\setminus\xi^-) e_\lambda(e^{-\psi(x-\cdot)}-1;\eta^+\setminus\xi^+) \\ &\times (G(\xi^+\setminus x\cup y,\xi^-) + G(\xi^+\cup y,\xi^-))\mathrm{d}y\end{aligned}$$

and on correlation functions by

$$\begin{aligned}(L^\Delta k)(\eta) = &\sum_{y\in\eta^+}\int_{\Gamma_0^2}\int_{\mathbb{R}^d} a(x-y)k(\eta\cup\xi\setminus y\cup x) e^{-E_\phi(x,\eta^-)} e^{-E_\psi(x,\eta^+\setminus y)} \\ &\times e_\lambda(e^{-\phi(x-\cdot)}-1;\xi^-) e_\lambda(e^{-\psi(x-\cdot)}-1;\xi^+)\mathrm{d}x\mathrm{d}\lambda^2(\xi) \\ &+ \sum_{x\in\eta^+}\sum_{y\in\eta^+\setminus x} a(x-y) e^{-E_\phi(y,\eta^-)} e^{-E_\psi(y,\eta^+\setminus y)} \\ &\times \int_{\Gamma_0^2} k(\eta\cup\xi\setminus y) e_\lambda(e^{-\phi(x-\cdot)}-1;\xi^-) e_\lambda(e^{-\psi(x-\cdot)}-1;\xi^+)\mathrm{d}\lambda^2(\xi).\end{aligned}$$

Hence M_α can be chosen as

$$\begin{aligned}M_\alpha(\eta) = &\beta_\psi(\alpha^+)\beta_\phi(\alpha^-)\sum_{y\in\eta^+}\int_{\mathbb{R}^d} a(x-y) e^{\psi(x-y)} e^{-E_\phi(x,\eta^-)} e^{-E_\psi(x,\eta^+)}\mathrm{d}x \\ &+ e^{-\alpha^+}\beta_\psi(\alpha^+)\beta_\phi(\alpha^-)\sum_{x\in\eta^+}\sum_{y\in\eta^+\setminus x} a(x-y) e^{-E_\phi(y,\eta^-)} e^{-E_\psi(y,\eta^+\setminus y)},\end{aligned}$$

where $\beta_\psi(\alpha^+) = \exp\Big(e^{\alpha^+}\int_{\mathbb{R}^d}(1-e^{-\psi(x)})\mathrm{d}x\Big)$ and $\beta_\phi(\alpha^-) = \exp\Big(e^{\alpha^-}\int_{\mathbb{R}^d}(1-e^{-\phi(x)})\mathrm{d}x\Big)$. Rescaling $a\to\varepsilon a$, $\phi\to\varepsilon\phi$, $\psi\to\varepsilon\psi$, putting $\frac{1}{\varepsilon}$ in front of the generator and renormalizing, we arrive at

$$\begin{aligned}&(\widehat{L}_{\varepsilon,ren}G)(\eta) \\ &= \sum_{\xi\subset\eta}\sum_{x\in\xi^+} e^{-\varepsilon E_\phi(x,\xi^-)} e^{-\varepsilon E_\psi(x,\xi^+\setminus x)} \int_{\mathbb{R}^d} a(x-y) e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\eta^-\setminus\xi^-\right) \\ &\quad\times e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\eta^+\setminus\xi^+\right)(G(\xi^+\setminus x\cup y,\xi^-) + \varepsilon G(\xi^+\cup y,\xi^-))\mathrm{d}y\end{aligned}$$

and on correlation functions at

$$
\begin{aligned}
&(L^{\Delta}_{\varepsilon,ren}k)(\eta)\\
&= \sum_{y\in\eta^+}\int_{\Gamma_0^2}\int_{\mathbb{R}^d} a(x-y)k(\eta\cup\xi\backslash y\cup x)e^{-\varepsilon E_\phi(x,\eta^-)}e^{-\varepsilon E_\psi(x,\eta^+\backslash y)}\\
&\qquad\times e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\xi^-\right)e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\xi^+\right)\mathrm{d}x\mathrm{d}\lambda^2(\xi).\\
&+\varepsilon\sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a(x-y)e^{-\varepsilon E_\phi(y,\eta^-)}e^{-\varepsilon E_\psi(y,\eta^+\backslash y)}\\
&\qquad\times\int_{\Gamma_0^2} k(\eta\cup\xi\backslash y)e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\xi^-\right)e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\xi^+\right)\mathrm{d}\lambda^2(\xi).
\end{aligned}
$$

This yields

$$
N_\alpha(\eta)=\exp(e^{\alpha^+}\langle\psi\rangle+e^{\alpha^-}\langle\phi\rangle)\Big(\langle ae^{\psi}\rangle|\eta^+|+e^{-\alpha^+}\sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x}a(x-y)\Big).
$$

Taking the limit $\varepsilon\to 0$, we arrive at

$$
\begin{aligned}
(\widehat{L}_V G)(\eta) &= \sum_{\xi\subset\eta}\sum_{x\in\xi^+}\int_{\mathbb{R}^d} a(x-y)e_\lambda(-\phi(x-\cdot);\eta^-\backslash\xi^-)\\
&\qquad\times e_\lambda(-\psi(x-\cdot);\eta^+\backslash\xi^+)G(\xi^+\backslash x\cup y,\xi^-)\mathrm{d}y
\end{aligned}
$$

and

$$
\begin{aligned}
(L^{\Delta}_V k)(\eta) &= \sum_{y\in\eta^+}\int_{\Gamma_0^2}\int_{\mathbb{R}^d} a(x-y)k(\eta\cup\xi\backslash y\cup x)\\
&\qquad\times e_\lambda(-\phi(x-\cdot);\xi^-)e_\lambda(-\psi(x-\cdot);\xi^+)\mathrm{d}x\mathrm{d}\lambda^2(\xi)
\end{aligned}
$$

and hence $N^V_\alpha(\eta)=\exp(e^{\alpha^+}\langle\psi\rangle+e^{\alpha^-}\langle\phi\rangle)\langle a\rangle|\eta^+|$. Thus the kinetic description is given by

$$
\frac{\partial\rho_t^+}{\partial t}(x)=(a*\rho_t^+)(x)e^{-(\psi*\rho_t^+)(x)}e^{-(\phi*\rho_t^-)(x)},\qquad \frac{\partial\rho_t^-}{\partial t}(x)=0.
$$

Example 17. Another possibility is, where each $-$-cell creates a new $+$-cell independent of all other cells. Such free branching is described by the formal Markov generator

$$
(LF)(\gamma)=\sum_{x\in\gamma^-}\int_{\mathbb{R}^d}a(x-y)(F(\gamma^+\cup y,\gamma^-)-F(\gamma))\mathrm{d}y.
$$

On quasi-observables it is described via

$$
\begin{aligned}
(\widehat{L}G)(\eta) &= \sum_{x\in\eta^-}\int_{\mathbb{R}^d}a(x-y)G(\eta^+\cup y,\eta^-\backslash x)\mathrm{d}y\\
&\quad+\sum_{x\in\eta^-}\int_{\mathbb{R}^d}a(x-y)G(\eta^+\cup y,\eta^-)\mathrm{d}y
\end{aligned}
$$

and on correlation functions via

$$(L^{\Delta}k)(\eta) = \sum_{y\in\eta^+}\int_{\mathbb{R}^d} a(x-y)k(\eta^+\backslash y, \eta^-\cup x)\mathrm{d}x + \sum_{x\in\eta^-}\sum_{y\in\eta^+} a(x-y)k(\eta^+\backslash y, \eta^-).$$

Hence the functions $M_\alpha = N_\alpha$ can be chosen as $M_\alpha(\eta) = e^{-\alpha^+ + \alpha^-}\langle a\rangle|\eta^+| + e^{-\alpha^+}\sum\limits_{x\in\eta^-}\sum\limits_{y\in\eta^+} a(x-y)$. After scaling we arrive at

$$(\widehat{L}_V G)(\eta) = \sum_{x\in\eta^-}\int_{\mathbb{R}^d} a(x-y)G(\eta^+\cup y, \eta^-\backslash x)\mathrm{d}y$$

and

$$(L^{\Delta}_V k)(\eta) = \sum_{y\in\eta^+}\int_{\mathbb{R}^d} a(x-y)k(\eta^+\backslash y, \eta^-\cup x)\mathrm{d}x$$

so that $M_\alpha^V(\eta) = e^{-\alpha^+ + \alpha^-}\langle a\rangle|\eta^+|$. Finally the kinetic description is given by

$$\frac{\partial\rho_t^+}{\partial t}(x) = (a*\rho_t^-)(x), \quad \frac{\partial\rho_t^-}{\partial t}(x) = 0.$$

Example 18. Let us investigate here the case of jumping particles. For simplicity let us only consider the case of additive intensities, i.e.

$$(LF)(\gamma) = \sum_{x\in\gamma^+} E(x,\gamma^-)\int_{\mathbb{R}^d} c(x-y)(F(\gamma^+\backslash x\cup y, \gamma^-) - F(\gamma))\mathrm{d}y,$$

where $0 \le a \in L^1(\mathbb{R}^d)$ is symmetric. In such case the operator on quasi-observables is given by

$$\begin{aligned}
&(\widehat{L}G)(\eta)\\
&= -\langle c\rangle \sum_{x\in\eta^+}\sum_{w\in\eta^-} a(x-w)G(\eta) - \langle c\rangle \sum_{x\in\eta^+}\sum_{w\in\eta^-} a(x-w)G(\eta^+, \eta^-\backslash w)\\
&\quad + \sum_{x\in\eta^+}\sum_{w\in\eta^-} a(x-w)\int_{\mathbb{R}^d} c(x-y)(G(\eta^+\cup y\backslash x, \eta^-\backslash w) + G(\eta^+\cup y\backslash x, \eta^-))\mathrm{d}y
\end{aligned}$$

and on correlation functions by

$$\begin{aligned}
(L^{\Delta}k)(\eta) = &-\langle c\rangle \sum_{x\in\eta^+}\sum_{w\in\eta^-} a(x-w)k(\eta) - \langle c\rangle \sum_{x\in\eta^+}\int_{\mathbb{R}^d} a(x-w)k(\eta^+, \eta^-\cup w)\mathrm{d}w\\
&+ \sum_{y\in\eta^+}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x-w)c(x-y)k(\eta^+\backslash y\cup x, \eta^-\cup w)\mathrm{d}x\mathrm{d}w\\
&+ \sum_{y\in\eta^+}\sum_{w\in\eta^-}\int_{\mathbb{R}^d} a(x-w)c(x-w)k(\eta^+\backslash y\cup x, \eta^-)\mathrm{d}x,
\end{aligned}$$

thus $M_\alpha = N_\alpha$ with

$$M_\alpha(\eta) = \langle c\rangle \sum_{x\in\eta^+}\sum_{w\in\eta^-} a(x-w) + 2\langle c\rangle\langle a\rangle e^{\alpha^-}|\eta^+| + \sum_{x\in\eta^+}\sum_{w\in\eta^-}(a*c)(x-w).$$

Scaling the potentials means $a \to \varepsilon a$ and after renormalization and limit transition $\varepsilon \to 0$ we arrive at

$$\begin{aligned}(\widehat{L}_V G)(\eta) = &-\langle c\rangle \sum_{x\in\eta^+}\sum_{w\in\eta^-} a(x-w)G(\eta^+, \eta^-\backslash w)\\ &+ \sum_{x\in\eta^+}\sum_{w\in\eta^-} a(x-w)\int_{\mathbb{R}^d} c(x-y)G(\eta^+\cup y\backslash x, \eta^-)\mathrm{d}y\end{aligned}$$

and

$$\begin{aligned}(L_V^\Delta k)(\eta) = &-\langle c\rangle \sum_{x\in\eta^+}\int_{\mathbb{R}^d} a(x-w)k(\eta^+, \eta^-\cup w)\mathrm{d}w\\ &+ \sum_{y\in\eta^+}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x-w)c(x-y)k(\eta^+\backslash y\cup x, \eta^-\cup w)\mathrm{d}x\mathrm{d}w,\end{aligned}$$

so $N_\alpha^V(\eta) = 2\langle c\rangle\langle a\rangle e^{\alpha^-}|\eta^+|$. Therefore the kinetic description is given by

$$\frac{\partial\rho_t^+}{\partial t}(x) = (c*((a*\rho_t^-)\rho_t^+))(x) - \langle c\rangle(a*\rho_t^-)(x)\rho_t^+(x), \quad \frac{\partial\rho_t^-}{\partial t}(x) = 0.$$

Let us now look at interactions, where it is allowed to change the type of cells. We will only investigate the change from + to − cells, whereas the other case can be obtained, by simply exchanging all + with − and vice versa.

Example 19. In the simplest case, the intensity to change from + to − is constant, here $q > 0$. In such case the Markov generator has the form

$$(LF)(\gamma) = q\sum_{x\in\gamma^+}(F(\gamma^+\backslash x, \gamma^-\cup x) - F(\gamma)).$$

It is not difficult to see, that in this case the operator on quasi-observables will have the form

$$(\widehat{L}G)(\eta) = -q|\eta^+|G(\eta) + q\sum_{x\in\eta^+} G(\eta^+\backslash x, \eta^-\cup x)$$

and on correlation functions it will be given by

$$(L^\Delta k)(\eta) = -q|\eta^+|k(\eta) + q\sum_{x\in\eta^-} k(\eta^+\cup x, \eta^-\backslash x),$$

so $M_\alpha(\eta) = N_\alpha(\eta) = N_\alpha^V(\eta) = q|\eta^+| + qe^{\alpha^+-\alpha^-}|\eta^-|$. Since on scaling is necessary here, we immediately obtain the kinetic description

$$\frac{\partial\rho_t^+}{\partial t}(x) = -q\rho_t^+(x), \quad \frac{\partial\rho_t^-}{\partial t}(x) = q\rho_t^+(x).$$

Example 20. Let us consider density dependent changes of types, where the intensity depends on the same type of particles, in such case the Markov generator is given by

$$(LF)(\gamma) = \sum_{x\in\gamma^+} E(x,\gamma^+)(F(\gamma^+\backslash x, \gamma^- \cup x) - F(\gamma)).$$

The generator on quasi-observables is given by

$$\begin{aligned}(\widehat{L}G)(\eta) = -E(\eta^+)G(\eta) &- \sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a(x-y)G(\eta^+\backslash y, \eta^-)\\ &+ \sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a(x-y)G(\eta^+\backslash x\backslash y, \eta^-\cup x)\\ &+ \sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a(x-y)G(\eta^+\backslash x, \eta^-\cup x)\end{aligned}$$

where $E(\eta^+) = \sum\limits_{x\in\eta^+}\sum\limits_{y\in\eta^+\backslash x} a(x-y)$. Similarly we can compute the operator for correlation functions and obtain

$$\begin{aligned}(L^{\Delta}k)(\eta) = -E(\eta^+)G(\eta) &- \sum_{x\in\eta^+}\int_{\mathbb{R}^d} a(x-y)k(\eta^+\cup y, \eta^-)\mathrm{d}y\\ &+ \sum_{x\in\eta^-}\int_{\mathbb{R}^d} a(x-y)k(\eta^+\cup x\cup y, \backslash x)\mathrm{d}y\\ &+ \sum_{x\in\eta^-}\sum_{y\in\eta^+} a(x-y)k(\eta^+\cup x, \eta^-\backslash x),\end{aligned}$$

which implies $M_\alpha = N_\alpha$ given by

$$M_\alpha(\eta) = E(\eta^+) + e^{\alpha^+}\langle a\rangle|\eta^+| + e^{2\alpha^+-\alpha^-}\langle a\rangle|\eta^-| + e^{\alpha^+-\alpha^-}\sum_{x\in\eta^-}\sum_{y\in\eta^+} a(x-y).$$

Scaling $a \to \varepsilon a$ and renormalizing we arrive at

$$\begin{aligned}(\widehat{L}_V G)(\eta) = &- \sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a(x-y)G(\eta^+\backslash y, \eta^-)\\ &+ \sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a(x-y)G(\eta^+\backslash x\backslash y, \eta^-\cup x)\end{aligned}$$

and

$$\begin{aligned}(L_V^{\Delta}k)(\eta) = &- \sum_{x\in\eta^+}\int_{\mathbb{R}^d} a(x-y)k(\eta^+\cup y, \eta^-)\mathrm{d}y\\ &+ \sum_{x\in\eta^-}\int_{\mathbb{R}^d} a(x-y)k(\eta^+\cup x\cup y, \eta^-\backslash x)\mathrm{d}y,\end{aligned}$$

so $N_\alpha^V(\eta) = e^{\alpha^+}\langle a\rangle|\eta^+| + e^{2\alpha^+-\alpha^-}\langle a\rangle|\eta^-|$. Therefore the kinetic description is given by

$$\frac{\partial\rho_t^+}{\partial t}(x) = -\rho_t^+(x)(a*\rho_t^+)(x), \quad \frac{\partial\rho_t^-}{\partial t}(x) = \rho_t^+(x)(a*\rho_t^+)(x).$$

Example 21. In this case the intensity to change the type dependent on the collection of cells of different type, here the Markov generator has the form

$$(LF)(\gamma) = \sum_{x\in\gamma^+} E(x,\gamma^-)(F(\gamma^+\backslash x, \gamma^-\cup x) - F(\gamma)).$$

Some computations yield

$$\begin{aligned}(\widehat{L}G)(\eta) = &- \sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)G(\eta) - \sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)G(\eta^+,\eta^-\backslash y)\\ &+ \sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)G(\eta^+\backslash x, \eta^-\cup x\backslash y)\\ &+ \sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)G(\eta^+\backslash x, \eta^-\cup x)\end{aligned}$$

and

$$\begin{aligned}(L^\Delta k)(\eta) = &- \sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)k(\eta) - \sum_{x\in\eta^+}\int_{\mathbb{R}^d} a(x-y)k(\eta^+,\eta^-\cup y)\mathrm{d}y\\ &+ \sum_{x\in\eta^-}\int_{\mathbb{R}^d} a(x-y)k(\eta^+\cup x, \eta^-\cup y\backslash x)\mathrm{d}y\\ &+ \sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x} a(x-y)k(\eta^+\cup y, \eta^-\backslash y).\end{aligned}$$

This yields $M_\alpha = N_\alpha$ with

$$M_\alpha(\eta) = \sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y) + e^{\alpha^-}\langle a\rangle|\eta^+| + e^{\alpha^+}\langle a\rangle|\eta^-| + e^{\alpha^+-\alpha^-}\sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x} a(x-y).$$

Scaling $a\to\varepsilon a$ and renormalizing we obtain

$$\begin{aligned}(\widehat{L}_V G)(\eta) = &- \sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)G(\eta^+,\eta^-\backslash y)\\ &+ \sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)G(\eta^+\backslash x, \eta^-\cup x\backslash y)\end{aligned}$$

and

$$\begin{aligned}(L^\Delta k)(\eta) = &- \sum_{x\in\eta^+}\int_{\mathbb{R}^d} a(x-y)k(\eta^+,\eta^-\cup y)\mathrm{d}y\\ &+ \sum_{x\in\eta^-}\int_{\mathbb{R}^d} a(x-y)k(\eta^+\cup x, \eta^-\cup y\backslash x)\mathrm{d}y,\end{aligned}$$

so $N_\alpha^V(\eta) = e^{\alpha^-}\langle a\rangle|\eta^+| + e^{\alpha^+}\langle a\rangle|\eta^-|$. Finally the kinetic equation is given by

$$\frac{\partial\rho_t^+}{\partial t}(x) = -\rho_t^+(x)(a*\rho_t^-)(x), \quad \frac{\partial\rho_t^-}{\partial t}(x) = \rho_t^+(x)(a*\rho_t^-)(x).$$

Example 22. Let us take here exponential decaying intensities for changing the type. More precisely the Markov generator is given by

$$(LF)(\gamma) = \sum_{x\in\gamma^+} e^{-E_\phi(x,\gamma^-)} e^{-E_\psi(x,\gamma^+\backslash x)} (F(\gamma^+\backslash x, \gamma^- \cup x) - F(\gamma)).$$

The operator on quasi-observables is given by

$$\begin{aligned}(\widehat{L}G)(\eta) = &\sum_{\xi\subset\eta}\sum_{x\in\xi^+} e^{-E_\phi(x,\xi^-)} e^{-E_\psi(x,\xi^+\backslash x)} (G(\xi^+\backslash x, \xi^- \cup x) - G(\xi)) \\ &\times e_\lambda(e^{-\phi(x-\cdot)} - 1; \eta^-\backslash\xi^-) e_\lambda(e^{-\psi(x-\cdot)} - 1; \eta^+\backslash\xi^+)\end{aligned}$$

and on correlation functions by

$$\begin{aligned}&(L^\Delta k)(\eta) \\ &= \sum_{x\in\eta^-} e^{-E_\phi(x,\eta^-\backslash x)} e^{-E_\psi(x,\eta^+)} \\ &\quad \times \int_{\Gamma_0^2} e_\lambda(e^{-\phi(x-\cdot)} - 1; \xi^-) e_\lambda(e^{-\psi(x-\cdot)} - 1; \xi^+) k(\eta^+ \cup \xi^+ \cup x, \eta^- \cup \xi^-\backslash x) \mathrm{d}\lambda^2(\xi) \\ &\quad - \sum_{x\in\eta^+} e^{-E_\phi(x,\eta^-)} e^{-E_\psi(x,\eta^+\backslash x)} \\ &\quad \times \int_{\Gamma_0^2} e_\lambda(e^{-\phi(x-\cdot)} - 1; \xi^-) e_\lambda(e^{-\psi(x-\cdot)} - 1; \xi^+) k(\eta \cup \xi) \mathrm{d}\lambda^2(\xi),\end{aligned}$$

which implies

$$\begin{aligned}M_\alpha(\eta) = &e^{\alpha^+ - \alpha^-} \beta_\psi(\alpha^+)\beta_\phi(\alpha^-) \sum_{x\in\eta^-} e^{-E_\psi(x,\eta^+)} e^{-E_\phi(x,\eta^-\backslash x)} \\ &+ \beta_\psi(\alpha^+)\beta_\phi(\alpha^-) \sum_{x\in\eta^+} e^{-E_\psi(x,\eta^+\backslash x)} e^{-E_\phi(x,\eta^-)}\end{aligned}$$

with

$$\beta_\psi(\alpha^+) = \exp\Bigg(e^{\alpha^+} \int_{\mathbb{R}^d} (1 - e^{-\psi(x)}) \mathrm{d}x\Bigg), \quad \beta_\phi(\alpha^-) \exp\Bigg(e^{\alpha^-} \int_{\mathbb{R}^d} (1 - e^{-\phi(x)}) \mathrm{d}x\Bigg).$$

Scaling $\phi, \psi \to \varepsilon\phi, \varepsilon\psi$ and renormalize we obtain

$$\begin{aligned}(\widehat{L}_{\varepsilon,ren}G)(\eta) = &\sum_{\xi\subset\eta}\sum_{x\in\xi^+} e^{-\varepsilon E_\phi(x,\xi^-)} e^{-\varepsilon E_\psi(x,\xi^+\backslash x)} (G(\xi^+\backslash x, \xi^- \cup x) - G(\xi)) \\ &\times e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)} - 1}{\varepsilon}; \eta^-\backslash\xi^-\right) e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)} - 1}{\varepsilon}; \eta^+\backslash\xi^+\right)\end{aligned}$$

and

$$
\begin{aligned}
&(L^{\Delta}_{\varepsilon,ren}k)(\eta) \\
&= \sum_{x\in\eta^-} e^{-\varepsilon E_\phi(x,\eta^-\setminus x)} e^{-\varepsilon E_\psi(x,\eta^+)} \\
&\quad \times \int_{\Gamma_0^2} e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\xi^-\right) e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\xi^+\right) \\
&\quad \times k(\eta^+\cup\xi^+\cup x, \eta^-\cup\xi^-\setminus x)\mathrm{d}\lambda^2(\xi) \\
&\quad - \sum_{x\in\eta^+} e^{-\varepsilon E_\phi(x,\eta^-)} e^{-\varepsilon E_\psi(x,\eta^+\setminus x)} \\
&\quad \times \int_{\Gamma_0^2} e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\xi^-\right) e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\xi^+\right) k(\eta\cup\xi)\mathrm{d}\lambda^2(\xi)
\end{aligned}
$$

so that

$$N_\alpha(\eta) = \exp(e^{\alpha^+}\langle\psi\rangle + e^{\alpha^-}\langle\phi\rangle)(e^{\alpha^+-\alpha^-}|\eta^-| + |\eta^+|) = N_\alpha^V(\eta).$$

In the limit $\varepsilon \to 0$ we arrive at

$$
\begin{aligned}
&(\widehat{L}_V G)(\eta) \\
&= \sum_{\xi\subset\eta}\sum_{x\in\xi^+} (G(\xi^+\setminus x, \xi^-\cup x) - G(\xi)) e_\lambda(-\phi(x-\cdot);\eta^-\setminus\xi^-) e_\lambda(-\psi(x-\cdot);\eta^+\setminus\xi^+)
\end{aligned}
$$

and

$$
\begin{aligned}
&(L_V^{\Delta}k)(\eta) \\
&= \sum_{x\in\eta^-}\int_{\Gamma_0^2} e_\lambda(-\phi(x-\cdot);\xi^-) e_\lambda(-\psi(x-\cdot);\xi^+) k(\eta^+\cup\xi^+\cup x, \eta^-\cup\xi^-\setminus x)\mathrm{d}\lambda^2(\xi) \\
&\quad - \sum_{x\in\eta^+}\int_{\Gamma_0^2} e_\lambda(-\phi(x-\cdot);\xi^-) e_\lambda(-\psi(x-\cdot);\xi^+) k(\eta\cup\xi)\mathrm{d}\lambda^2(\xi)
\end{aligned}
$$

and hence the kinetic description is given by

$$\frac{\partial\rho_t^+}{\partial t}(x) = -\rho_t^+(x) e^{-(\phi*\rho_t^-)(x)} e^{-(\psi*\rho_t^+)(x)}, \quad \frac{\partial\rho_t^-}{\partial t}(x) = \rho_t^+(x) e^{-(\phi*\rho_t^-)(x)} e^{-(\psi*\rho_t^+)(x)}.$$

5.1 Cell-death model

Let us start with the analysis of the first model stated in the context of two-component systems, the heuristic Markov generator is given by, c.f. (23),

$$(LF)(\gamma^+,\gamma^-) = (AF)(\gamma^+,\gamma^-) + (BF)(\gamma^+,\gamma^-) + (VF)(\gamma^+,\gamma^-).$$

The first operator A is the Contact Model for usual cells and has the form

$$(L_{CM}F)(\gamma^+,\gamma^-) = m_0 \sum_{x\in\gamma^+} (F(\gamma^+\backslash x, \gamma^- \cup x) - F(\gamma^+,\gamma^-)) + \lambda \sum_{x\in\gamma^+} \int_{\mathbb{R}^d} a(x-y)(F(\gamma^+ \cup y, \gamma^-) - F(\gamma^+,\gamma^-))\mathrm{d}y.$$

The operator B describes the evolution of $-$ cells, which can only disappear from the system, so it has the simple form

$$(BF)(\gamma^+,\gamma^-) = m_1 \sum_{x\in\gamma^-} (F(\gamma^+,\gamma^-\backslash x) - F(\gamma^+,\gamma^-)).$$

The last part describes the interaction of both types and is assumed to be of the form

$$(VF)(\gamma^+,\gamma^-) = \lambda^- \sum_{x\in\gamma^+}\sum_{y\in\gamma^-} \varphi(x-y)(F(\gamma^+\backslash x, \gamma^- \cup x) - F(\gamma^+,\gamma^-)).$$

The intensities $m_0, m_1, \lambda, \lambda^-$ are strictly positive and the potentials $0 \le a, \varphi \in L^1(\mathbb{R}^d)$ are symmetric and normalized to 1. In [9] the general form of $\widehat{L} = \widehat{A} + \widehat{B} + \widehat{V}$ was computed for $G \in B_{bs}(\Gamma_0^2)$. In this special case we get

$$(\widehat{A}G)(\eta^+,\eta^-) = -m_0|\eta^+|G(\eta^+,\eta^-) + m_0 \sum_{x\in\eta^+} G(\eta^+\backslash x, \eta^- \cup x) + \lambda \sum_{x\in\eta^+} \int_{\mathbb{R}^d} a(x-y)G(\eta^+\backslash x \cup y, \eta^-)\mathrm{d}y + \lambda \sum_{x\in\eta^+} \int_{\mathbb{R}^d} a(x-y)G(\eta^+ \cup y, \eta^-)\mathrm{d}y$$

for the first part

$$(BG)(\eta^+,\eta^-) = -m_1|\eta^-|G(\eta^+,\eta^-)$$

for the second part, and finally

$$(\widehat{V}G)(\eta^+,\eta^-) = \lambda^- \sum_{x\in\eta^+}\sum_{y\in\eta^-} \varphi(x-y)(G(\eta^+\backslash x, \eta^- \cup x\backslash y) - G(\eta^+,\eta^-\backslash y)) + \lambda^- \sum_{x\in\eta^+}\sum_{y\in\eta^-} \varphi(x-y)(G(\eta^+\backslash x, \eta^- \cup x) - G(\eta^+,\eta^-)).$$

Let us first realize this operator on the Banach space $\mathbb{B}_\alpha$.

Lemma 5.1. *The corresponding function $M_\alpha = M_\alpha^A + M^B + M_\alpha^V$ is given by*

$$M_\alpha^A(\eta^+,\eta^-) = (m_0 + e^{\alpha^+-\alpha^-}m_0 + \lambda)|\eta^+| + \lambda e^{-\alpha^+} \sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a(x-y)$$

$$M^B(\eta^+,\eta^-) = m_1|\eta^-|$$

$$M_\alpha^V(\eta^+,\eta^-) = \lambda^-(e^{\alpha^-}|\eta^+| + e^{\alpha^+}|\eta^-|) + \lambda^- e^{\alpha^+-\alpha^-} \sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x} \varphi(x-y) + \lambda^- \sum_{x\in\eta^+}\sum_{y\in\eta^-} \varphi(x-y).$$

If $a, \varphi \in L^\infty(\mathbb{R}^d)$, then $\widehat{L} \in L(\mathbb{B}_\alpha, \mathbb{B}_{\alpha'})$ for any $\alpha' < \alpha$.

Proof. Let $G \in D(M_\alpha)$, then clearly $\widehat{A}G, \widehat{B}G \in \mathbb{B}_\alpha$, so we will only check $\widehat{V}G \in \mathbb{B}_\alpha$, which follows from

$$\begin{aligned}
&\int_{\Gamma_0^2} \sum_{x\in\eta^+} \sum_{y\in\eta^-} \varphi(x-y)|G(\eta^+\backslash x, \eta^- \cup x\backslash y)| e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}\lambda(\eta^+,\eta^-)\\
&= e^{\alpha^+ + \alpha^-} \int_{\Gamma_0^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x-y)|G(\eta^+, \eta^- \cup x)| e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}x\mathrm{d}y\mathrm{d}\lambda(\eta^+,\eta^-)\\
&= e^{\alpha^+} \int_{\Gamma_0^2} \sum_{x\in\eta^-} \int_{\mathbb{R}^d} \varphi(x-y)|G(\eta^+, \eta^-)| e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}y\mathrm{d}\lambda(\eta^+,\eta^-)\\
&= e^{\alpha^+} \int_{\Gamma_0^2} |\eta^-||G(\eta^+,\eta^-)| \mathrm{d}\lambda^2(\eta^+,\eta^-)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Gamma_0^2} \sum_{x\in\eta^+} \sum_{y\in\eta^-} \varphi(x-y)|G(\eta^+\backslash x, \eta^- \cup x)| e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}\lambda(\eta^+,\eta^-)\\
&= e^{\alpha} \int_{\Gamma_0^2} \sum_{y\in\eta^-} \int_{\mathbb{R}^d} \varphi(x-y)|G(\eta^+, \eta^- \cup x) e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}x\mathrm{d}\lambda(\eta^+,\eta^-)\\
&= \int_{\Gamma_0^2} \sum_{x\in\eta^-} \sum_{y\in\eta^-\backslash x} \varphi(x-y)|G(\eta^+, \eta^-)| e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}\lambda(\eta^+,\eta^-).
\end{aligned}$$

The contributions from the negative parts can be dealt in the same way and the estimate for $\|\widehat{L}\|_{\alpha\alpha'}$ can be shown like in the one-component case. □

Again the computation of the operator L^Δ was done for a more general case in [9] which shows that for $|k(\eta)| \leq |\eta|!C^{|\eta|}$ for some $C > 0$ the operator $L^\Delta = A^\Delta + B^\Delta + V^\Delta$ is given by

$$\begin{aligned}
(A^\Delta k)(\eta^+,\eta^-) = -m_0|\eta^+|k(\eta^+,\eta^-) &+ \sum_{x\in\eta^-} k(\eta^+ \cup x, \eta^-\backslash x)\\
&+ \lambda \sum_{x\in\eta^+} \int_{\mathbb{R}^d} a(x-y)k(\eta^+\backslash x \cup y, \eta^-)\mathrm{d}y\\
&+ \lambda \sum_{x\in\eta^+} \sum_{y\in\eta^+\backslash x} a(x-y)k(\eta^+\backslash x, \eta^-)
\end{aligned}$$

and

$$(Bk)(\eta^+,\eta^-) = -m_1|\eta^-|k(\eta^+,\eta^-)|$$

and

$$
\begin{aligned}
(V^{\Delta}k)(\eta^+,\eta^-) = \lambda^- \sum_{x\in\eta^-} \int_{\mathbb{R}^d} &\varphi(x-y)k(\eta^+\cup x,\eta^-\cup y\backslash x)\mathrm{d}y \\
&- \lambda^- \sum_{x\in\eta^+} \int_{\mathbb{R}^d} \varphi(x-y)k(\eta^+,\eta^-\cup y)\mathrm{d}y \\
&+ \lambda^- \sum_{x\in\eta^-} \sum_{y\in\eta^-\backslash x} \varphi(x-y)k(\eta^+\cup x,\eta^-\backslash x) \\
&- \lambda^- \sum_{x\in\eta^+} \sum_{y\in\eta^-} \varphi(x-y)k(\eta^+,\eta^-).
\end{aligned}
$$

As before (43) can be used to realize L^{Δ} on $\mathbb{B}_{\alpha}^*$.

Scaling

For scaling let us scale the potentials a,φ to εa and $\varepsilon\varphi$, then the renormalized operator will have the form $\widehat{L}_{\varepsilon,ren} = \widehat{L}_V + \varepsilon C$ given by

$$
\begin{aligned}
(\widehat{L}_V G)(\eta^+,\eta^-) = &-m_0|\eta^+|G(\eta^+,\eta^-) - m_1|\eta^-|G(\eta^+,\eta^-) \\
&+ \sum_{x\in\eta^+} G(\eta^+\backslash x,\eta^-\cup x) + \lambda \sum_{x\in\eta^+} \int_{\mathbb{R}^d} a(x-y)G(\eta^+\backslash x\cup y)\mathrm{d}y \\
&+ \sum_{x\in\eta^+} \sum_{y\in\eta^-} \varphi(x-y)(G(\eta^+\backslash x,\eta^-\cup x\backslash y) - G(\eta^+,\eta^-\backslash y))
\end{aligned}
$$

and

$$
\begin{aligned}
(CG)(\eta) = \lambda \sum_{x\in\eta^+} \int_{\mathbb{R}^d} &a(x-y)G(\eta^+\cup y,\eta^-)\mathrm{d}y \\
&+ \lambda^- \sum_{x\in\eta^+} \sum_{y\in\eta^-} \varphi(x-y)(G(\eta^+\backslash x,\eta^-\cup x) - G(\eta^+,\eta^-))
\end{aligned}
$$

Therefore the function the function N_{α} is given by M_{α}. Concerning convergence of the generators we obtain the following.

Theorem 5.2. *For each $G \in D(N_{\alpha})$*

$$\widehat{L}_{\varepsilon,ren}G \to \widehat{L}_V G, \quad \varepsilon \to 0$$

in $\mathbb{B}_{\alpha}$. If $a,\varphi \in L^{\infty}(\mathbb{R}^d)$, then for all $\alpha' < \alpha$

$$\|\widehat{L}_{\varepsilon,ren} - \widehat{L}_V\|_{\alpha\alpha'} \to 0, \quad \varepsilon \to 0$$

holds.

The dual operators can be simply computed and are given by:

$$(L_V^\Delta k)(\eta^+,\eta^-) = -m_0|\eta^+|k(\eta^+,\eta^-) - m_1|\eta^-|k(\eta^+,\eta^-) + \sum_{x\in\eta^-} k(\eta^+\cup x,\eta^-\backslash x)$$
$$+ \lambda \sum_{x\in\eta^+} \int_{\mathbb{R}^d} a(x-y)k(\eta^+\backslash x\cup y,\eta^-)\mathrm{d}y$$
$$+ \lambda^- \sum_{x\in\eta^-} \int_{\mathbb{R}^d} \varphi(x-y)k(\eta^+\cup x,\eta^-\cup y\backslash x)\mathrm{d}y$$
$$- \lambda^- \sum_{x\in\eta^+} \int_{\mathbb{R}^d} \varphi(x-y)k(\eta^+,\eta^-\cup y)\mathrm{d}y$$

and

$$(C^\Delta k)(\eta^+,\eta^-) = -\lambda^- \sum_{x\in\eta^+}\sum_{y\in\eta^-} \varphi(x-y)k(\eta^+,\eta^-)$$
$$+ \lambda^- \sum_{x\in\eta^-}\sum_{y\in\eta^-} \varphi(x-y)k(\eta^+\cup x,\eta^-\backslash x)$$
$$+ \lambda \sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a(x-y)k(\eta^+\backslash x,\eta^-).$$

If $a,\varphi\in L^\infty(\mathbb{R}^d)$, then

$$\|L^\Delta_{\varepsilon,ren} - L^\Delta_V\|_{\alpha'\alpha} \to 0, \quad \varepsilon\to 0.$$

Let us finally compute $L_V^\Delta e_\lambda(\rho^+)e_\lambda(\rho^-)$ and derive from this the kinetic description.

$$(L_V^\Delta e_\lambda(\rho^+)e_\lambda(\rho^-))(\eta^+,\eta^-)$$
$$= \sum_{x\in\eta^-}\int_{\mathbb{R}^d} \varphi(x-y)\rho^+(x)\rho^-(y)e_\lambda(\rho^+;\eta^+)e_\lambda(\rho^-;\eta^-\backslash x)$$
$$- \sum_{x\in\eta^+}\int_{\mathbb{R}^d} \varphi(x-y)\rho^+(x)e_\lambda(\rho^+;\eta^+\backslash x)\rho^-(y)e_\lambda(\rho^-;\eta^-)$$
$$- \sum_{x\in\eta^+} m_0\rho^+(x)e_\lambda(\rho^+;\eta^+\backslash x)e_\lambda(\rho^-;\eta^-)$$
$$- \sum_{x\in\eta^-} m_1\rho^-(x)e_\lambda(\rho^-;\eta^-\backslash y)e_\lambda(\rho^+;\eta^+)$$
$$+ \sum_{x\in\eta^+} \lambda\int_{\mathbb{R}^d} a(x-y)\rho^+(y)\mathrm{d}y e_\lambda(\rho^+;\eta^+\backslash x)e_\lambda(\rho^-;\eta^-)$$
$$+ \sum_{x\in\eta^-} m_0\rho^+(x)e_\lambda(\rho^+;\eta^+)e_\lambda(\rho^-;\eta^-\backslash x)$$

and thus the system of equations for ρ_t^+ and ρ_t^- is given by, c.f. (24)

$$\begin{cases} \dfrac{\partial \rho_t^-}{\partial t}(x) = -m_1\rho_t^-(x) + \rho_t^+(x)(\varphi * \rho_t^-)(x) + m_0\rho_t^+(x) \\ \dfrac{\partial \rho_t^+}{\partial t}(x) = -(m_0 + (\varphi * \rho_t^-)(x))\rho_t^+(x) + (a * \rho_t^+)(x) \end{cases}.$$

5.2 Go-or-grow models

First model

Here the first model is given by $L = L_{CM} + L_{hop} + V$, where L_{CM} is given by

$$\begin{aligned}(L_{CM}F)(\gamma^+, \gamma^-) &= m \sum_{x\in\gamma^-} (F(\gamma^+, \gamma^-\backslash x) - F(\gamma^+, \gamma^-)) \\ &\quad + \lambda \sum_{x\in\gamma^-} \int_{\mathbb{R}^d} a(x-y)(F(\gamma^+, \gamma^- \cup x) - F(\gamma^+, \gamma^-))\mathrm{d}y\end{aligned}$$

and is describing the proliferation of the −-cells. The density independent intensity of death is given by $m > 0$ and the proliferation intensity by $\lambda > 0$. The kernel $0 \le a \in L^1(\mathbb{R}^d)$ is again symmetric and normalized to 1. The motion of the moving +-cells is described by

$$\begin{aligned}(L_{hop}F)(\gamma^+, \gamma^-) &= d \sum_{x\in\gamma^+} (F(\gamma^+\backslash x, \gamma^-) - F(\gamma^+, \gamma^-)) \\ &\quad + \sum_{x\in\gamma^+} \int_{\mathbb{R}^d} c(x-y)(F(\gamma^+\backslash x \cup y, \gamma^-) - F(\gamma^+, \gamma^-))\mathrm{d}y.\end{aligned}$$

Here we included also density independent mortality of the moving cells with intensity $d > 0$. The microscopic behaviour to change from one type (state) to another is given by

$$\begin{aligned}(VF)(\gamma^+, \gamma^-) &= q \sum_{x\in\gamma^+} (F(\gamma^+\backslash x, \gamma^- \cup y) - F(\gamma^+, \gamma^-)) \\ &\quad + \sum_{x\in\gamma^-} \left(p + \sum_{y\in\gamma^-\backslash x} \varphi(x-y) \right) (F(\gamma^+ \cup x, \gamma^-\backslash x) - F(\gamma^+, \gamma^-)).\end{aligned}$$

The operator for quasi-observables $\widehat{L} = \widehat{L}_{CM} + \widehat{L}_{hop} + \widehat{V}$ is given by, c.f. [9]

$$\begin{aligned}(\widehat{V}G)(\eta^+, \eta^-) &= q \sum_{x\in\eta^+} (G(\eta^+\backslash x, \eta^- \cup x) - G(\eta^+, \eta^-)) \\ &\quad + p \sum_{x\in\eta^-} (G(\eta^+ \cup x, \eta^-\backslash x) - G(\eta^+, \eta^-)) \\ &\quad + \sum_{x\in\eta^-} \sum_{y\in\eta^-\backslash x} \varphi(x-y)(G(\eta^+ \cup x, \eta^-\backslash x\backslash y) - G(\eta^+, \eta^-\backslash y)) \\ &\quad + \sum_{x\in\eta^-} \sum_{y\in\eta^-\backslash x} \varphi(x-y)(G(\eta^+ \cup x, \eta^-\backslash x) - G(\eta^+, \eta^-))\end{aligned}$$

and

$$(\widehat{L}_{hop}G)(\eta^+,\eta^-) = -d|\eta^+|G(\eta^+,\eta^-) + \sum_{x\in\eta^+}\int_{\mathbb{R}^d} c(x-y)(G(\eta^+\backslash x\cup y,\eta^-) - G(\eta^+,\eta^-))\mathrm{d}y.$$

The expression for $\widehat{L}_{CM}$ is similar to those before and is given by

$$(\widehat{L}_{CM}G)(\eta^+,\eta^-) = -m|\eta^-|G(\eta^+,\eta^-) + \lambda\sum_{x\in\eta^-}\int_{\mathbb{R}^d} a(x-y)G(\eta^+,\eta^-\backslash x\cup y)\mathrm{d}y + \lambda\sum_{x\in\eta^-}\int_{\mathbb{R}^d} a(x-y)G(\eta^+,\eta^-\cup y)\mathrm{d}y.$$

Lemma 5.3. *The function $M_\alpha = M_\alpha^{CM} + M^{hop} + M_\alpha^V$ is given by*

$$M_\alpha^{CM}(\eta^+,\eta^-) = (m+\lambda)|\eta^-| + \lambda e^{-\alpha^-}\sum_{x\in\eta^-}\sum_{x\in\eta^-\backslash y} a(x-y)$$

$$M^{hop}(\eta^+,\eta^-) = (d+2\langle c\rangle)|\eta^+|$$

$$\begin{aligned} M_\alpha^V(\eta^+,\eta^-) &= |\eta^+|(q + pe^{\alpha^- - \alpha^+} + e^{2\alpha^- - \alpha^+}\langle\varphi\rangle) \\ &\quad + |\eta^-|(p + qe^{\alpha^+ - \alpha^-} + e^{2\alpha^- - \alpha^+}\langle\varphi\rangle) \\ &\quad + \sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x}\varphi(x-y) + e^{\alpha^- - \alpha^+}\sum_{x\in\eta^+}\sum_{y\in\eta^-}\varphi(x-y). \end{aligned}$$

If $a,\varphi \in L^\infty(\mathbb{R}^d)$ then $\widehat{L} \in L(\mathbb{B}_\alpha,\mathbb{B}_{\alpha'})$ for any $\alpha' < \alpha$.

Proof. We will only compute the function M_α^V for three terms, the rest can be done in the same way.

$$\begin{aligned} &\int_{\Gamma_0^2}\sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x}\varphi(x-y)|G(\eta^+\cup x,\eta^-\backslash x\backslash y)|e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}\mathrm{d}\lambda(\eta^+,\eta^-) \\ &= e^{2\alpha^-}\int_{\Gamma_0^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(x-y)|G(\eta^+\cup x,\eta^-)|e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}\mathrm{d}x\mathrm{d}y\mathrm{d}\lambda(\eta^+,\eta^-) \\ &= e^{2\alpha^- - \alpha^+}\int_{\Gamma_0^2}\int_{\mathbb{R}^d}\sum_{x\in\eta^+}\varphi(x-y)|G(\eta^+,\eta^-)|e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}\mathrm{d}y\mathrm{d}\lambda(\eta^+,\eta^-) \\ &= e^{2\alpha^- - \alpha^+}\langle\varphi\rangle\int_{\Gamma_0^2}|\eta^+||G(\eta^+,\eta^-)|e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}\mathrm{d}\lambda(\eta^+,\eta^-) \end{aligned}$$

and

$$\int_{\Gamma_0^2}\sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x}\varphi(x-y)|G(\eta^+,\eta^-\backslash y)|e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}\mathrm{d}\lambda(\eta^+,\eta^-)$$

$$= e^{2\alpha^-} \int_{\Gamma_0^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x-y) |G(\eta^+, \eta^- \cup x)| e^{\alpha^+ |\eta^+|} e^{\alpha^- |\eta^-|} \mathrm{d}x \mathrm{d}y \mathrm{d}\lambda^2(\eta^+, \eta^-)$$

$$= e^{\alpha^-} \int_{\Gamma_0^2} \sum_{x \in \eta^-} \int_{\mathbb{R}^d} \varphi(x-y) |G(\eta^+, \eta^-)| e^{\alpha^+ |\eta^+|} e^{\alpha^- |\eta^-|} \mathrm{d}y \mathrm{d}\lambda^2(\eta^+, \eta^-)$$

$$\leq e^{\alpha^-} \langle \varphi \rangle \int_{\Gamma_0^2} |\eta^-| |G(\eta^+, \eta^-)| e^{\alpha^+ |\eta^+|} e^{\alpha^- |\eta^-|} \mathrm{d}\lambda(\eta^+, \eta^-)$$

and, finally,

$$\int_{\Gamma_0^2} \sum_{x \in \eta^-} \sum_{y \in \eta^- \backslash x} \varphi(x-y) |G(\eta^+ \cup x, \eta^- \backslash x)| e^{\alpha^+ |\eta^+|} e^{\alpha^- |\eta^-|} \mathrm{d}\lambda(\eta^+, \eta^-)$$

$$= e^{\alpha^-} \int_{\Gamma_0^2} \sum_{y \in \eta^-} \int_{\mathbb{R}^d} \varphi(x-y) |G(\eta^+ \cup x, \eta^-)| e^{\alpha^+ |\eta^+|} e^{\alpha^- |\eta^-|} \mathrm{d}x \mathrm{d}\lambda^2(\eta^+, \eta^-)$$

$$= e^{\alpha^- - \alpha^+} \int_{\Gamma_0^2} \sum_{x \in \eta^+} \sum_{y \in \eta^-} \varphi(x-y) |G(\eta^+, \eta^-)| e^{\alpha^+ |\eta^+|} e^{\alpha^- |\eta^-|} \mathrm{d}\lambda(\eta^+, \eta^-). \quad \square$$

Next we easily see that

$$(L^{\Delta}_{hop} k)(\eta^+, \eta^-) = -d|\eta^+| k(\eta^+, \eta^-) + \sum_{x \in \eta^+} \int_{\mathbb{R}^d} c(x-y)(k(\eta^+ \backslash x \cup y, \eta^-) - k(\eta^+, \eta^-)) \mathrm{d}y$$

and

$$\begin{aligned}(V^{\Delta} k)(\eta^+, \eta^-) = {} & q \sum_{x \in \eta^-} k(\eta^+ \cup x, \eta^- \backslash x) - q|\eta^+| k(\eta^+, \eta^-) \\ & + p \sum_{x \in \eta^+} k(\eta^+ \backslash x, \eta^- \cup x) - p|\eta^-| k(\eta^+, \eta^-) \\ & + \sum_{x \in \eta^+} \int_{\mathbb{R}^d} \varphi(x-y) k(\eta^+ \backslash x, \eta^- \cup x \cup y) \mathrm{d}y \\ & + \sum_{x \in \eta^+} \sum_{y \in \eta^-} \varphi(x-y) k(\eta^+ \backslash x, \eta^- \cup x) \\ & - \sum_{x \in \eta^-} \int_{\mathbb{R}^d} \varphi(x-y) k(\eta^+, \eta^- \cup y) \mathrm{d}y \\ & - \sum_{x \in \eta^-} \sum_{y \in \eta^- \backslash x} \varphi(x-y) k(\eta^+, \eta^-).\end{aligned}$$

Again under the conditions $a, \varphi \in L^\infty(\mathbb{R}^d)$ this expression can be well-defined as an element of $L(\mathbb{B}^*_{\alpha'}, \mathbb{B}^*_{\alpha})$ with the same norm estimate as $\|\widehat{L}\|_{\alpha\alpha'}$.

In previous section the kinetic description for each term contained in $\widehat{L}$ was derived, so let us give only a short outline how it works in this particular case.

Since the jumping part is free, c.f. $\phi = 0 = \psi$ from previous section, the operator L_{hop} will not change after renormalization. So let us scale the potential φ by $\varepsilon\varphi$. This will lead to the renormalized operator

$$\begin{aligned}(\widehat{V}_{\varepsilon,ren}G)(\eta^+,\eta^-) = q &\sum_{x\in\eta^+}(G(\eta^+\backslash x, \eta^-\cup x) - G(\eta^+,\eta^-))\\ &+ p\sum_{x\in\eta^-}(G(\eta^+\cup x, \eta^-\backslash x) - G(\eta^+,\eta^-))\\ &+ \sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x}\varphi(x-y)(G(\eta^+\cup x, \eta^-\backslash x\backslash y) - G(\eta^+,\eta^-\backslash y))\\ &+ \varepsilon\sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x}\varphi(x-y)(G(\eta^+\cup x, \eta^-\backslash x) - G(\eta^+,\eta^-))\end{aligned}$$

and thus we get.

Theorem 5.4. *For each $G \in D(M_\alpha)$ we have $\widehat{L}_{\varepsilon,ren}G \in \mathbb{B}_\alpha$ and*

$$\widehat{L}_{\varepsilon,ren}G \to \widehat{L}_V G, \quad \varepsilon \to 0$$

in $\mathbb{B}_\alpha$, where $\widehat{L}_V = \widehat{A} + \widehat{L}_{hop} + \widehat{V}_V$ is a superposition of the limiting part for the contact model, the operator $\widehat{L}_{hop}$ and

$$\begin{aligned}(\widehat{V}_V G)(\eta^+,\eta^-) = q &\sum_{x\in\eta^+}(G(\eta^+\backslash x, \eta^-\cup x) - G(\eta^+,\eta^-))\\ &+ p\sum_{x\in\eta^-}(G(\eta^+\cup x, \eta^-\backslash x) - G(\eta^+,\eta^-))\\ &+ \sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x}\varphi(x-y)(G(\eta^+\cup x, \eta^-\backslash x\backslash y) - G(\eta^+,\eta^-\backslash y)).\end{aligned}$$

Assume $a, \varphi \in L^\infty$, then for all $\alpha' < \alpha$

$$\|\widehat{L}_{\varepsilon,ren} - \widehat{L}_V\|_{\alpha\alpha'} \to 0, \quad \varepsilon \to 0.$$

and $\widehat{A}$ was given above.

The same result holds for correlation function operators with

$$\begin{aligned}(V_V^{\triangle}k)(\eta^+,\eta^-) = q &\sum_{x\in\eta^-} k(\eta^+\cup x, \eta^-\backslash x) - q|\eta^+|k(\eta^+,\eta^-)\\ = p &\sum_{x\in\eta^+} k(\eta^+\backslash x, \eta^-\cup x) - p|\eta^-|k(\eta^+,\eta^-)\\ &+ \sum_{x\in\eta^+}\int_{\mathbb{R}^d}\varphi(x-y)k(\eta^+\backslash x, \eta^-\cup x\cup y)\mathrm{d}y\\ &- \sum_{x\in\eta^-}\int_{\mathbb{R}^d}\varphi(x-y)k(\eta^+, \eta^-\cup y)\mathrm{d}y.\end{aligned}$$

Let us now compute $(\widehat{V}_V e_\lambda(\rho^+)e_\lambda(\rho^-))(\eta^+,\eta^-)$. This is given by

$$\begin{aligned}
&(\widehat{V}_V e_\lambda(\rho^+)e_\lambda(\rho^-))(\eta^+,\eta^-)\\
&= q\sum_{x\in\eta^-}\rho^+(x)e_\lambda(\rho^+;\eta^+)e_\lambda(\rho^-;\eta^-\backslash x)\\
&\quad - q\sum_{x\in\eta^+}\rho^+(x)e_\lambda(\rho^+;\eta^+\backslash x)e_\lambda(\rho^-;\eta^-)\\
&\quad + p\sum_{x\in\eta^+}\rho^-(x)e_\lambda(\rho^-;\eta^-)e_\lambda(\rho^+;\eta^+\backslash x)\\
&\quad - p\sum_{x\in\eta^-}\rho^-(x)e_\lambda(\rho^-;\eta^-\backslash x)e_\lambda(\rho^+;\eta^+)\\
&\quad + \sum_{x\in\eta^+}e_\lambda(\rho^+;\eta^+\backslash x)e_\lambda(\rho^-;\eta^-)\rho^-(x)\int_{\mathbb{R}^d}\varphi(x-y)\rho^-(y)\mathrm{d}y\\
&\quad - \sum_{x\in\eta^-}e_\lambda(\rho^+;\eta^+)e_\lambda(\rho^-;\eta^-)\rho^-(x)\int_{\mathbb{R}^d}\varphi(x-y)\rho^-(y)\mathrm{d}y
\end{aligned}$$

and hence the kinetic description is given by

$$\begin{aligned}
\frac{\partial\rho^+}{\partial t}(x) &= -(\langle c\rangle+q+d)\rho^+(x)+(c*\rho^+)(x)+p\rho^-(x)+\rho^-(x)(\varphi*\rho^-)(x)\\
\frac{\partial\rho^-}{\partial t}(x) &= -(m+p)\rho^-(x)+\lambda(a*\rho^-)(x)-\rho^-(x)(\varphi*\rho^-)(x)+q\rho^+(x).
\end{aligned}$$

Second model

Now let us investigate the second model. Here $L = L_{CM} + L_{hop} + V$ with the operator $V = V_1 + V_2$ slightly changed to

$$\begin{aligned}
(VF)(\gamma^+,\gamma^-) &= q\sum_{x\in\gamma^+}\exp\Big(-\sum_{y\in\gamma^-}\psi(x-y)\Big)(F(\gamma^+\backslash x,\gamma^-\cup y)-F(\gamma^+,\gamma^-))\\
&\quad + \sum_{x\in\gamma^-}\Big(p+\sum_{y\in\gamma^-\backslash x}\varphi(x-y)\Big)(F(\gamma^+\cup x,\gamma^-\backslash x)-F(\gamma^+,\gamma^-)).
\end{aligned}$$

and therefore the rate of changing from + to − cells is also density dependent. Clearly all results except these concerning V_1 still hold true, so let us only investigate this part. The expression for quasi-observables is given by

$$\begin{aligned}
&(\widehat{V}_1 G)(\eta^+,\eta^-)\\
&= \sum_{x\in\eta^+}\sum_{\xi^-\subset\eta^-}e^{-E_\psi(x,\xi^-)}e_\lambda(e^{-\psi(x-\cdot)}-1;\eta^-\backslash\xi^-)(G(\eta^+\backslash x,\xi^-\cup x)-G(\eta^+,\xi^-)).
\end{aligned}$$

Lemma 5.5. *The function M_α is given by $M_\alpha = M_\alpha^{CM} + M^{hop} + M_\alpha^V$, where M_α^{CM} and M^{hop} are given as in the Cell-death model and*

$$\begin{aligned} M_\alpha^V(\eta^+,\eta^-) = |\eta^+|(pe^{\alpha^- - \alpha^+} + e^{2\alpha^- - \alpha^+}\langle\varphi\rangle) + |\eta^-|(p + e^{2\alpha^- - \alpha^+}\langle\varphi\rangle) \\ + \sum_{x\in\eta^-}\sum_{y\in\eta^-\setminus x}\varphi(x-y) + e^{\alpha^- - \alpha^+}\sum_{x\in\eta^+}\sum_{y\in\eta^-}\varphi(x-y). \\ + q\beta_\psi(\alpha^-)e^{\alpha^+ - \alpha^-}\sum_{x\in\eta^-} e^{-E_\psi(x,\eta^-\setminus x)} + q\beta_\psi(\alpha^-)\sum_{x\in\eta^+} e^{-E_\psi(x,\eta^-)}, \end{aligned}$$

where

$$\beta_\psi(\alpha^-) = \exp\Big(e^{\alpha^-}\int_{\mathbb{R}^d}(1-e^{-\psi(x)})\mathrm{d}x\Big).$$

If $\varphi, a \in L^\infty(\mathbb{R}^d)$, then $\widehat{L} \in L(\mathbb{B}_\alpha, \mathbb{B}_{\alpha'})$ for all $\alpha' < \alpha$.

Proof. This follows from

$$\begin{aligned} &\int_{\Gamma_0^2}\sum_{x\in\eta^+}\sum_{\xi^-\subset\eta^-} e^{-E_\psi(x,\xi^-)}e_\lambda(|e^{-\psi(x-\cdot)}-1|;\eta^-\setminus\xi^-) \\ &\qquad\times|G(\eta^+\setminus x,\xi^-\cup x)|e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}\mathrm{d}\lambda^2(\eta^+,\eta^-) \\ &= e^{\alpha^+}\int_{\Gamma_0^2}\int_{\mathbb{R}^d}\int_{\Gamma_0} e^{-E_\psi(x,\xi^-)}e_\lambda(1-e^{-\psi(x-\cdot)};\eta^-) \\ &\qquad\times|G(\eta^+,\xi^-\cup x)|e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}e^{\alpha^-|\xi^-|}\mathrm{d}\lambda(\xi^-)\mathrm{d}x\mathrm{d}\lambda^2(\eta^+,\eta^-) \\ &= \beta_\psi(\alpha^-)e^{\alpha^+-\alpha^-}\int_{\Gamma_0^2}\Big(\sum_{x\in\xi^-}e^{-E_\psi(x,\xi^-\setminus x)}\Big)|G(\eta^+,\xi^-)|e^{\alpha|\eta^+|}e^{\alpha|\xi^-|}\mathrm{d}\lambda^2(\eta^+,\xi^-) \end{aligned}$$

and

$$\begin{aligned} &\int_{\Gamma_0^2}\sum_{x\in\eta^+}\sum_{\xi^-\subset\eta^-} e^{-E_\psi(x,\xi^-)}e_\lambda(|e^{-\psi(x-\cdot)}-1|;\eta^-\setminus\xi^-) \\ &\qquad\times|G(\eta^+,\xi^-)|e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}\mathrm{d}\lambda^2(\eta^+,\eta^-) \\ &= \int_{\Gamma_0^2}\sum_{x\in\eta^+}\int_{\Gamma_0} e^{-E_\psi(x,\xi^-)}e_\lambda(1-e^{-\psi(x-\cdot)};\eta^-) \\ &\qquad\times|G(\eta^+,\xi^-)|e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}e^{\alpha^-|\xi^-|}\mathrm{d}\lambda(\xi^-)\mathrm{d}\lambda^2(\eta^+,\eta^-) \\ &= \beta_\psi(\alpha^-)\int_{\Gamma_0^2}\Big(\sum_{x\in\eta^+}e^{-E_\psi(x,\xi^-)}\Big)|G(\eta^+,\xi^-)|e^{\alpha^+|\eta^+|}e^{\alpha^-|\xi^-|}\mathrm{d}\lambda^2(\eta^+,\xi^-). \quad\square \end{aligned}$$

Since ψ is non-negative we can skip the terms containing q in the definition of the domain, i.e. if $M_\alpha^V = M_\alpha^{V_1} + qM_\alpha^{V_2}$, then

$$D(M_\alpha) = \left\{G\in\mathbb{B}_\alpha \ : \ M^{hop}G, M_\alpha^{CM}G, M_\alpha^{V_1}G \in \mathbb{B}_\alpha\right\},$$

where $M_{alpha}^{V_1}$ contains the terms for switching $-$ to $+$ cells and V_2 corresponds to the switching of $+$ to $-$ cells. The operator for correlation functions is

$$\begin{aligned}&(V_2^{\Delta}k)(\eta^+,\eta^-)\\ &= \sum_{x\in\eta^-} e^{-E_\psi(x,\eta^-\backslash x)}\int_{\Gamma_0} e_\lambda(e^{-\psi(x-\cdot)}-1;\xi^-)k(\eta^+\cup x,\eta^-\cup\xi^-\backslash x)\mathrm{d}\lambda(\xi^-)\\ &\quad - \sum_{x\in\eta^+} e^{-E_\psi(x,\eta^-)}\int_{\Gamma_0} e_\lambda(e^{-\psi(x-\cdot)}-1;\xi^-)k(\eta^+,\eta^-\cup\xi^-)\mathrm{d}\lambda(\xi^-).\end{aligned}$$

The scaling $a,\varphi,\psi\to\varepsilon a,\varepsilon\varphi,\varepsilon\psi$ leads to the new renormalized expression for $\widehat{V}_{2,\varepsilon,ren}$

$$\begin{aligned}(\widehat{V}_{2,\varepsilon,ren}G)(\eta^+,\eta^-) = \sum_{x\in\eta^+}\sum_{\xi^-\subset\eta^-} & e^{-\varepsilon E_\psi(x,\xi^-)}e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\eta^-\backslash\xi^-\right)\\ &\times\left(G(\eta^+\backslash x,\xi^-\cup x)-G(\eta^+,\xi^-)\right)\end{aligned}$$

and thus to the limiting hierarchical operator

$$(\widehat{V}_{1,V}G)(\eta^+,\eta^-) = \sum_{x\in\eta^+}\sum_{\xi^-\subset\eta^-} e_\lambda(-\psi(x-\cdot);\eta^-\backslash\xi^-)(G(\eta^+\backslash x,\xi^-\cup x)-G(\eta^+,\xi^-)).$$

Theorem 5.6. *Assume $\psi\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, then for all $G\in D(M_\alpha)$ such that $\sum\limits_{x\in\xi^-}\sum\limits_{y\in\xi^-\backslash x}\psi(x-y)G\in\mathbb{B}_\alpha$ and $\sum\limits_{x\in\eta^+}\sum\limits_{y\in\xi^-}\psi(x-y)G\in\mathbb{B}_\alpha$ the convergence $\widehat{L}_{\varepsilon,ren}G\to\widehat{L}_VG$ for $\varepsilon\to0$ holds in $\mathbb{B}_\alpha$. If in addition $a,\varphi,\psi\in L^\infty(\mathbb{R}^d)$ then for all $\alpha'<\alpha$*

$$\|\widehat{L}_{\varepsilon,ren}-\widehat{L}_V\|_{\alpha\alpha'}\to0,\quad\varepsilon\to0.$$

Proof. Let us first estimate

$$\begin{aligned}&|(\widehat{V}_{1,\varepsilon,ren}G)(\eta)-(\widehat{V}_{1,V}G)(\eta)|\\ &\le\sum_{x\in\eta^+}\sum_{\xi^-\subset\eta^-}|G(\eta^+\backslash x,\xi^-\cup x)-G(\eta^+,\xi^-)|\\ &\qquad\times|e^{-\varepsilon E_\psi(x,\xi^-)}e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\eta^-\backslash\xi^-\right)-e_\lambda(-\psi(x-\cdot);\eta^-\backslash\xi^-)|\end{aligned}$$

and then the modulus in the sum by

$$\begin{aligned}&|e^{-\varepsilon E_\psi(x,\xi^-)}e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\eta^-\backslash\xi^-\right)-e_\lambda(-\psi(x-\cdot);\eta^-\backslash\xi^-)|\\ &\le|1-e^{-\varepsilon E_\psi(x,\xi^-)}|e_\lambda\left(\frac{|e^{-\varepsilon\psi(x-\cdot)}-1|}{\varepsilon};\eta^-\backslash\xi^-\right)\\ &\quad+\left|e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\eta^-\backslash\xi^-\right)-e_\lambda(-\psi(x-\cdot);\eta^-\backslash\xi^-)\right|\\ &\le\varepsilon E_\psi(x,\xi^-)e_\lambda(\psi(x-\cdot);\eta^-\backslash\xi^-)\end{aligned}$$

$$+ \sum_{w\in\eta^-\backslash\xi^-} \left| \frac{e^{-\varepsilon\psi(x-w)}-1}{\varepsilon} + \psi(x-w) \right| e_\lambda(\psi(x-\cdot);\eta^-\backslash\xi^-\backslash w)$$

$$\leq \varepsilon E_\psi(x,\xi^-)e_\lambda(\psi(x-\cdot);\eta^-\backslash\xi^-) + \varepsilon \sum_{w\in\eta^-\backslash\xi^-} \psi(x-w)^2 e_\lambda(\psi(x-\cdot);\eta^-\backslash\xi^-\backslash w).$$

Integrating over Γ_0^2 with respect to $e^{\alpha'^+|\eta^+|}e^{\alpha'^-|\eta^-|}\mathrm{d}\lambda^2(\eta^+,\eta^-)$ we obtain for the part containing $G(\eta^+\backslash x, \xi^- \cup x)$

$$\varepsilon e^{\alpha'^+} \int_{\Gamma_0^2}\int_{\mathbb{R}^d}\int_{\Gamma_0} |G(\eta^+,\xi^-\cup x)| E_\psi(x,\xi^-)e_\lambda(\psi(x-\cdot);\eta^-)$$

$$\times e^{\alpha'^-|\xi^-|}e^{\alpha'^-|\eta^-|}e^{\alpha'^+|\eta^+|}\mathrm{d}x\mathrm{d}\lambda^3(\eta,\xi^-)$$

$$\leq \varepsilon e^{e^{\alpha'^-}\langle\psi\rangle} \int_{\Gamma_0^2} \Big(\sum_{x\in\eta^-} E_\psi(x,\xi^-\backslash x) \Big) |G(\eta^+,\xi^-)| e^{\alpha'^+|\eta^+|}e^{\alpha'^-|\xi^-|}\mathrm{d}\lambda^2(\eta^+,\xi^-)$$

and for the second term

$$\varepsilon e^{\alpha'} \int_{\Gamma_0^2}\int_{\Gamma_0} |G(\eta^+,\xi^-)| e^{\alpha'|\eta^+|}e^{\alpha'|\xi^-|}$$

$$\times \sum_{x\in\eta^+}\int_{\mathbb{R}^d} \psi(x-w)^2 e_\lambda(e^{\alpha'}\psi(x-\cdot);\eta^-)\mathrm{d}w\mathrm{d}\lambda(\eta^+,\eta^-,\xi^-)$$

$$\leq \varepsilon e^{\alpha'\langle\psi\rangle}\langle\psi^2\rangle e^{\alpha'} \int_{\Gamma_0^2} |\eta^+||G(\eta^+,\xi^-)| e^{\alpha'|\eta^+|}e^{\alpha'|\xi^-|}\mathrm{d}\lambda(\eta^+,\xi^-).$$

Similar estimations for the parts containing $G(\eta^+,\xi^-)$ show together with above computations the first part of the assertion. The second part follows from $E_\psi(x,\xi) \leq \|\psi\|_{L^\infty}|\xi|$. □

The operator for correlation function is changed only at the new operator V^Δ and the rescaled version has the form

$$(V^\Delta_{1,\varepsilon,ren}k)(\eta^+,\eta^-)$$

$$= \sum_{x\in\eta^-} e^{-\varepsilon E_\psi(x,\eta^-\backslash x)} \int_{\Gamma_0} e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\xi^- \right) k(\eta^+\cup x, \eta^-\cup\xi^-\backslash x)\mathrm{d}\lambda(\xi^-)$$

$$- \sum_{x\in\eta^+} e^{-\varepsilon E_\psi(x,\eta^-)} \int_{\Gamma_0} e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\xi^- \right) k(\eta^+, \eta^-\cup\xi^-)\mathrm{d}\lambda(\xi^-)$$

and the limiting operator

$$(V^\Delta_{1,V}k)(\eta^+,\eta^-) = \sum_{x\in\eta^-}\int_{\Gamma_0} e_\lambda(-\psi(x-\cdot);\xi^-)k(\eta^+\cup x,\eta^-\cup\xi^-\backslash x)\mathrm{d}\lambda(\xi^-)$$

$$- \sum_{x\in\eta^+}\int_{\Gamma_0} e_\lambda(-\psi(x-\cdot);\xi^-)k(\eta^+,\eta^-\cup\xi^-)\mathrm{d}\lambda(\xi^-).$$

Again if $a, \varphi, \psi \in L^\infty(\mathbb{R}^d)$, then the convergence

$$\|L^\Delta_{\varepsilon,ren} - L^\Delta_V\|_{\alpha'\alpha} \to 0, \quad \varepsilon \to 0$$

holds. Computing $V_{1,V} e_\lambda(\rho^+) e_\lambda(\rho^-)$ one sees that the equations for the local densities will have the prescribed form (18),(19).

Last two models

Here the changes of types are density independent, i.e. $\varphi = \psi = 0$, but the proliferation is changed either to density dependent mortality or to density dependent birth. Both models were analysed in the one-component case. Since the changes of types are prescribed by constant intensities they do not influence the construction of an evolution and only contribute by additional terms in the kinetic description. It is not difficult to combine all results and derive from them the corresponding kinetic description stated before.

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Short history of stock markets and stochastic finances

Yu. Mishura[1]

Abstract. This text will be published as a part of the Introduction to the book *Financial mathematics*, 1st Edition, ISTE PRESS, Elsevier, 2016.

Financial markets are sometimes identified with stock markets; sometimes they are differentiated with the view that on the financial markets we can trade only securities and on the stock market we can trade other values, such as real estate, property and currency. The stock market is also called stock exchange. A stock exchange is a market for different kinds of securities, including stocks, bonds, shares as well as payment documents. As for the randomness, the situation is such that the prices in the financial market, more specifically, on the stock exchange, are affected by many external factors that cannot be predicted in advance and cannot be controlled completely. This is mainly a consequence of economic circumstances, for example, of the state of the world economy and of the local economy, production levels in some sectors, and the balance between supply and demand. It may be the weather and climate factors affecting, for example, a certain type of agricultural products, or it may be the activities of large exchange speculators. Since stock prices at any given time are random, over time they accordingly become random processes. Of course, the same situation occurred even in those days when exchange existed, but the theory of random processes has not yet been established. Recall that the Chicago Stock Exchange began operating 21 March 1882.

As for the theory of random processes, curiously enough, its founder was not a mathematician but botanist Robert Brown, who in 1827 discovered under a microscope the process of chaotic motion of flower pollen in water. The nature of this phenomenon remained unclear for long time, and only in the late 19th—early 20th Century it was realized that it is one of the manifestations of the thermal motion of atoms and molecules, and to explore this phenomenon we need methods of probability theory. Appropriate random process was eventually called the Brownian motion, and then Wiener process, according to the name of the famous mathematician Norbert Wiener who not only constructed integral with respect to this process but also wrote hundreds of articles on probability theory and mathematical statistics, Fourier series and integrals, potential theory, number theory and generalized harmonic analysis. He is also called the "father of cybernetics" for his book "Cybernetics: or Control and Communication in the Animal and the Machine" [1], first published in 1948.

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He helped to develop a system of air defense of the USA. Note that the initial framework for the analysis of randomness in the change in stock prices was established by French mathematician and economist Louis Bachelier, who in 1900 in his doctoral thesis "Théorie de la spéculation" [2] made the attempt to describe the stock price by means of stochastic process $S = S_t, t \geq 0$ with the increments $\Delta S_t = S_{t+\Delta t} - S_t$ of order $\sqrt{\Delta t}$, in some probabilistic sense. Such a process is a prototype of the Wiener process but the Bachelier's model had a crucial disadvantage: the prices in this model could be negative. In fact, Bachelier's model can be described as $S_t = S_0 + \mu t + \sigma W_t$, where W is a Wiener process. Nevertheless, Bachelier's discovery of the "effect of Δt" in fluctuations of the value of shares under a large number of economic factors and due to the central limit theorem became later the key point in the construction of the general theory of random diffusion processes. Although for some time Bachelier's works had been forgotten, after many years they were rightly remembered and highly appreciated, and now the main representative congresses on financial mathematics are named World Congresses of the Bachelier Finance Society. At the beginning, the mathematical study of the Brownian motion (Wiener process) was produced in the papers of physicists, namely Albert Einstein and Marian Smoluchowski, and then it was widely studied by mathematicians, including Norbert Wiener.

A very interesting person in economic theory and, to some extent, in financial mathematics is Russian mathematician Leonid Kantorovich, a specialist in functional analysis. In 1938, he provided advice to plywood plant how to use their machines in the most effective way to minimize the waste of plywood. Over time, Kantorovich realized that such very particular problem can be generalized to the problem of maximization of the linear form depending on many variables and containing a large number of restrictions that have the form of linear equalities and inequalities. He also realized that the enormous number of economic issues can be reduced to the solution of such problems. In 1939, Kantorovich published the paper "Mathematical Methods of Organizing and Planning Production" [3], describing the problems of the economy that can be solved by his method and thus laid the foundation of mathematical programming. His contribution directly into financial mathematics is that he found such an interesting coincidence: the best prices, including the prices of financial assets, are at the same time the prices supplying market equilibrium. Then his conclusions were obtained independently by US economists, and in 1975 he received the Nobel Prize in economics together with Tjalling C. Koopmans "for their contribution to the theory of optimal allocation of resources".

Financial mathematics has received a new impetus for development in 1965, when at the initiative of mathematician and economist Leonard Savage, who "rediscovered" Bachelier's work, American economist Paul Samuelson, who would also subsequently become the winner of the Nobel Prize in economics, has suggested to describe share prices with the help of geometric Brownian motion $S_t = S_0 e^{\mu t} e^{\sigma W_t - \sigma^2 t/2}$, whose advantage is to be non-negative and even strictly positive with probability 1 [4]. Over time, the model of geometric Brownian motion was substantially generalized. In particular, we can consider jump-diffusion process or Levy process, that is homogeneous process with independent increments, or semimartingale, instead of the Wiener process.

Finally, in 1968, there was a significant economic and financial event: the prices of gold and other precious metals were "released". The history of this issue is as follows: from 1933 to 1976, the official price of gold was under control of the Department of the Treasury of the United States federal government. Now it is managed, in a certain sense, by the London Stock Exchange. In 1944, the price of gold was at the level of 35 USD per troy ounce (31.1034768 g) and from time to time increased or decreased under the influence of the devaluation of the dollar, world crises or wars. The price of gold increased due to the increasing of the demand for gold as a raw material for production of electronics and radio-technics, the jewelry industry, medicine and other purposes. But often the price of gold grew as the result of speculative transactions on the stock exchange and as the result of the creation of the highly liquid assets by central banks of different countries. In 1961, Western Europe countries created a "golden pool", which included central banks of the UK, Germany, France, Italy, Belgium, The Netherlands, Switzerland and the Bank of New York. This pool was created in order to stabilize the world prices for gold, but in 1968, after the devaluation of the British pound, UK spent 3,000 tons of gold to regulate interior prices for gold, and after this the gold pool collapsed. From that time, the price of gold is determined by the market, i.e., by the demand and the supply.

Free gold prices led to additional random components in the financial markets, and the stochastic finance theory started to develop very intensively both as a theoretical science and as a tool for the daily management of banking and stock exchange activities. An additional factor that contributed to its development was the opening of the first stock exchange in 1973 on which option contracts were traded. In the same year, two works that led to the revolution in financial calculations of option prices were published. It was the paper of Fischer Black and Myron Scholes, "The Pricing of Options and Corporate Liabilities" [5], and the paper of Robert Merton, "Theory of Rational Option Pricing" [6]. In October 1997, R. Merton (Harvard University) and M. Scholes (Stanford University) were awarded the Nobel Prize in economics. (F. Black died in 1995, and the Nobel Foundation awards prizes only to living scientists). Briefly, the Black-Scholes formula evaluates "fair" option price. The Black-Scholes-Merton model is very useful in making investment decisions, but principally does not guarantee profit without risk. Conceptually, the Black-Scholes formula can be explained as follows: the option price equals the expected future asset price minus the expected cash price, or as the difference of two binary options: an asset-or-nothing call minus a cash-or-nothing call. The concept of fair price is based on the concept of arbitrage-free market. We should pay attention to the point that the real market can be modeled in various ways, and its properties will be different in different models. For example, the same market can be modeled as complete and incomplete, but the only way to determine which model suits the best is to verify them in practice. Typically, the construction of several models of the market and the consideration of several trading strategies are expensive problems, and the art of a financial analyst consists, in particular, of choosing the correct model. Note also that the models constructed for financial mathematics are not situated aside all other science and practice. Indeed, they are used in biology, weather forecasting, climatology and the study of changes in the mobile electrical circuits communication because the processes in these fields very often have the same features.

The description of modern financial models is based both on the theory of random processes and stochastic analysis (theory of martingales, stochastic integration, Itô formula, Girsanov's theorem, theory of stochastic differential equations, martingale representations and elements of Malliavin calculus) and on basic facts of functional analysis (topological, Banach and Hilbert spaces, linear functionals, Hahn-Banach theorem etc.).

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Complexity, Acceleration, Globalization — A Challenge for Democracy

L. Streit[1]

Abstract. This paper is dedicated to the memory of the late Glória Cravo who first invited me to discuss these issues at the "Conversas da dentro e de fora" of the Universidade da Madeira, in 2014.

Democracy

Democracy — the rule by the people — was invented more than 2000 years ago to organize the social life of communities scattered around the Mediterranean, typically much smaller than present day Switzerland. Life then was dominated by the rhythm of celestial bodies: time was measured in terms of years, seasons, months — not much would happen in a day.

When industrialization began to change the world and civil society was born, democracy was reborn with her, and was later contemplated by Winston Churchill[2], saying "Many forms of Government have been tried, and will be tried in this world of sin and woe. No one pretends that democracy is perfect or all-wise. Indeed, it has been said that democracy is the worst form of Government except for all those other forms that have been tried from time to time." Remarkably the first two sentences are quoted much less frequently than the last.

And in fact, nowadays any doubt about the superior quality of democratic government is considered politically unacceptable.

How did the world look 200 years ago when democracy began to spread through Europe in the wake of the French revolution?

Vast, endless, with open space to conquer and to explore or to get lost in, white, uncharted areas were abundant on the world maps. News of an Asian earthquake or tsunami would arrive at our shores many months after the event had happened and would be no more than an item of curiosity without much further effect on our lives.

Now, 200 years later, democratic rule has not only taken hold in most of the developed world but has acquired the status of an ethical postulate. It is considered as an essential attribute of superior social organization and serves to "justify" military intervention such as e.g. in the 2nd Iraq war.

And how does our world look now?

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[2]Speech in the House of Commons, 11 November 1947.

Invented for those small city states of ancient Greece, democracy now has to face the challenges of the "global village" where news travel around the globe in fractions of a second, where events are tightly connected all over the world and effects can spread with lightning speed, and where for example the effect of a tsunami in Japan was able within days to impact the national German energy production portfolio in a profound way from which it has not recovered yet.

How did this happen? What innovation had the most important impact in shaping this world we live in today?

The advent of the PC

On August 12, 1981, IBM rolled out its first "Personal Computer"; digital computing entered the offices and living rooms. Mainframe computers had been around much earlier, they were born in the forties as a twin of the atom bomb. It is the global availability of PC power that made the difference. It made us all see and do things we would not have imagined earlier.

As an example there is the famous Mandelbrot fractal. As we zoom into its finer and finer details, and so on ad infinitum there appears a beautiful dream world of forms and colors. A complex structure indeed. My point here is: the mathematical formula behind it had been catching dust for a hundred years or so, but until the PC came, nobody had the slightest idea of the complex pattern that lay behind it[1].

Richard Feynman, one of the greatest minds of 20$^{\text{th}}$ century physics, once said that we lack an intuition for nonlinear phenomena and that perhaps *the emergence of such an intuition might mark a new intellectual awakening of mankind.*

This "new awakening" sounds like science fiction, something out of Kubrick's film "Space Odyssey 2001". Kubrick's film from 1968 was speculating that by 2001, a computer would attempt to get the upper hand over us. The takeover has not happened yet, but we do feel the impact[2]. Even kids to-day have seen the Mandelbrot fractal somewhere or another, and terms from complex systems such as the famous "butterfly effect", have become buzzwords in the socio-economics discourse.

The end of reductionism

What is this impact of omnipresent computing power?

For two millennia, scientific problems had to be drastically simplified before being "understood". The dynamics of protons and neutrons is too hard to handle? OK, so let us describe them by a bunch of harmonic oscillators because those are simple enough to deal with. In physics, but as well in other disciplines such as e.g. economics, simplifications and massive complexity reductions were

[1] Hear Benoît Mandelbrot himself about the wonders of this computer-based breakthrough: `http://www.ted.com/talks/benoit\_mandelbrot\_fractals\_the\_art\_of\_roughness/transcript?language=en\#t-586141`

[2] A dramatic acccount by one who is considered "the father of virtual reality technology" can be found in J. Lanier: "You Are Not a Gadget." Knopf, NY, 2010.

all over the place. In particular collective behavior was often beyond the reach of reliable modeling

Now, with our exponentially growing computing power, complex systems, and in particular the collective behavior of large systems can be studied, and the intuition which Feynman called a New Awakening, is rapidly developing. The tools were there at the dawn of the new millennium, the time was right.

But the computer not only changed our understanding of the world, it has changed our own world, dramatically.

With millions of PCs linked in the Web, our cultural, economic and social world has become a closely connected global network.

Information flies at almost the speed of light. The effect of local disasters or interventions is felt almost instantaneously around the world.

As one of the consequences, Asian workers are now in a very direct competition with their colleagues in the Americas or Europe, the work force has been globalized, and employers go for the cheapest offer worldwide. In the Manila metropolitan region alone , there are more than one million employees working in call-centers for American, Australian and other companies and even universities.

All of this has become an incredible complex socio-economic system, with enormous risks of sudden global destabilization.

What is Complexity?

As an example let us have a look at the Malthusian growth of populations. In this model the new population is assumed to be proportional to the size of the previous generation,

$$P_{n+1} = f \cdot P_n,$$

where f is the fertility rate (Malthus model, 1798), and whenever the fertility f is larger than one, the population will grow; in fact it then grows exponentially.

But the real world is more complex: the effect of competition is felt when populations become big and can be described by

$$P_{n+1} = f \cdot P_n - c \cdot P_n^2.$$

The new population is proportional to the size of the previous one, minus the effect of competition for limited resources, encoded in the parameter c.

Clear, isn't it? Now we expect that the population will saturate at a limiting size, before the quadratic second term becomes too big..

But this is not all that can happen. There can also be oscillations and even chaos as time progresses from one generation to the next.

We should note here two important features of our model:

1. All these scenarios can occur if one varies the fertility rate f or competition c just very slightly! In this sense the evolution of populations becomes *practically unpredictable* since we can never be sure of those parameters with the necessary precision to exclude one or the other scenario.

2. Even if the formula itself is extremely simple, its less obvious consequences were only readily available in the age of the PC.

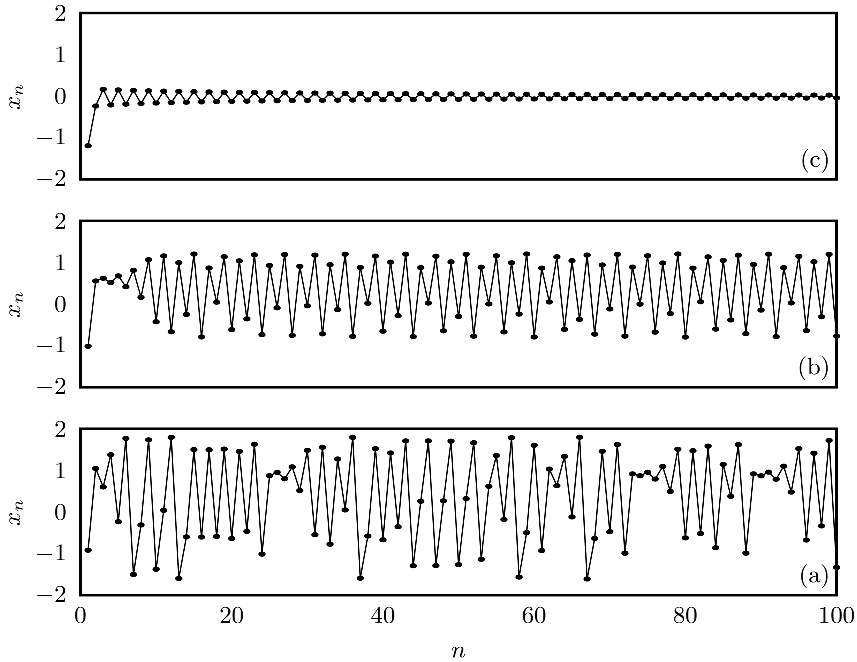

Fig. 1: Three Scenarios: Saturation, Oscillations, Chaos

What then is Complexity?

A concept of such generality necessarily has several definitions. They will be context dependent, but not only. In the context of computation for example, complexity may mean a very complex computer program, or one that takes a long time to run, or one that uses a lot of memory, or one that is simple in all these respects but produces surprising results, like our population model.

For our purposes, we may just characterize complex systems by two properties:

1. Their evolution is difficult to predict. Weather and climate are well known examples of this difficulty.

2. It is equally difficult to predict their reaction to even small variations or interventions ("**The Butterfly Effect**")[1].

Consequences

Crucially important is the lesson that — all the way from the micro-economic, business level to the macroeconomic level of national and international policy

[1]This practical unpredictability does not invalidate quantitative mathematical modeling. Such models, while not furnishing specific predictions, will display, and alert for, possible types of evolution and outcomes of interventions. As J. Gleick writes: "...a new generation of scientists has come along.... They know that a complex, dynamical system can get freaky. They know, when it does that, that you can still look it in the eye and take its measure." J. Gleick: "Chaos: Making a New Science". Revised edition, Penguin 2008.

making — planning and strategies have to be continuously readjusted to counteract the surprises of the butterfly effect and of random exterior influences.

According to Lenin, the whole national economy should be organized on the lines of the postal service.

In practical Soviet terms, this meant governance through centralized plans, based on an equilibrium model of the economy. The result is well known. When East Germany was irreversibly bankrupt, the Berlin Wall fell and the communist bloc crumbled.

In a modern highly complex economy, long term planning must clearly be replaced by continuous monitoring and feedback loops. The — admittedly rather sophisticated — techniques to compensate random disturbances on the fly go back to Cold War technologies, they were invented under the name of optimal control or dynamic programming, mainly to keep ICBMs on target during their flight so that they would surely hit New York or Moscow in spite of any in-flight perturbations. Economists and more generally people from the humanities tend to be allergic to the notion of control theory — it sounds too repressive to their tender ears. But now economists have discovered a nice new label, that of "adaptive management", it is the same thing, and now increasingly popular, a new buzzword in economics.

Acceleration

Of course the problems of socio-economic complexity have been extremely aggravated by globalization and acceleration. While economic news used to need months to travel around the globe before the advent of steamships, information now flies around the globe with the velocity of light, and hectic reactions are a potential source of serious instability

As an example, financial transaction strategies with the potential to produce crises of a global dimension take place in milliseconds, powered by computer algorithms, without the possibility of adequate human intervention. There have been instances where a large scale financial crash was only avoided by literally pulling the plug of those computers at the New York stock exchange.

Clouds on the Horizon

Like with other revolutionary inventions there is a definite dark side to our exponentially increasing computing and data-handling power.

Paradoxically the net and digital data-handling have produced a new kind of reductionism: to see this, compare the rich and manifold traditional meaning of the word friendship with its reduction to a mouse click in the Facebook world, or the rich and unique spectrum of a Stradivari violin with its reduction under MP3.

Then there is the loss of privacy and freedom of choice: You "pay" for the web with the loss of your privacy, and the web then defines your identity. Google et al. will decide what you will want to see, what to know, what to buy.

And we have to be concerned with a cyberspace beyond the reach of the law. Google and Microsoft were reported to be planning to move activities to the "off-shore".

Larry Page[1], co-founder of Google, said in 2013:

"There's many, many exciting and important things you could do that you just can't do because they're illegal ... we should have some safe places where we can try out some new things and figure out what is the effect on society, what's the effect on people..."

Democracy — Rule by the People?

The democratic decision-making process requires:

- Transparency of the issues
- Transparency of, and confidence in the effectiveness of the strategies available
- Enough time to present and discuss these issues and strategies, and often to test their legal issues in court
- On the other hand, nowadays, an adaptive political management is needed, responding quickly and promptly to unexpected threats and challenges
- We lack sufficient regulatory capacity to deal with banks "too big to fail", to deal with an economy that does not know borders and evades national laws.

We have to cope with the anti-intuitive behavior of complex systems: they are highly non-linear — double in will not produce double out. Raising a tax may produce less instead of more tax revenue because business moves to other markets. Modifications in the health care system may produce utterly unforeseeable effects, and so on and so forth everywhere in government. Politicians will cite their favorite experts, but even experts are at a loss for reliable predictions, To handle the financial crisis competently the German government had to resort to the advice of Germany's leading banker Joseph Ackermann. This was admittedly a blatant case of conflict of interest, but government simply did not have the necessary competence to understand the problems and necessities and to arrive at an independent judgment. In 2009, time of the global banking crisis, the German economics minister asked a cabinet of specialized lawyers to draft a law concerning the administration of ailing banks. Criticized for giving this task to a law firm with important ties to international banking and the ensuing risk of conflict of interest, he protested that the ministry itself simply did not have the expertise to handle this highly complex matter.

In view of present-day complexity and acceleration, it is no surprise that the lifespan of regulations become shorter and shorter before they need to be rewritten[2].

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[1] Quoted from the Open Letter of M. Doepfner to Google's Eric Schmidt, `https://www.axelspringer.de/dl/433625/LetterMathiasDoepfnerEricSchmidt.pdf`

[2] For more on the causes of legislative acceleration see e.g. J.-C. Michéa: *The Realm of Lesser Evil.* Polity Press, Cambridge UK, 2009.

Complexity, Acceleration, Globalization — Some of the Challenges

The list is long — Demography, social disparity, public debt, sustainable energy, ecology, protection of privacy, terrorism, massive immigration, ...

These are undoubtedly giant questions, touching upon the determinants of society, — and even the man in the street begins to sense the incapacities of the democratic political system.

A Closer Look at Some Recent Issues

After the collapse of the Lehman Brothers investment bank in New York, an ultra-fast and high intensity dialogue between governments and central banks was needed to prevent a global collapse of the financial system.

Then there was the crisis of public debt: bankruptcy of Greece and other countries was avoided through loans and guarantees from richer EU countries. These allocations amounted to a significant fraction of their national budgets.

Most recently, Europe is literally overwhelmed with a seemingly uncontrollable wave of immigration.

In all these cases parliaments were — at best — informed by the executive that the situation was far from transparent, but large financial commitments were without alternative, and were so urgent there was no time for any proper discussion on them.

Very complex and dramatic issues, and parliaments are told that there is no time to discuss them at any length.

It should be recalled that the authority over the national budget, about how to allocate taxpayer money, is the central and most important privilege of Parliaments. Or was it...?

Post-Democracy

No wonder that the public feels at a loss with regard to the big political questions. Low voter turnout is the consequence.

Colin Crouch[1] observes "...Even if elections take place and continue to influence governments, the electoral debate is a tightly controlled show, rival groups led by experienced professionals in the techniques of persuasion practice on a limited number of questions selected from these groups. The mass of citizens plays a passive, acquiescent, even apathetic role, merely reacting to the signals it receives. Apart from the spectacle of the election campaign, policy is decided in private by the integration between elected governments and elites...". [2]

One just has to look at the propaganda before elections. The enormous gap between the complexity of the current problems, and the extreme simplicity of the "slogans" of political parties highlights the helplessness of the political class to communicate properly with the electorate, and effectively dissuades the people from exercising its democratic right of voting.

[1]Colin Crouch: *Post-Democracy*. Polity Press, Cambridge 2004.

[2]For more on this see also Jürgen Habermas: in "Critique et communication: les tâches de la philosophie". Esprit 2015/8, p. 40–54.

Public reactions to this disillusionment of the electorate tend to be politically correct: characteristically, the ensuing discussion focuses almost entirely on somehow keeping democracy going by working on the symptoms. The Secretary General of Germany's oldest and most venerable political party recently proposed to locate polling stations in shopping malls or big railway stations and to keep them open for a week to revive voting interest, instead of facing the delicate question whether the democratic process in its present form might be fundamentally inadequate.

Democracy — Two Imperfect Alternatives

There is Singapore with no natural resources, and a delicate balance between three ethnic groups of very different cultures — Chinese, Malay, and Indian.

Rising from great poverty and massive unemployment it is now one of the richest societies in the world, with very low crime rate and corruption, reasonable human rights, but with a very limited political participation.

Then there is China, often cited for human rights deficits, and with extremely low political participation, but quickly becoming an industrial superpower, global, financial and military, feared by its neighbors of becoming the bully in the region, but also with an impressive record in eliminating poverty. An 88% drop of the number of people living in poverty within the 23 years from 1978 to 2001[1] shows how China is winning the war against poverty, and thus the approval and gratitude of many.

These two models cannot and should not be copied. But it would also be a big mistake not to study them carefully with regard to their important successes.

Summarizing: Questions, Doubts, and Challenges

The most important feature of democracy, the one that is essential to be maintained and protected especially in troubled times, is of course its control of the political process. The problem we need to address here when we look at our role as voters is threefold:

1. Ex ante control is insufficient in view of *Complexity*, which entails intransparence of the issues and unpredictability of outcomes.

2. Ex post control often comes much too late in view of *Acceleration.*

3. Both have limited impact because of the transnational nature of many issues due to *Globalization.*

Can the democratic process effectively deal with increasingly important issues which are — at best — understood only by a small group of experts, taking into account that they occur so fast that there is no time to consult the legislative, much less the electorate, and that they require a considerable amount of continuous adaptive management and imprevisible, ad hoc government intervention?

[1]See e.g. M. Jacques: "When China Rules the World" Penguin, NY, 2009, p. 162

Or else:

Is it advisable — or is it possibly too risky — to seek and study possible adaptations or alternatives of the democratic process as we know it?

What could be these modifications? How can they ensure quality of the governmental system and civic control of its performance? *What do you think?*

Acknowledgements: My thanks go to my late colleague Glória Cravo, Funchal, for her encouragement, and to Josef Fröhlich, Vienna, for his stimulating suggestions and "question marks".

Космологія як синтез науки, філософії та мистецтва

Cosmology as Synthesis of Science, Philosophy, and Art

Interdisciplinary Studies of Complex Systems
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«Из-за поля Хиггса...»

Ю. И. Манин[1]

От редакции: Юрий Иванович Манин один из самых известных современных математиков, член.-корр. РАН, член Королевской академии наук Нидерландов, Французской академии наук, Американской академии искусства и науки, почетный доктор Сорбонны, университета Осло, Геттингена... До 2005 г. директор Института Макса Планка в Бонне, с 2005 г. заслуженный профессор Института Макса Планка.

Характерной особенностью научной деятельности Юрия Ивановича Манина является активный интерес к новейшим открытиям в математике и физике. Сфера его научных интересов широка: алгебраическая геометрия, диофантовы уравнения, интегрируемые системы, квантовые струны, теория вычислимости...

Среди последних работ Ю.И. Манина работы в области космологии в соавторстве с итальянским математиком Матильдой Марколли — «Big Bang, Blow Up and Modular Curves: Algebraic Geometry in Cosmology» и «Symbolic Dynamics, Modular Curves and Bianchi IX Cosmologies».

В своей работе «Time between Real and Imaginary: What geometries describe Universe near Big Bang?» Юрий Манин пишет: «Космология имеет свое собственное, исключительное место в научном познании, такую же полезность для понимания Вселенной имеют философия, поэзия, вера». Философия, психология, поэзия так же являются сферами интереса Юрия Манина. Здесь следует упомянуть его книгу «Математика как Метафора», в которую вошли «нематематические» тексты и поэзии, написанные за многие годы. В предисловии к этой книги Юрий Манин пишет: «Математика, прекрасное ремесло, которым я занимаюсь всю жизнь, служит здесь не только поводом для нематематических размышлений, но и метафорой человеческого существования. Не следует понимать эту фразу эзотерически. Математиков мало в каждом поколении, и они общаются часто над головами современников и через прошедшие десятилетия и столетия, как это делают поэты, музыканты, философы».

Ниже мы знакомим читателей с рядом стихов и поэтических переводов Юрия Манина, это философские размышления о жизни и смерти, о человеке в бескрайней Вселенной.

Комментарии к стихам и поэтическим переводам — автора.

[1] Max Planck Institute for Mathematics, Bonn, Germany. manin@mpim-bonn.mpg.de

Памяти Иосифа Бродского

....
Из-за поля Хиггса на берег Стикса
Выбираться, теряя остатки смысла,
Да и голоса, словно бельмо на глотке,
Так что не докричаться гребца и лодки,
Над водой, над которой еще светлеет
Слабый свет. Постепенно и он слабеет,
Потому что, подрагивая, уплывает с сетчатки на дно
Золотой пятак, медный обол, пятно...

Квантовое поле Хиггса — причина, по которой в ранней Вселенной изначально безмассовые частицы приобрели массу. В результате люди сделаны не из света, как ангелы, а из тяжелой материи. Выдумка трех нобелевских лауреатов, эквивалент первородного греха.

«На сетчатке моей — золотой пятак. Хватит на всю длину потемок» — последние две строки двенадцатой Римской элегии Бродского.

Ю. М.

Пространство-время как диагноз

Межреберная ностальгия:
тоска по времени, когда
я тосковал по пространству.

Пространство было закрыто.

Сейчас

пространство открылось, но почти закрылось

время.

Из Альфреда Бренделя

Ящички

Перед Большим Взрывом
были в основном выдвижные ящички
Мир до Большого Взрыва
не считая нескольких шариков
состоял из ящичков
В ящичках помещался
миллион с лишним световых лет
а больше ровно ничего
Потом при невыясненных обстоятельствах
ящички
медленно но верно
стали наполняться динамитом
Мир перед Большим Взрывом
был в большом порядке
По временам даже слышался чей-то смех

Альфред Брендель — знаменитый пианист, родившийся в Австрии, ныне живущий в Лондоне. Опубликовал несколько книг стихов, о которых говорит: «Все это приснилось мне по-немецки, потому что сны мне снятся по-немецки, и многие стихи зачинались в том состоянии между сном и явью, где смысл сливается с бес-смыслием и порядок с хаосом».

Из Дурса Грюнбайна.

Париж. Эйфория

Аритмия сердца при взгляде на Нотр-Дам...
На подносе Ситэ оставил белый скелет
допотопной рыбы
                неведомый мэтр-д'
черных Средних веков и лет.
Взгляд плывет. Запрокинутая голова
кружиться вместе с Парижем, влекома осью,
проходящей сквозь око портала. Небо, листва —
все прозрачно. Осень,
девятое октября. Колокола Сен-Дени
славят Блаженного. Воздух полон блаженств:
чист, прозрачен, звенит, звон уплывает в зенит,
каждый миг — совершенный жест.
Париж глубоко дышит. В Люксембургском саду
над травой и щебнем ветер пьянит, как наркотик.
Ни одной свободной скамейки нет на виду.
Сетка улиц растянута в подкорке.
Как прекрасно все преходящее. Все, что с лица
мирозданья исчезнет, когда исчезнешь ты, бренный.
От Большого Взрыва до твоего конца
кружится вальс со слезой
                Вселенной.
Закружись и ты. По бульварам, где чередой
фланируют экспатриды, изгнанники, эмигранты.
У букинистов вдохни над водой
аромат нераскрытых за сотни лет фолиантов.

Дурс Грюнбайн один из самых известных современных поэтов и переводчиков Германии, родился 9 октября 1962 г. в Дрездене. Приехал в Париж из ГДР и был поражен открытием новой вселенной как и я, когда приехал впервые в Париж в 1967 г.

Из Ганса-Магнуса Энценсберга

Научная теология

Вероятно, он лишь один из многих.
Иногда устает,
глаза в разные стороны. Работка — не приведи...
Все эти несчетные попытки... Ну да,
в принципе он всезнающ,
но ведь нет никакой возможности
все время входить во все детали.
Темная материя
никак не желает светиться.
Матрица рассеяния
только рассеивает внимание.
Нас много, а он один.
Проходит вечность,
и вот он снимает пробу.
В огромных глазах
отражается вся наша Вселенная.
А нас уже нет.
Жаль. Может быть — в чисто научном плане —
мы бы его заинтересовали.
Все же новинка.
Ну, скоропортящаяся,
за другими делами
и не заметишь.
Этот Бог нас проспал.

Ганс-Магнус Энценсбергер немецкий поэт, писатель, переводчик. Родился в 1929 г. в Баварии.

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Дискусійний клуб ○ Discussion Club

Космология как синтез религии, философии и науки

Н. В. Кондратьева[1]

Я полагаю, что космология — самая близкая к религии наука.

Поль Дирак
(Из дискуссии с кардиналом Жоржем Леметром, президентом Папской Академии Наук)

Все будет в руках будущих людей, — все науки, гипотезы, верования, техника, телепатия... и ничем будущее знание не станет пренебрегать, как пренебрегаем мы — данными веры, творениями философов, писателей и ученых древности, фактами, наблюдениями. Даже вера в Перуна и та пригодится. И она будет нужна для создания истинной картины мира.

К. Э. Циолковский

Введение

Сегодня мы являемся свидетелями небывалого в истории человечества научно-технического прогресса. Еще сто пятьдесят лет назад скорости, доступные человечеству, определялись наличием колеса и лошадей, и карета определяла скорость распространения информации. А сто лет назад уже был телеграф, телефон, автомобиль и электрическая лампочка. Сегодня люди работают в космосе, интернет изменил наше представление о пространстве, скорость жизни как количество событий в единицу времени резко возросла и наше представление о жизни на планете и в космосе быстро меняется. Большое значение в нашем новом понимании космических процессов имеет космология.

Космология — это наука о Вселенной. Человечество может исследовать Вселенную с помощью телескопов (оптических, нейтринных, радио- и инфракрасных,...), запущенных с Земли космических аппаратов, с помощью различных современных приборов. Поэтому, космология — часть физики и астрономии. В своих теоретических построениях космология является частью математики и теоретической физики. Научная космология

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родилась в начале XX века. До этого космология была предметом религии, философии и поэзии. Повзрослев, математика, физика, химия посчитали себя независимыми от схоластики, астрологии, алхимии и других средневековых наук; все, что нельзя было измерить и проверить, не считалось научным, религиозные верования и философские идеи не являлись научными аргументами.

Разрыв между религией и наукой произошел по многим причинам. Западная церковь с раннехристианских времен отстаивала свою доктрину о Творце, который все породил, все видит и всех судит. Церковь боялась астрономических исследований, т. к. ее доктрина была логичной для Земли как единственного и главного центра мира, но эта доктрина подверглась бы сомнению в случае, если бы Земля оказалась бы маленькой планетой в одной из бесчисленных Солнечных систем. Вряд ли бы в такой Вселенной кто-то бы занимался конкретным человеком, — там уже должен был действовать Закон. Закон как смысловое ядро Вселенной, в соответсвии с которым возможно познать связи всего. Инквизиция ставила барьер между религией и наукой. Позже разрыву между религией и наукой способствовало распространение грубого материализма, что в свою очередь привело к падению моральных принципов и использованию научных достижений в корыстных целях, противоречащих целям человеческой эволюции.

Однако полного отстранения науки от «ненаучного» никогда не происходило и в последнее время все больше взаимного интереса возникает между представителями науки, религии и философии. Это обусловлено тем, что новые научные открытия, с одной стороны, все чаще подтверждают философско-религиозные утверждения, с другой стороны, влияют на их развитие.

Можно было бы предположить, что в Советском Союзе, стране победившего материализма, марксистско-ленинской философии, стране, где религия считалась чем-то вроде «опиума для народа», ведущие ученые должны были бы не думать о единстве религии, философии и науки. Но, почему-то, именно самые известные советские ученые, оставили нам свои мысли об этом единстве.

Академик А.Д. Сахаров (физик): «Я не могу представить себе Вселенную и человеческую жизнь без какого-то осмысливающего их начала, без источника духовной теплоты, лежащего вне материи и ее законов. Вероятно, такое чувство можно назвать религиозным». «Для меня Бог — не управляющий миром, не творец его законов, а гарант смысла бытия — смысла вопреки видимому бессмыслию...» [1].

Академик Б.В. Раушенбах (физик, один из основателей космонавтики): «Религиозное переживание — это сфера эмоционального. Но существует ведь и богословие, совершенно логическое построение наподобие философских систем — суховатое, строгое, как математика; оно держится на логике... Существование логически строгого богословия наряду с глубоко интимным религиозным переживанием и красота математических доказательств свидетельствуют, что на самом деле разрыва нет, что есть целостное восприятие мира» [2].

Академик А.Д. Александров (математик): из автографа своеобразного собственного «Евангелия»: «Не говорю Вам — веруйте слепо, а устремляйтесь к пониманию того, что есть и как оно есть, и не ставьте впереди того предубеждения свои. Ибо в том, что есть и как оно есть премудрость Господня, а в предубеждениях твоих — только твоя. Так как же ты поставишь себя выше Господа.» [3].

Академик В.И. Вернадский (естествоиспытатель, основатель биогеохимии), из дневниковых записей: «Я считаю себя глубоко религиозным человеком. Могу очень глубоко понимать значение и силу религиозных исканий, религиозных догматов. Великая ценность религии для меня ясна, не только в том утешении в тяжестях жизни, в каком она часто оценивается. Я чувствую ее как глубочайшее проявление человеческой личности. Ни искусство, ни наука, ни философия ее не заменят, и эти человеческие переживания ее касаются тех сторон, которые составляют ее удел. А между тем для меня не нужна церковь и не нужна молитва. Мне не нужны слова и образы, которые отвечают моему религиозному чувству».

Академик Н.Н. Моисеев (механик, математик) в главе «О Боге, философии и науке» в своих мемуарах «Как далеко до завтрашнего дня...» пишет: «Я думаю, что сочетание веры в нечто высшее и способности к научному творчеству делает человека по-настоящему счастливым... . Я знал людей, которые жили преимущественно в мире чувств и веры и видел, что они были неизмеримо счастливее меня и всех тех, кто жил в другой, непересекающейся плоскости. И, поэтому, когда мне очень плохо, я иногда произношу ту кощунственную молитву, которую придумал еще в ранней юности: «Господи, если Ты есть, помоги мне уверовать в Тебя».

Можно продолжит эти примеры.

Бог Ньютона, Бог Эйнштейна, Бог Вернадского — у каждого свой Бог.

У Вивекананды однажды спросили почему он употребляет устаревшее слово «Бог». Вивекананда ответил: «Потому, что в нем сосредоточены все надежды и стремления человечества. Ныне стало уже невозможно изменить это слово. Вначале оно было выковано великими душами, ощущавшими его силу и понимавшими его значение. Затем, по мере того как оно обращалось в человеческом обществе им овладели невежды и уничтожили его дух... Слово «Бог» с незапамятных времен употреблялось для выражения Космического Разума и всего, что с ним связывали великого и святого..., оно было связано с бесчисленным множеством величественных и могучих идей, миллионы человеческих душ отождествляли его со всем, что есть самого высокого и самого лучшего, со всем, что разумно, со всем, что достойно любви, со всем, что есть героического и возвышенного в человеческой природе...». Понятие религии так же не является понятием однозначным. Мировых религий несколько и их религиозно-философские учения отличаются друг от друга, однако, в своих высших проявлениях они едины, — это познание Бога. Бога как Высшей Истины, как Законов мироздания, Сущности (Абсолюта), лежащего в основе всякого бытия; Света, в котором жизнь видится такой, какова она есть, ... И мы видим, что в своих Высших проявлениях цели религии и науки совпадают.

Как рождаются и умирают планеты, звезды, галактики; что за излучения пронизывают нашу планету, и что за «темная» энергия заполняет Космос; как «засыпает» и «просыпается» Вселенная и что значит «как на Небе, так и на земле»? Космологии принадлежит важная роль в создании новой научной парадигмы.

Науку космологию творят люди с их достоинствами и недостатками, драматическими событиями жизни и героической самоотверженностью, мучительными переживаниями и радостью познания Вселенной. Сегодня, когда развитие научно-технического прогресса стремительно продолжается, ряд ученых и философов видят необходимость в синтезе религиозно-философских и научных теорий для выработки нового миропонимания, другая часть ученых продолжает рассматривать науку как независимую от религии и философии сферу познания. В поисках аргументов pro et contra мы обращаемся к опыту ученых, которые внесли большой вклад в космологию, являясь при этом философами или религиозными деятеля.

Елена Блаватская и Роджер Пенроуз

В следующем году исполнится 185 лет со дня рождения Елены Петровны Блаватской. В Днепропетровске сохранился дом в котором родилась Елена Петровна, сегодня в нем находится Музей Е.П.Блаватской.

Среди учеников Елены Петровны были знаменитые ученые и изобретатели ее времени. Известно, так же, что основной труд Блаватской «Тайная Доктрина» была настольной книгой Альберта Эйнштейна, который внес значительный вклад в развитие научной космологии.

«Тайная Доктрина» была настолько новой и трудной для понимая в XIX веке книгой, что очень мало кто увидел в этом труде идеи, дававшие невиданный импульс развитию науки, и это причинило много страданий автору этого труда.

В первом томе «Тайной Доктрины», который называется «Синтез религии, философии и науки. Космогенез», Елена Блаватская изложила философские положения древнеиндийской религии, ранее неизвестные западному миру, дала комментарии к ним, рассмотрела их в контексте принятых научных положений своего времени.

Космология «Тайной Доктрины» рассматривает существование трех Миров, — двух непроявленных и одного проявленного, а также циклическую Вселенную, — последовательно рождающуюся, проявляющуюся и уходящую опять в мнимость. В этом труде, впервые, были даны подробные описания процессов предшествующих Большому взрыву и следующих за ним, как они были изложены в древних индийских книгах, хранившихся в тибетских монастырях. Станцы «Книги Дзиан» говорили на своем языке: «Времени не было, оно покоилось в бесконечных недрах продолжительности... Едина Тьма наполняла беспредельное Все, ибо Отец, Мать и Сын еще раз были воедино, и Сын не пробудился еще для нового колеса и странствий на нем... Вселенная — необходимости сын — была погружена в абсолютное совершенство... Причины существования исчезли; бывшее видимое и сущее невидимое покоилось в вечности не-бытия — единого бытия...» [4].

Елена Блаватская писала: «...центральная точка, из которой все возникает, вокруг которой и к которой все тяготеет и на которой висит вся ее философия, есть Божественная Субстанция — Принцип, Единая Начальная Причина. ... Субстанция — Принцип становится Субстанцией на плане проявленной Вселенной...». Далее Блаватская пишет о периодическом проявлении Вселенной, о Вселенной в ее трех аспектах: Пре-существующая, эволюционирующая из Вечно существующей, и Феноминальная, как отражение первых.

«Вселенная вырабатывается и устремлена изнутри наружу. Как вверху, так и внизу, как на Небе, так и на Земле, и человек, микрокосм и миниатюрная копия Макрокосма, есть живой свидетель этому Вселенскому Закону и его способу действия. Мы видим, что каждое внешнее движение, действие или жест ... производится и предваряется внутренним чувством или эмоцией, волею или желанием, мыслью или умом». [5]

Более чем через сто лет математик из Оксфорда Роджер Пенроуз написал о трех мирах: физическом мире («В нем находятся настоящие столы и стулья, телевизоры и автомобили, люди,...»); мире восприятий сознания («В этом мире есть счастье, боль и цвет. В нем любовь, понимание..., а также невежество и мстительность. ...»); платоновском мире математических форм («В этом мире мы встретим электромагнитные уравнения Максвелла и гравитационные — Эйнштейна, равно как и бесчисленные удовлетворяющие им теоретические пространства — времена... Именно здесь пребывают математические модели столов и стульев, которыми можно воспользоваться в «виртуальной реальности», а так же модели черных дыр и ураганов») [6].

Роджер Пенроуз также изложил свою версию циклической Вселенной в книге «Круги Времени». Согласно этой версии, эоны циклической Вселенной отделены друг от друга событием Большого взрыва. Коллапс сверхмассивных черных дыр перед Большим взрывом производит возмущения в виде гравитационных волн, которые, передают информацию от эона к эону. Ученый объехал многие университеты мира с лекцией «Круги времени. Можно ли сквозь Большой взрыв разглядеть предыдущую Вселенную?». И если некоторые ученые, выходя из лекционного зала, добродушно позволяли себе заметить: «Ну, конечно, это Пенроуз, он может себе позволить пофантазировать...», то часть других явно вдохновлялась услышанным.

Можно констатировать, что за сто лет отношение к ряду религиозных идей стало более толерантным и что наука уже «амнистировала» некоторые философские концепции в космологии. Во многом этому способствовало развитие научной космологии в XX веке у истоков которой стояли Альберт Эйнштейн, Александр Фридман, Жорж Леметр, Павел Флоренский, Константин Циолковский.

Альберт Эйнштейн и Александр Фридман

В 1949 г. в честь 70-летнего юбилея Эйнштейна вышел сборник «Альберт Эйнштейн. Философ-ученый». Эйнштейн не только писал работы по философии, философией он поверял научные работы. Он разделял теософские взгляды Блаватской и Спинозы об отрицании Бога как личности (индивидуальности) и рассматривал мир как Субстанцию, наделенную Разумом, который вырабатывает Идеи (Законы). Согласно философии Спинозы, которая была близка Эйнштейну, процесс познания мира состоял в формулировке аксиом и получении всех остальных положений путем логических следствий, что гарантировало истинность выводов в случае истинности аксиом.

О своих религиозных чувствах Эйнштейн писал: «Самое прекрасное и глубокое переживание, выпадающее на долю человека — это ощущение таинственности, оно лежит в основе религии и всех наиболее глубоких тенденций в искусстве и науке. Тот, кто не испытал этого ощущения, кажется мне, если не мертвецом, то во всяком случае слепым.

Способность воспринимать то непостижимое для нашего разума, что скрыто под непосредственными переживаниями, чьи красота и совершенство доходят до нас в виде косвенного отзвука — это и есть религиозность. В этом отношении я религиозен. Я довольствуюсь тем, что строю догадки об этих тайнах и смиренно пытаюсь мысленно создать далеко не полную картину совершенной структуры всего сущего... Если говорить о том, что вдохновляет современные научные исследования, то я считаю, что в области науки все наиболее тонкие идеи берут свое начало из глубокого религиозного чувства и что без такого чувства эти идеи не были бы столь плодотворными» [11].

В этом году научное сообщество отмечает 100 лет общей теории относительности Альберта Эйнштейна. Эта работа сыграла большую роль в развитии научной космологии.

При создании общей теории относительности, Альберт Эйнштейн обнаружил, что кроме обычного вещества и излучения, источником гравитации может служить особый член в правой части его уравнения. В своих первоначальных работах по космологии Эйнштейн придавал большое значение этому члену и получил статическую модель Вселенной, но не смог (впрочем, и не хотел!) найти нестационарные модели — это противоречило его философским убеждениям [7].

Модели расширяющейся Вселенной были найдены русским ученым А.А. Фридманом.

В 1922 г. Александр Фридман опубликовал в журнале «Zeitschrift fur Physik» работу, в которой на основании исследования общей теории относительности Эйнштейна, сделал вывод о том, что Вселенная должна расширяться. Эйнштейн дал отрицательный отзыв об этой работе, т. к. считал Вселенную стационарной. Однако, в XX веке в систему аксиом (научных теорий) и логических выводов активно вторгся эксперимент и, вскоре, Эйнштейн признал, что по поводу работы Фридмана ошибался. Тем не менее, обнаруженный особый член в уравнении, космологическую постоянную, считал «самой большой ошибкой своей жизни». Сегодня мы знаем, что Вселенная не статична и космологическая постоянная не равна нулю. Так что «ошибка» Эйнштейна оказалась его большим открытием.

Американский физик В. Вайскопф замечает, что «вещество», которое описывает космологическая постоянная весьма близко к имеющемуся в Библии выражению «тоху вабоху» — земле, которая была «безвидна и пуста» и которая существовала согласно книге Бытия до сотворения света [8]. Речь идет о «темной энергии», которая может занимать до 70% вещества Вселенной.

Эйнштейн всегда подчеркивал, что приоритет модели расширяющейся Вселенной принадлежит Александру Фридману.

Александр Фридман родился в Санкт-Петербурге в 1888 году, умер от тифа в 1925году и похоронен на православном кладбище в Санкт-Петербурге. Гениальный изобретатель и ученый, Фридман за 37 лет успел сделать очень много в авиаприборостроении, метеорологии, космологии. О мире его философских мыслей, чувств и верований известно очень мало. Во время Первой мировой войны Александр Фридман воевал, был летчиком, участвовал в воздушной разведке. Александр Фридман — Георгиевский кавалер.

Имя Александра Фридмана тесно связано с именем Жоржа Леметра. Бельгиец Жорж Леметр тоже героически воевал во время Первой мировой войны и был награжден Военным Крестом (Croix de guerre).

Аббат Жорж Леметр пришел к модели расширяющейся Вселенной независимо от Александра Фридмана. Поэтому часто говорят об уравнениях и моделях Леметра-Фридмана.

Католический священник Жорж Леметр и православный священник Павел Флоренский

Аббат Жорж Леметр и священник Павел Флоренский были учеными. Павел Флоренский закончил физико-математический факультет Московского университета, Жорж Леметр получил физико-математическое образование в университете Лувена, в Бельгии. Им обоим приходилось решать задачи синтеза религии, философии и науки. Это были разные решения, тем интереснее их рассмотреть.

После защиты докторской диссертации по математике Леметр поступил в семинарию архиепископии Малины. В сентябре 1923 г. он был рукоположен в сан священника и непосредственно после этого отправился в в Кембридж на постдокторскую программу под руководством А. Эдингтона. Затем, после получения степени доктора философии Массачусетского института технологии в 1927 г., Леметр был назначен на должность профессора Католического университета Лувена. В том же году он сделал свой ключевой вклад в космологию, опубликовав статью «Однородная Вселенная постоянной массы и увеличение радиуса в зависимости от радиальной скорости удаления галактик». Во время написания этой статьи Леметр не знал о том, что А. Фридман предвосхитил его на пять лет. Леметр имел хорошую научную интуицию и утверждал, что космологическая постоянная отлична от нуля и играет существенную роль. Он так же предложил гипотезу о «первичном атоме», которая позже получила название теории Большого взрыва [9].

В 1960 г. Жорж Леметр стал Президентом Папской академии наук в Ватикане.

Согласно уставу 1936 г., цель Академии — способствовать прогрессу математических, физических и других естественных наук и изучению связанных с ними гносеологических проблем. Членство в Академии не связано с какими бы то ни было ограничению по этническому или религиозному признаку. В ее состав входили М. Планк, Э. Резерфорд, Н. Бор, Э. Шредингер, В. Вольтерра, ...

Есть версия, что во время Первой мировой войны Леметр пережил мистический опыт, заставивший его верить непоколебимо. Вера в Бога была абсолютна, как и вера в то, что Богу угодно, чтобы люди самостоятельно познавали устройство мира и Его мысли. Это убирало всякое противоречие между религией и наукой. Леметр постоянно подчеркивал значительную концептуальную дистанцию, которая пролегает между двумя путями познания истины,- наукой и религией. С его точки зрения науки, включая космологию, не имели прямого отношения к религии, субъекту, чьей областью были души, а не галактики. Леметр, конечно, знал и разделял позицию Галилея, который писал великой герцогине Христине:

«Намерение Святого Духа в том, чтобы научить нас, как взойти на небеса, а не тому, как небеса движутся». В то же время Леметр писал: «По мере того, как наука проходит простую стадию описания, она становится истинной наукой. Также она становится более религиозной. Математики, астрономы и физики, например, являются очень религиозными людьми, за немногими исключениями. Чем глубже они проникают в тайну вселенной, тем глубже становится их убеждение, что сила, стоящая за звездами, электронами и атомами, есть закон и благость». И это означало, что религиозные и метафизические ценности становились важными для ученого на более высоком уровне постижения истины, стоящем над описанием и методами.

В разговоре с Полем Дираком, который утверждал, что самая близкая к религии наука это космология, Жорж Леметр возразил и высказал мнение, что самой близкой к религии наукой является психология. Здесь мне хочется сделать небольшое отступление и обратиться к работам Владимира Александровича Лефевра, который связал воедино космологию и психологию и показал, что в равной степени правы и Дирак и Леметр.

Лефевр построил свою модель Вселенной, во многом гипотетическую, в которую ввел разумный космический субъект наделенный рефлексией и совестью. (Рефлексия как самосознание и способность определять сознания других субъектов и совесть как способность различать «добро» или «зло»). Эта, не строго научная модель тем не менее позволила сделать очень интересные предположения: главная цель космических субъектов — достижение вечного существования, т. е. бессмертия; материей совершенных космических субъектов может быть магнито-плазменное образование; совершенные космические субъекты в своем поведении неукоснительно подчиняются высшим морально-этическим законам, которые обеспечивают существование Вселенной [12]. Лефевр также показал, что в основе совести и натуральных музыкальных интервалов лежат сходные алгебраические структуры и сделал вывод, что для обнаружения внеземных разумных субъектов следует искать музыкальные структуры в доходящих до нас космических сигналах.

Синтез психологии и космологии особенно присущ восточным религиям. Гаутаме Будде приписывают следующие слова: «Не принимайте мое учение просто из веры. Подобно тому как купец на базаре при покупке золота проверяет его: нагревает, плавит, режет — чтобы убедится в его подлинности, так же проверяйте и мое учение». Веками последователи Будды «нагревали» и «плавили», накапливали опыт достижения озарений путем созерцания, сосредоточения и медитации. И сегодня буддийский опыт часто называют наукой о сознании. С другой стороны, буддизм развил свою космологию — буддийскую философскую доктрину о чередовании проявленной и непроявленной Вселенной, и периодах возникновения и разрушения миров, измеряемых в кальпах (кальпа — 4,32 млрд. лет).

Библия так же говорит о времени сотворения мира. Жорж Леметр делал свои измерения — это время у него исчислялось в 4,5 млрд. лет.

В недавно изданной книге Сандера Вайса «Во славу науки» со ссылкой на В. Вайскопфа приводится случай, который произошел во время чтения лекций Леметром по релятивистской космологии в Геттингене. Студенты спросили Леметра зачем он занимается исчислением возраста Земли, разве он не доверяет Библии? Леметр ответил: «Просто для того, чтобы убедить себя, что Бог не сделал ни единой ошибки» [9], [13].

Павел Флоренский закончил физико-математический факультет Московского университета, затем Московскую духовную академию, в 1911 г. получил сан священника и сделался настоятелем домовой церкви Общины Красного Креста в Сергиевом Посаде. После революции 1917 г. занимался проблемами электрических полей и диэлектриков при «Главэлектро». В 1928 г. был первый раз арестован, в 1933 второй и сослан на Дальний Восток в Сковородино. Там Флоренский изучал возможность строительства на вечной мерзлоте. Далее Соловецкий лагерь особого назначения, где Флоренский работал на лагерном заводе йодной промышленности и запантетовал 10 научных открытий... Это очень поверхностная биографическая справка о человеке, который как никто в начале XX века воплощал в своем творчестве синтез религии, философии, науки и искусства.

Философ Н.Лосский писал о Флоренском: «Он был прекрасным музыкантом, проницательным поклонником Баха и полифонической музыки... Флоренский был полиглотом, в совершенстве владевшим латинским и древнегреческим и большинством современных европейских языков, а так же языками Кавказа, Ирана и Индии...» (П.А. Флоренский: Pro et contra, с. 395) Исследователь наследия Флоренского И. Исупов дал ему очень важную характеристику, он писал, что с Флоренским в культуру пришел новый тип личности, новизна которого определялась нестандартным устройством памяти и структурой внутреннего пространства ума Флоренского. В этом пространстве не было центра, как в Космосе — он был везде.

Флоренский оставил нам религиозно-философскую работу «Столп или утверждение истины», работы в области искусства, философии, религии и науки «Иконостас» и «Неправильная перспектива», работу в области науки, философии и религии «Мнимости в геометрии». Флоренский не только явил в этих работах синтез религии,философии, науки и искусства, он показал новые положения, следующие из этого синтеза. В работе «Мнимости в геометрии» Флоренский определил скорость света как границу между этим светом и Тем светом, (Землей и Небом): «Что собственно значит предельность величины скорости света?

Это значит вовсе не невозможность скоростей равных и больших C, а лишь появление вместе с ними вполне новых, пока нами наглядно непредставимых, если угодно, — трансцендентных нашему земному, кантовскому опыту, условий жизни; но это вовсе не значит, чтобы таковые условия немыслимы, а может быть, с расширением области опыта, — и представимы. Иначе говоря: при скоростях, равных С и тем более — больше С, мировая жизнь качественно отлична от того, что наблюдается при скоростях меньших С, и переход между областями этого качественного различия мыслим только прерывным...

На границе Земли и Неба длина всякого тела делается равной нулю, ... тело утрачивает свою протяженность, переходит в вечность и приобретает абсолютную устойчивость. Разве это не есть пересказ в физических терминах — признаков идеи, по Платону — бестельных, непротяженных вечных сущностей? Разве это не аристотелевские чистые формы? Или, наконец, разве это не воинство небесное,- созерцаемое с Земли, но земным свойствам чуждое? ...Область мнимостей реальна, постижима, ... Все пространство мы можем представить ДВОЙНЫМ, составленным из действительных и из совпадающих с ними мнимых гауссовых координатных поверхностей...» (1922, 3/17, Сергиев Посад).

Флоренский понял двойственность Мироздания, две части вселенной: мнимую вечную и временную реальную. Он раскрывал смысл слов Христа: «Царство Мое не от мира сего», «Я есмь путь и истина и жизнь», «Возьми свой крест и следуй за Мной». Этот путь Флоренский понимал как путь расширение сознания, одухотворение и утончение материи, путь от объекта к субъекту эволюции. Объект превращается в субъект когда путь становиться осознанным и возникает потребность в творчестве. Это становится возможным при вмещении сознанием синтеза. Как разность потенциалов порождает движение, так и эта великая двойственность Мироздания порождает вечное движение, беспредельность и бессмертие. Человечество тоскует и стремится в мир горний, а достигнув его возвращается за новым опытом в мир косной материи.

Здесь я хочу сделать небольшое отступление и привести цитату из одной станцы архаичной «Книги Дзиан», которая касалась предвоплощению Вселенной и которую комментировала Елена Петровна Блаватская: «Из лучезарности света — луча вечной тьмы — устремились в пространстве энергии ... три, один, четыре, один, пять — дважды семь, сумма всего. И эти суть естества, пламена, начала, строители,...». Блаватская, комментируя эту станцу, отмечала: «Это относится к кругу и цифрам и равнозначно словам о том, что цифры 3 1 4 1 5 все имеют отношение к окружности и диаметру круга». Понятно, что речь идет о числе пи, которое задает отношение кривизны к прямолинейности для любой сферы и что это число играет особую роль при формировании Вселенной. Если теперь мы вспомним мнимую единицу i (по известному высказыванию Лейбница: «Дух божий нашел тончайшую отдушину в этом чуде анализа, двойственной сущности, находящейся между бытием и небытием, которую мы называем мнимым корнем из отрицательной единицы») и число e, то можем констатировать, что признанная самой красивой математической формулой, — формула Ейлера есть также формула космологическая.

Рассматривая творческий процесс в живописи как геометризацию пространства в работах «Иконостас» и «Неправильная перспектива», Флоренский приходит к выводу: «Миропонимание есть пространствопонимание. В живописи именно пространство определяет не только стиль художественного построения, но и отражает миропонимание самого творца... Строение пространства есть КРИВИЗНА его... Вся культура может быть истолкована как деятельность организации пространства» [10]. Исследуя икону Андрея Рублева «Троица», Флоренский показал, что перспектива на иконе не такая как у художников — реалистов, она обратная. Пространство на иконе искривлено, оно искривлено согласно четвертому измерению.

Работы Флоренского, его записи, это океан, в котором волны научных и философских мыслей, религиозных и эстетических переживаний накатываются одна на другую и растворяются в едином вселенском пространстве.

Павел Флоренский был расстрелян в Соловецком лагере в 1937 году и похоронен в общей могиле для заключенных под Ленинградом.

Людмила Шапошникова в своей книге «Вселенная Мастера» одну главу посвятила Павлу Флоренскому. В этой главе Шапошникова приводит интересный эпизод: «...Январь 2002 года был в Италии удивительно теплым. Я бродила по Ватикану, переходя из одного храма в другой, из одного музея в другой... Собираясь уходить, я еще раз зашла в залы Древнего Рима. Я стояла у древнего саркофага, рассматривая его удивительно искусно сделанные барельефы, когда возле меня остановился человек и сказал : «Простите за назойливость, но Вы были в часовне Божией Матери?» «Право, я затрудняюсь ответить, столько храмов я посмотрела, что запуталась. Возможно я и была в ней», — сказала я. «Но если Вы были, то так бы не ответили.», — и отошел, направляясь прямо к выходу.

А я, заинтересованная этим странным разговором, пошла искать часовню. Нашла я ее не скоро, а войдя в нее, ничего странного и таинственного не обнаружила, пока мой взгляд не задержался на стене, где была мозаика, сделанная, я бы сказала, в современном стиле. Я подошла поближе и увидела три фигуры. Судя по надписи, сделанной на русском языке, одна из них изображала Флоренского. Он стоял с краю, распластав руки, похожие на крылья. Тогда же я узнала, что все трое были признаны католической церковью новомученниками».

Возможно, Жорж Леметр посещал эту часовню.

Константин Циолковский

Константин Циолковский говорил: «Я — чистейший материалист. Ничего не признаю, кроме материи» [14]. Циолковский подчеркивал свою нерелигиозность, что не помешало ему создать свою философскую теорию, очень напоминающую философию индуизма и буддизма. Свою философскую теорию Циолковский назвал теорией космических эр или лучистого человечества. Он говорил, что все изобретения и теория ракетостроения были разработаны им лишь как приложения к его философским изысканиям.

Мы многое знаем о жизни и творчестве Циолковского благодаря его ученику и другу Александру Чижевскому — одному из основателей космической биологии, «Леонардо XX века».

Удивительно,что в начале XX века в провинциальном городе Калуге, где и автомобиль был в диковинку, два великих ученых и изобретателя обсуждали жизнь Вселенной, изучение космического пространства реактивными аппаратами и прозревали будущее человечества.

Чижевский сохранил записи этих обсуждений и сегодня мы можем познакомиться с ними.

Циолковский делился с Чижевским своими раздумьями: «Мы уже много раз говорили с вами о передачи мысли на расстоянии, молниеносно, мгновенно. Мгновенность — это самое удивительное. Мгновенность и проницаемость. Последнее качество обязательно сопровождает первое. Но есть еще одно качество телепатии — это повсюдность, т. е. Проницаемость повсюду. Мозговое общение есть мировое явление. ... Но пойдем далее. Минковский вообразил «мировую линию». Мы уже говорили о «мировом мозге». Пока его нет. Не видно! Но если телепатическая функция перейдет со времени в «самое существо мира», а это, очевидно, неизбежно, то тогда ... Космос станет единым мозгом. Эту эру я называю для краткости «лучистой». ... Неясным остается скорость распространения пелепатического поля, но я думаю, эта скорость превосходит скорость распространения света. ... Человек постепенно перерождается — из жалкого просителя он становиться в воинственную позу и начинает требовать: дескать, выкладывай, мать-природа, всю истину. Так заявляет о себе новая космическая эра, к которой мы подходим, медленно подходим, но верно. ... Через многие миллиарды лет «лучистая» эра Космоса снова превратится в корпускулярную, но более высокого уровня, чтобы все начать сначала: возникнут Солнца, туманности, созвездия, планеты, но по более совершенному закону, и снова в Космос придет новый, еще более совершенный человек, чтобы перейти через долгие миллиарды лет и погаснуть снова, превратившись в сверхлучевое или сверхтелепатическое состояние, но уже более высокого уровня. Пройдут миллиарды лет, и опять из лучей возникнет материя...».

Как и Флоренский Циолковский рассматривал процесс эволюции как движение от объекта к субъекту, от косной материи к энергии сознания. Двойственность Вселенной (проявленной и лучистой) он воспринимал как разность потенциалов, порождающую движение — бесконечную эволюцию Вселенной.

Чижевский писал, что за внешне спокойной жизнью Циолковского скрывалась большая драма жизни, непонимание и умышленное игнорирование его. Да и самому Чижевскому пришлось пройти через лагерную жизнь, труды его определялись как «ненаучные» и воспоминания о Циолковском ему так и не удалось опубликовать при жизни. В 1963 году рукопись Чижевского о Циолковском передали Главному конструктору космических кораблей Сергею Павловичу Королеву с просьбой подписать отрицательный отзыв. Королев отказался подписывать такой отзыв и пытался даже помочь с изданием рукописи, тем не менее рукопись была издана только в 1995 году.

Константин Циолковский умер в 1935 году, его похоронили как атеиста. Через 31 год православный священник Александр Мень провел обряд отпевания над его могилой.

Достижения научной космологии в XX–XXI веках оказали большое влияние на формирование современных философско-религиозных мировоззрений.

«С эпохи Ренессанса сложилась концепция, пропагандировавшаяся Джордано Бруно, о «бесконечной» Вселенной как некотором бесконечном «складе» различных вещей, неизвестно как появившихся... Наша Земля и люди на ней — это тоже неизвестно как и зачем взявшиеся случайные объекты этого гигантского «склада». Эту точку зрения Ж. Леметр назвал «кошмаром бесконечности». Окруженный этим пространственно-временным кошмаром, человек мог найти единственное утешение в некотором «закрывании глаз» — сосредоточении на «здесь и сегодня» — сиюминутных интересах и сознательном отказе от постановки фундаментальных вопросов мироздания» [5].

Современная космология утверждает, что возраст и радиус Вселенной можно вычислить.

«Вселенная — это громадный «дом», в котором все было готово для рождения человечества и где хранятся «фотографии» его прошлого» [6]. «Кошмар бесконечности» сменился устремлением к бесконечному познанию мыслей Бога, среди которых закон вечного движения, двойственности Вселенной, закон цикличности с последовательным «разбрасыванием камней» и «собиранием камней», закон причины-следствия... Человечество долго пребывало и остается до сих пор в очевидности трехмерного пространства с прямыми направлениями влево — вправо, вверх — вниз; в очевидности несправедливости, т. к. оперирует очень малыми периодами времени; в невежестве уверенности в абсолютном праве и владении своими мыслями и желаниями; ...

Сегодня, когда Стандартная модель уже недостаточна для описания получаемых учеными результатов и стали говорить о расширении Стандартной модели (хотя не все ученые с этим согласны), космология вплотную подошла к изучению черных дыр, темной материи и темной энергии или Того света о котором писал Флоренский. Мысль как энергия, информация передаваемая со скоростью больше скорости света, передача ее по магнитным «туннелям» и общение между мирами, ... — это все уже предметы исследования науки.

Мы стоим на пороге того, что наука докажет, что ложь, воровство, зависть, честолюбие, невыгодны, т.к. человек это процесс эволюции и его жизнь не ограничивается 80 — 100 годами земной жизни и на больших временных и пространственных интервалах законы божеские о справедливости, гармонии и разумности сущего неукоснительно соблюдаются. Наука, и космология как ее часть, поможет развить и повысить моральные ориентиры общества и тем возьмет на себя часть задач религии, но новые тайны мироздания встанут перед человечеством и вера вместе с научно-философскими теориями и научными исследованиями принесет людям великую радость и сделает их сотрудниками эволюции.

Об этом 29 ноября 1996 г. писал Папа Иоанн Павел II обращаясь с посланием к Папской академии: «К счастью, Церковь и научное сообщество могут сегодня рассматривать друг друга как партнеров в общем стремлении ко все более совершенному пониманию Вселенной, той сцены, по которой человек идет сквозь время навстречу своему трансцендентному предназначению. ... Яркий пример общего интереса науки и религии, более того, их нужды друг в друге — тема вашего нынешнего собрания:

«Возникновение структуры во Вселенной на уровне галактик». Этой конференцией вы завершаете общий обзор физического космоса.

Потрясающе, что с помощью сложной современной технике вы «видите» не только обширность Вселенной, но и невообразимую энергию и динамизм, пронизывающий ее. Еще более поразительно то, что поскольку сигналы от ее самых дальних областей передаются светом с конечной скоростью, вы способны «заглянуть» в отдаленные прошлые эпохи, а не только описывать процессы происходящие сегодня. ... Вы, люди науки, внимая огромной пульсирующей Вселенной и разгадывая ее тайны, осознаете, что в некоторых точках наука, видимо, достигает той таинственной границы, у которой ее вопрошание соприкасается со сферами метафизики и теологии. В результате этого нужда в диалоге и сотрудничестве науки и веры становится все более животрепещущей и многообещающей» [15].

5 октября 2015 г.

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My Spacetime: Linking Past, Present and Future in an Unbroken Chain of Eternity

A. Azarova-Antonova[1]

They say living in the present is the only way to be happy. If so, why do I so often tend to yearn for my past, trying to revive emotions and events that turned to cold ashes a long time ago. What is it in my past afterwards that makes it so attractive I can't or even don't want to let go of it... Or, what is the past itself? Is it just my sweet memories, my unfulfilled wishes, my unexpressed potential or something else?..

To understand this phenomenon, I start to analyse what I truly feel when dwelling on the past — or rather on its brightest moments. First of all, I have to admit that I feel comfortable, warm and safe — yes, I feel like being at a safe distance, far away from the harsh modern day reality. Well, to some extent, my past gives me a momentary solace and relief, which help me cope with a day-to-day boring routine. The past helps me balance with the present.

However, there is something more to it than a perfunctory and brief relief. The thing is that I had (or I seem to have had) a very abundant life in my past, full of different kinds of events and experiences that would come and go, with every day bringing in something new. Then I was young and therefore eager to learn things, to absorb — even glutton — any piece of information coming from everywhere. Perhaps, it is my adolescence that made my perception too sharp, too acute, and too sensitive, no matter what I experienced — grief or joy, pain or excitement. Perhaps, due to my acute perception and open-mindedness, I thought I had been doing a big job in my life, implementing some important tasks (the Mission?) that mattered, in a way, both to the other people and to my spiritual progress.

Those days are all gone now...

However, memories still stay here to help me go through hardships and ordeals that come up my way now and then. They help me face the current days, full of frustrations, sadness and bitterness. When I recall my past happiness in times of misery, it gives me strength and energy to go on. It gives me a sense of meaning and purpose of my being. I say to myself — if I was happy once, then I can be happy again, I can again enjoy the taste of life, I can again feel important, individual and confident.

Well, I am talking about recollections of mind, and this is just the first layer of memory. However, there is the second — deeper — layer of memory, which entails reminiscences of Soul, recollections of its previous incarnations, previous journeys and previous experiences with their effects still reflected in my nature, character, and the circumstances that surround my relations...

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I think this 'déjà vu' feeling is familiar to you all — for example, when you walk through a brand new place and feel like you have been here before. Or you meet someone for the first time in your (current) life and suddenly feel like having strong and close ties with him/her. Our paths have crossed before, now they are crossing again. And I ask myself - what does it mean to me now? What for does this person appear in my life again? Shall I avoid or invite him/her in my life to complete my unfinished tasks? Do we have to continue our story or put an end to it? What roles shall we play now?

Memories of Soul can do some other tricks as well.

Sometimes I plunge in the past so deeply it becomes even more real than reality itself. It is a bizarre yet exciting feeling — like going through a layer of various dimensions, like being here and there and somewhere else just at the same moment. I feel like being someone else when still remaining Me, Myself. Perhaps this is a moment when I am truly awakened to some parts of my inner world, which were hidden (rooted) in my unconscious to come out as a sudden revelation. Each of this part also has its own past life memories to give me an idea who and what I am or was or can be or had better not be...

This multifaceted and multidimensional state of mind is equally as dangerous and destructive as conducing and healing. (It is just a tool, and the key issue is to learn how to use it in a proper way. Now I have arrived at the very core of my analyses, this is the cornerstone of the whole story…) This can be utterly ruinous only in case I start comparing my present and my past, with a distinct preference to the latter (while indulging myself in a sweet memory pie). In this manner I pump the vital energy out of my present and give it away to my past, thus stripping my future of its great potential. As a result, subconsciously, I know that nothing good awaits me in future, and consequently I feel upset, depressed and frustrated. Then the vicious cycle starts rotating…

This is how this foul scheme works — but only if I resort to comparative assessment. Although, when I don't make evaluative assessments, when I acknowledge what I feel without judgment, my past memories (deprived of energy now) remain just…memories, which like fragments fall into one picture — my rich, wonderful and precious experience.

That is it. It is purely my experience and I have to treat it in a right way. My life is the process of going through a combination of states, of being and becoming. If I focus only on the past memories, I fail to pay attention to what's going on in the present. If I dream of the future, I will get lost in fantasies leading me nowhere.

The best compass to guide me to the lighthouse is embracing whatever happens in my life: the past with its mighty history, the present with its poignant reality, and the future with its promising perspectives. I have to embrace it all to keep things going and write a new chapter of my life.

When I take my past as it is with all its lessons useful for the present, it gives me energy and hope for the future. Here is where the circle of LIFE starts rotating to produce the beautiful formula of my personal Spacetime: **"Trapped in the Past = Fearful Hopeful of the Future".**

Here is where the Eternity steps in…

Circles Across Universe

I am the Rock, hidden under the inevitable dust,
(I wonder if ever the celestial sphere will condescend to embrace me?)
It might have been the Lord who flung me away,
Or perhaps it was Lucifer who kicked me off ...
Though scorched by the sun, still I remain cold,
Though worn away by the constant water dropping, and subdued by the seasonal snow,
Still I can feel, I can breath and I can see,
I am the Rock, I am the Faith coming from above.
I know what it feels like carrying the burden of the terrestrial gravity
Against the arrogant wind trying my patience with hundreds of its tricks,
I am the groaning of the Tablet, which gives birth to the Word,
I am the Rock — I am alive! So are the purple streaks throbbing on my skin
And shaping an intricate pattern of my future and past lives...
Constrained within myself, I am absorbing the outer Space,
I am the Rock, I am the flame concealed in a stone goblet.
I am Creator's first attempt, his initial design,
And his incipient premonition, which pre-shaped the first ever human heart...
I am grey, and therefore invisible when lying on the grey ground,
However, this is what makes my fragile carcass firmer and safer...
I am the Rock wrapped in a limp cocoon,
I am the moss-covered, worm-eaten mystery of the Being,
Call me a tombstone, a wall-stone, a cornerstone!
I am a block bearing the Universe,
I am the living soul of the co-creation process...
I am the Wanderer, who does not move,
But keeps going along the path of repentance —
At the end of my journey I will be freed (with a knowing grin on my face)
And I will sink in the starry Heavens
Sending circles across the Universe...
I am the Rock, I am the Rock...

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Зміст

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Contents

Тематика та мета журналу

«Міждисципрлінарні дослідження складних систем» — це рецензований журнал із вільним доступом, що публікує дослідницькі статті, огляди, повідомлення, дискусійні листи, історичні та філософські студії в усіх областях теорії складних систем для впровадження взаємодії між науковцями з різних галузей математики, фізики, біології, хімії, інформатики, соціології, економіки та ін. Ми бажаємо запропонувати істотне джерело актуальної інформації про світ складних систем. Журнал має стати частиною наукового форуму, відкритого та цікавого як для експертів з різних областей, так і для широкої аудиторії читачів: від студентів до досвідчених дослідників. Журнал надає можливість для науковців з різних галузей презентувати нові ідеї, гіпотези, піонерські дослідження. Особливо запрошуються до публікації автори наукових статей та (але не тільки) наукових оглядів, проте статті з історії та філософії науки, інформації про наукові події, дискусійні повідомлення також вітаються.

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Журнал друкує оригінальні статті, огляди, повідомлення українською, російською, англійською та німецькою мовами. Статті українською та російською мовами мають містити переклад англійською назви статті, анотації та прізвищ авторів.

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Aims and Scope

"Interdisciplinary Studies of Complex Systems" is a peer-reviewed open-access journal, which publishes research articles, reviews, letters, discussions, historical and philosophical studies in all areas of the complex systems theory in order to provide the interaction between scientists working in different areas of Mathematics, Physics, Biology, Chemistry, Computer Science, Sociology, Economics etc. We would like to promote the significant source of up-to-date information on complex systems worldwide. The journal shall be a part of the scientific forum, open and interesting for experts from several areas and for a broad audience from students to senior researchers. The journal shall give a possibility for scientists from different disciplines to present new ideas, conjectures and pioneering developments. The research papers and (but not only) reviews are especially encouraged. At the same time, papers in the history and philosophy of science, information about scientific events, discussion papers will welcome.

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