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Billiards, invisibility, and perfectly streamlining objects

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1 Introduction

In this paper we shall describe recent applications of billiards in aerodynamics and optics. More precisely, we shall explain how to construct perfectly streamlining bodies in the framework of Newtonian aerodynamics and invisible objects in geometric optics. The methods we shall use are quite elementary and accessible to students of the high school; they include focal properties of curves of the second order and unfolding of a billiard trajectory.

2 Perfectly streamlining bodies in aerodynamics

To start with, let us consider a rigid body moving through a rarefied medium of point particles. The medium has zero absolute temperature; this means that the particles are initially at rest. When hitting the body, particles are reflected in the perfectly elastic manner. The medium is so rarefied that particles never hit each other.

The (generalized) Newton aerodynamic problem consists in finding the best streamlining body from a given class of bodies. This means that the force of resistance exerted by the medium on the body is minimal in this class of bodies. This problem was solved by Newton himself in the class of convex axially symmetric bodies with fixed length and width [13], and by several authors in various classes of bodies, provided that each particle hits the body at most once [2–9, 11, 12].

In a reference system connected with the body one observes a flow of medium particles with equal velocities incident on the body at rest. Choose the reference system in such a way that the velocity of the flow is $(0, 0, -1)$. If the body surface turned to the flow is the graph of a function $z = u(x, y)$ and each particle hits the body only once, the projection of the resistance force of the body on the z-axis $R(u)$ (which will be referred to as *resistance* in the

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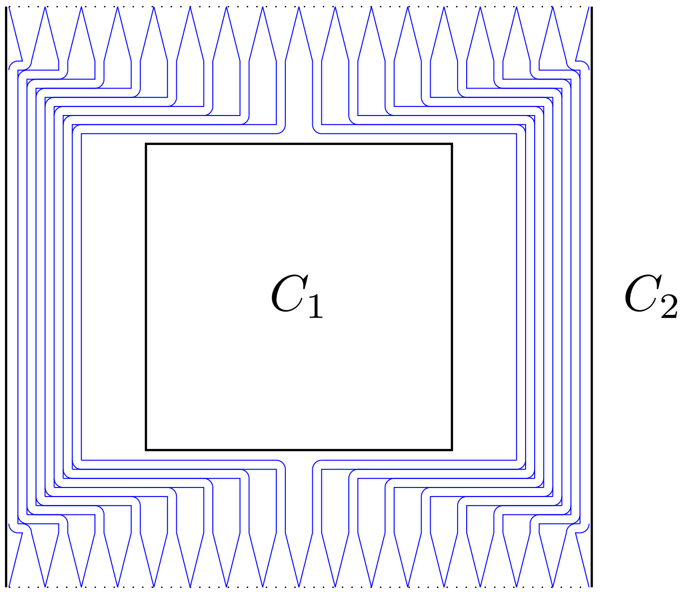

Fig. 1: Two concentric squares with a built-in channel system.

sequel) can be written down in a comfortable analytical form

$$F(u) = \iint \frac{1}{1 + |\nabla u(x,y)|^2}\, dx\, dy.$$

If one allows multiple reflections of particles, the formula of resistance is more implicit. Let $B \subset \mathbb{R}^3$ be the body under consideration, and let the particle of the flow that moves according to $(x, y, -t)$ for t sufficiently small, after several reflections from the body move freely with the velocity $v_B(x,y) = (v_B^x(x,y),\, v_B^y(x,y),\, v_B^z(x,y)) \in S^2$. The resistance equals

$$R(B) = \frac{1}{2} \iint (1 + v_B^z(x,y))\, dx\, dy.$$

Note that in the particular case when the condition of single reflection is satisfied and the front part od the body surface is given by $z = u(x,y)$, one has $F(u) = R(B)$.

If multiple reflections of the particles are allowed, and therefore the theory of billiards is applicable, one comes to some very surprising conclusions. First, in the class of bodies that contain a bounded convex body C_1 and are contained in another bounded convex body C_2 (where $C_1 \subset C_2$ and $\partial C_1 \cap C_2 = \emptyset$) the infimum of resistance is zero [15]. In other words, the resistance of a convex body can be made as small as we please by small perturbation of the body near its boundary. Let us illustrate this in the case when C_1 and C_2 are rectangular parallelepipeds with the edges parallel to the coordinate axes.

[*The construction with two rectangles follows. An explanation of motion in channels should be given. The 3D construction is obtained by making a "sandwich" whose layers are as in the 2D construction.*]

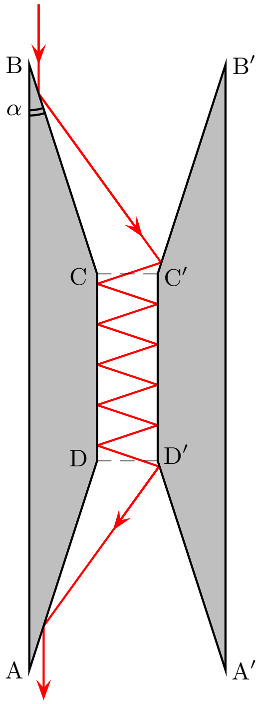

Fig. 2: Union of two trapezoids.

Let us now consider modification of this construction. Consider a rectangle with a single built-in channel (see Fig. 2). As first conjectured by E. Lakshtanov, under a certain condition (to be specified below) on the parameters of the figure, the final velocity of a particle of the parallel flow will always be equal to the initial velocity of the flow (vertical in Fig. 2).

Indeed, let E be the point of intersection of the lines BC and $B'C'$, and let $\beta = \measuredangle BOB'$. Consider the broken line formed by the segment CC' and its rotations by the angles β, $-\beta$, $2\beta, -2\beta, \ldots$ (while the modulus of the angle is smaller than π), and assume that the lines AB and $A'B'$ touch this broken line (see Fig. 3). The initial velocity of the particle is $v = (0, -1)$; let us prove that its final velocity is also v.

[*The proof (based on unfolding of a billiard trajectory) follows.*]

It follows that the union of two trapezoids in Fig. 2, when moving in the vertical direction, has zero resistance. Now it is easy to obtain 3D bodies with zero resistance. First, one can rotate the union of trapezoids about its vertical symmetry axis. Second, one can translate it in the direction orthogonal to the plane of the figure.

Let us further simplify the construction. Let C and D coincide and $\alpha = \pi/6$ in Fig. 2; then we obtain a union of two triangles, as shown in Fig. 4. This is probably the simplest figure of zero resistance.

Remarkably, we have found perfectly streamlined bodies. This means that they can move perpetually in a homogeneous rarefied medium without slowing down the velocity. However, the medium will resist to attempts of maneuvering.

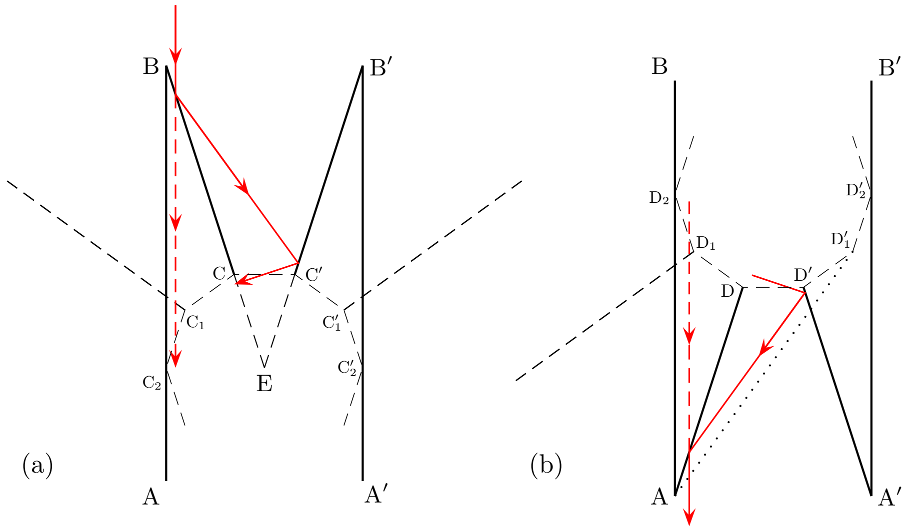

Fig. 3: Unfolding of a billiard trajectory.

Additionally, the resistance is nonzero, if the medium is not homogeneous. For instance, the body will slow down when getting into a homogeneous cloud, and will recover its original velocity when going away.

3 Invisible objects

The ideas of the previous section can be used in geometric optics when constructing invisible objects. Indeed, put together two bodies of zero resistance mutually symmetric with respect to a plane orthogonal to the direction of the flow; as a result we will obtain an object invisible in this direction (see, e.g., Fig. 5).

[*The explanation.*]

Now when we have constructed an object invisible in a direction, it is natural to ask, if there exist objects invisible from a point. They really exist, and the underlying construction is based on focal properties of curves of the second order.

The following geometrical statement plays an important role in problems of Newtonian aerodynamics [1, 14]. It allows one to build "invisible object" like the curvilinear triangle ABC shown in fig. 9 at the end of this paper. In this note we are going to prove this statement.

Theorem. *Let F_1F_2C be a right triangle with the right angle at F_2, and let $\mathcal{E}$ and $\mathcal{H}$ be the confocal, with foci at F_1 and F_2, ellipse and hyperbola through C. (We consider only the branch of the hyperbola $\mathcal{H}$ that contains C.) Consider a ray with the vertex at F_1, which intersects the ellipse $\mathcal{E}$ and the branch of the hyperbola $\mathcal{H}$ at some points A and B. Then the segment F_2C forms equal angles with the segments F_2A and F_2B: $\alpha = \beta$ (see Fig. 6).*

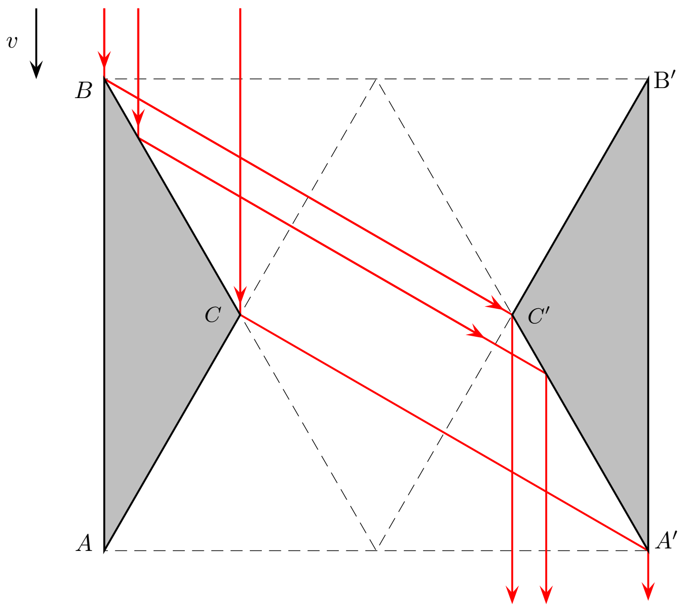

Fig. 4: A body of zero resistance: basic construction.

Notice the following property, which is a direct consequence of the theorem.

Corollary. *Let A_1 be the point of intersection of the ray F_2A with the branch of the hyperbola $\mathcal{H}$, and let the ray F_1A_1 intersect the ellipse at B_1 (Fig. 6). Then, according to the theorem, the points B, B_1, and F_2 lie on the same straight line. In other words, each of the triples, F_1AB, $F_1A_1B_1$, F_2A_1A, and F_2B_1B, is collinear.*

The proof of the theorem makes use of the following characteristic property of an angle bisector in a triangle.

Lemma. *Consider a triangle ABC and a segment BD joining the vertex B with a point D lying on the opposite side AC. Denote $a_1 = AB$, $a_2 = BC$, $b_1 = AD$, $b_2 = DC$, and $f = BD$ (see Fig. 7). The segment BD is the bisector of the angle B, if and only if $(a_1 + b_1)(a_2 - b_2) = f^2$.*

Proof. Let $f = BD$ be the bisector of the angle B to the side AC. Let us prove the following relations on the values a_1, a_2, b_1, b_2, and f:

1. $a_1/a_2 = b_1/b_2$;
2. $a_1a_2 - b_1b_2 = f^2$;
3. $(a_1 + b_1)(a_2 - b_2) = f^2$.

The equalities 1 and 2 are well known; each of them is a characteristic property of triangle bisector as well.

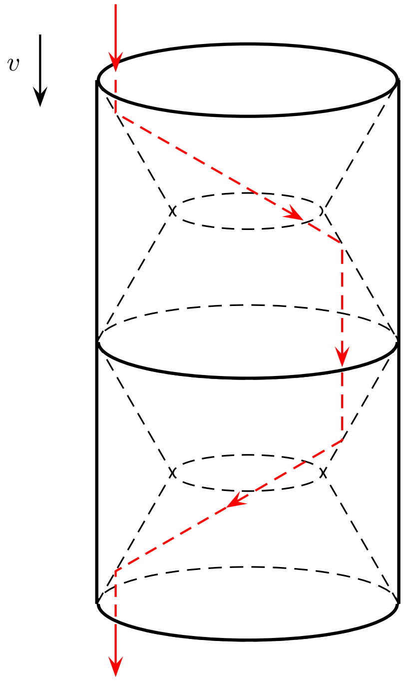

Fig. 5: A body invisible in the direction v. It is obtained by taking 4 truncated cones out of the cylinder.

The first property is a consequence of the following formula that compares areas of triangles:

$$\frac{a_1}{a_2} = \frac{\frac{1}{2}\, a_1 f \sin\alpha}{\frac{1}{2}\, a_2 f \sin\alpha} = \frac{S_{ABD}}{S_{BCD}} = \frac{\frac{1}{2}\, b_1 h}{\frac{1}{2}\, b_2 h} = \frac{b_1}{b_2}\ , \tag{1}$$

where $\alpha = \measuredangle ABD = \measuredangle CBD$, and h is the height put from the vertex B on the side AC.

The second property of the bisector is based on the notion of "degree" of a point relative to a circumference. Let us circumscribe the circumference ω around the triangle ABC. Take a chord through a point D inside a circumference ω; this chord is divided by D into two segments. The product of the lengths of these segments is called the *degree of the point* D (all such products are equal for the given point D). Denoting $DE = g$, we get for the point D that $b_1 b_2 = fg$ (Fig. 7).

Note that $\triangle ABE$ is similar to $\triangle DBC$ by two angles:

$$\measuredangle ABE = \measuredangle DBC = \alpha \quad \text{and} \quad \measuredangle AEB = \measuredangle ACB = \frac{1}{2}\ \overset{\frown}{AB}\ .$$

Therefore

$$\frac{a_1}{f+g} = \frac{f}{a_2},$$

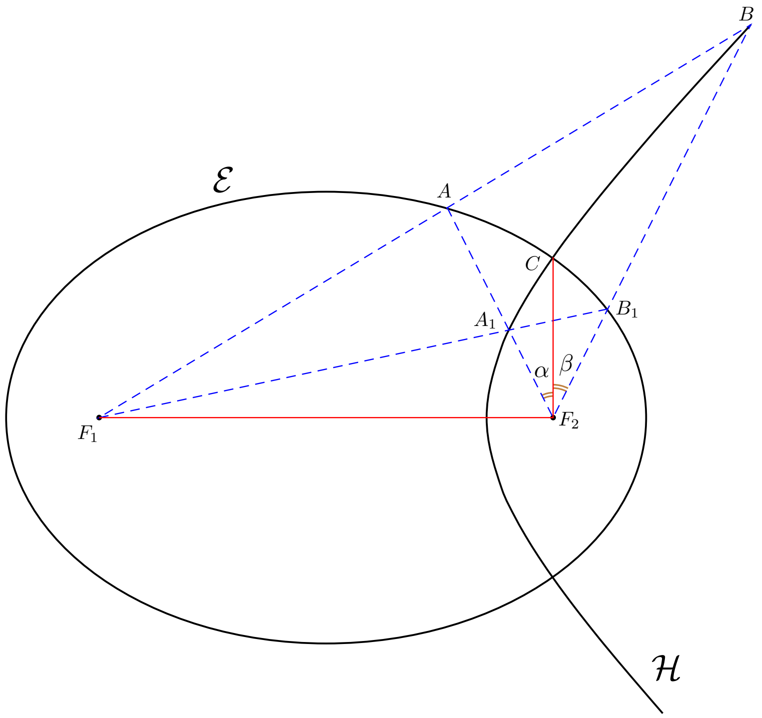

Fig. 6: $\alpha = \beta$.

whence

$$a_1 a_2 = f^2 + fg \quad \Rightarrow \quad f^2 = a_1 a_2 - fg = a_1 a_2 - b_1 b_2,$$

Q.E.D.

Let us now prove that the bisector f satisfies the equality 3, and vice versa, a segment BD satisfying this equality is the bisector. Notice that we are unaware of any mentioning of this property in the literature.

One easily sees that the algebraic relations 1, 2, and 3 are "linearly dependent": any two of them imply the third one. Therefore the properties 1 and 2 of the bisector imply the direct statement: the bisector f satisfies the property 3.

In order to derive the inverse statement, we need to apply the sine rule and some trigonometry. Denote $\alpha = \measuredangle ABD$, $\beta = \measuredangle CBD$, and $\gamma = \measuredangle BDC$ (see Fig. 7 (b)). We are going to prove the equality $\alpha = \beta$. Applying the sine rule to $\triangle ABD$, we have

$$\frac{a_1}{\sin\gamma} = \frac{b_1}{\sin\alpha} = \frac{f}{\sin(\gamma - \alpha)},$$

and applying the sine rule to $\triangle BDC$, we have

$$\frac{a_2}{\sin\gamma} = \frac{b_2}{\sin\beta} = \frac{f}{\sin(\gamma + \beta)}.$$

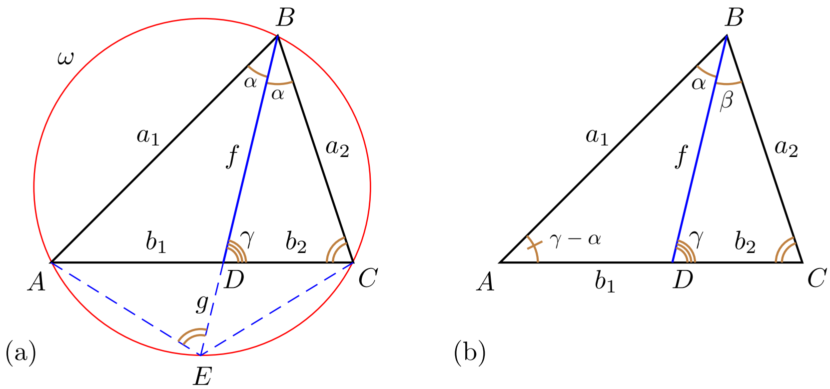

Fig. 7: The proof of the direct (a) and inverse (b) statements on the bisector.

This implies that

$$a_1 + b_1 = \frac{f}{\sin(\gamma - \alpha)} (\sin\gamma + \sin\alpha) = f\, \frac{\sin\dfrac{\gamma+\alpha}{2}}{\sin\dfrac{\gamma-\alpha}{2}},$$

$$a_2 - b_2 = \frac{f}{\sin(\gamma + \beta)} (\sin\gamma - \sin\beta) = f\, \frac{\sin\dfrac{\gamma-\beta}{2}}{\sin\dfrac{\gamma+\beta}{2}},$$

and using the condition 3, one gets

$$f^2\, \frac{\sin\dfrac{\gamma+\alpha}{2}\, \sin\dfrac{\gamma-\beta}{2}}{\sin\dfrac{\gamma-\alpha}{2}\, \sin\dfrac{\gamma+\beta}{2}} = f^2,$$

whence

$$\sin\frac{\gamma+\alpha}{2}\, \sin\frac{\gamma-\beta}{2} = \sin\frac{\gamma-\alpha}{2}\, \sin\frac{\gamma+\beta}{2},$$

$$\cos\frac{\alpha+\beta}{2} - \cos\Big(\gamma + \frac{\alpha-\beta}{2}\Big) = \cos\frac{\alpha+\beta}{2} - \cos\Big(\gamma - \frac{\alpha-\beta}{2}\Big),$$

$$\cos\Big(\gamma + \frac{\alpha-\beta}{2}\Big) = \cos\Big(\gamma - \frac{\alpha-\beta}{2}\Big).$$

The last equation and the conditions $0 < \alpha,\ \beta,\ \gamma < \pi$ imply that $\alpha = \beta$, Q.E.D. □

Let us now proceed to the proof of the theorem.

Extend the segment BF_2 until the second intersection with the ellipse at a point A'. Denote

$$f = F_1F_2,\ \ c = F_2C,\ \ a_1 = F_1A',\ \ b_1 = F_2A',\ \ a_2 = F_1B \text{ and } b_2 = F_2B$$

(see Fig. 8). Let the second point of intersection of the ellipse with the branch

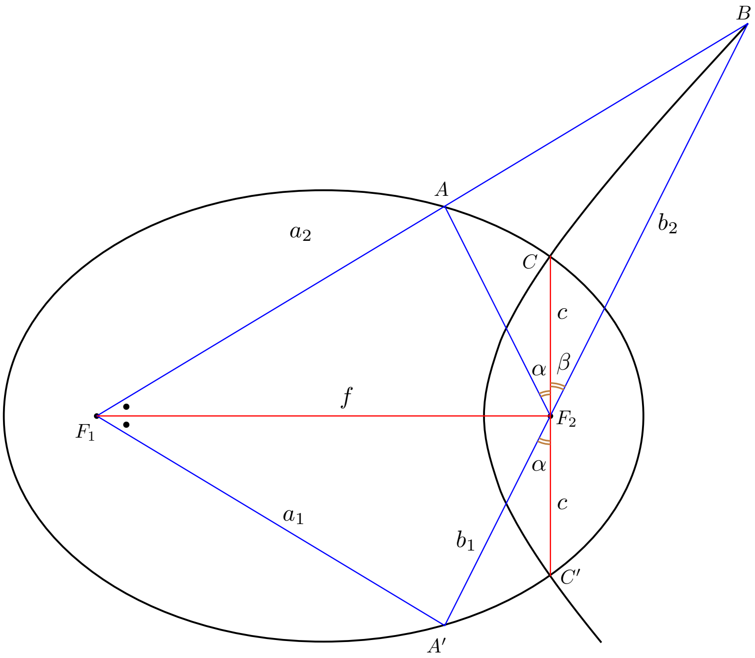

Fig. 8: Auxiliary construction.

of the hyperbola $\mathcal{H}$ be denoted by C'. By the focal property of the ellipse, one has the equality

$$F_1A' + F_2A' = F_1C' + F_2C',$$

that is,

$$a_1 + b_1 = \sqrt{f^2 + c^2} + c. \tag{2}$$

Further, by the focal property of the hyperbola we have

$$F_1B - F_2B = F_1C - F_2C,$$

that is,

$$a_2 - b_2 = \sqrt{f^2 + c^2} - c. \tag{3}$$

Multiplying the left hand sides and the right hand sides of (2) and (3), one gets

$$(a_1 + b_1)(a_2 - b_2) = f^2,$$

and taking into account the lemma, one concludes that F_1F_2 is the bisector of the angle F_1 in the triangle $A'F_1B$. In turn, this means that A' is symmetric to A with respect to the straight line F_1F_2, and by symmetry one has

$$\measuredangle AF_2C = \measuredangle A'F_2C'. \tag{4}$$

O the other hand, the angles $\measuredangle BF_2C$ and $\measuredangle A'F_2C'$ are vertical, and therefore, are equal:

$$\measuredangle BF_2C = \measuredangle A'F_2C'. \tag{5}$$

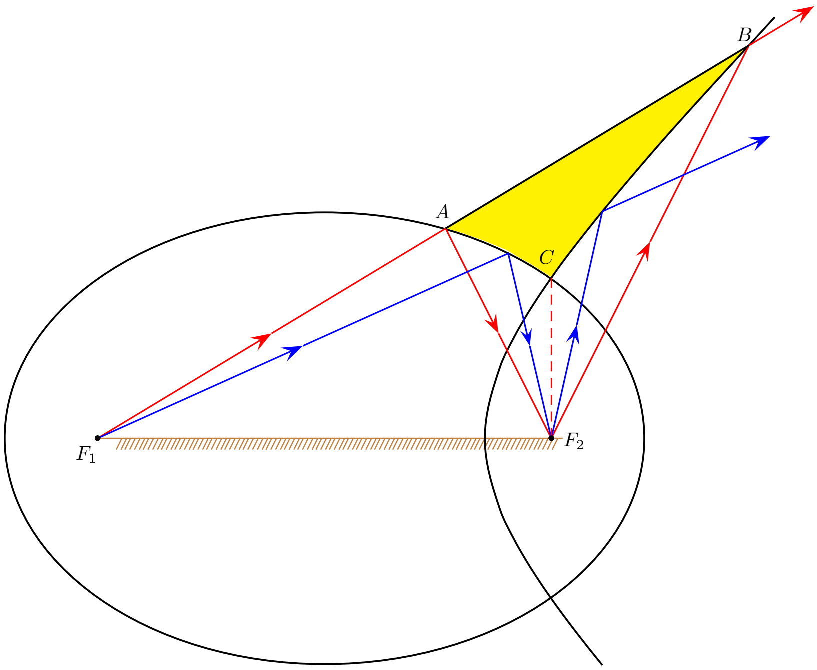

Fig. 9: The curvilinear triangle ABC is invisible from the focus F_1: all the rays of light emanated from F_1 go round the obstacle ACB in such a way, as if it was absent at all.

The equations (4) and (5) imply that

$$\measuredangle AF_2C = \measuredangle BF_2C,$$

therefore $\alpha = \beta$. The theorem is proved.

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