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Stochastic models of tumour development and related mesoscopic equations

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Abstract. We consider different mathematical models inspired by the problems of medicine, in particular, the tumour growth and the related topics. We demonstrate how to starting from an individual-based (microscopic) description, which characterizes cells' behaviour, derive the so-called kinetic (mesoscopic) equations, which describe the approximate system density. Properties of the solutions to the mesoscopic equations (in particular, their long-time behaviour) reflect statistical characteristics of the whole system and demonstrate the corresponding dependence on the system parameters.

1 Introduction

1.1 Mathematical description

Within the microscopic description of cells, the framework of interacting particle systems in continuum and their possibility of deriving rigorously a kinetic description, also called mesoscopic description, in terms of non-local and in general non-linear equations plays a crucial role. Here we start from some (simple) stochastic microscopic (heuristic) description of a cell model and derive from that rigorously the kinetic equations for the density of this system. Such approach can be interpreted similarly to the mean field limit in Physics, where one scales the dynamics in a proper way and obtains from that in the limit a deterministic equation for the density of the system. We assume in general, that each cell is determined uniquely by its position and no other properties will be tracked. Note that it is also possible to introduce marks within such de-

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scriptions and extend the microscopic stochastic dynamics to a situation, where more complicated effects may be covered. For instance one can introduce age-dependence of cells, i.e. each cell has an individual age, which influences the microscopic interactions. Here we always assume that the number of cell-types is small compared to the number of cells within each type. Therefore such stochastic treatment is adequate.

The main difference to other cell biological models is, that we start with a stochastic description, which incorporates individual cell behaviour stochastically. The choice of the individual stochastic behaviour incorporates cell intern effects and can be used to model a wide class of cells. Ignoring the internal structure necessary leads to randomness, but also leads to new methods describing such large interacting systems.

Heuristically the evolution of a system is described via its elementary Markov events like birth, death and jumps of cells. In this framework the evolution is assumed to be Markovian, which is a reasonable approximation of reality. Note, that this approach could also be extended to non-Markovian structures in order to include dependencies on cell intern processes like aging or may lead to some sort of cell-memory. Nevertheless assuming Markovian behaviour already leads to many non-trivial examples and effects, which have to be studied more intensively.

The microscopic description and its analysis can answer questions about asymptotic clustering of the system, invariant states and ergodicity of the system. The precise formulation of clustering will be explained more extended in the next part of the article. Usually it is not possible to measure the microscopic quantities to full extend, so in order to have a practically useful description it is necessary to describe the system also via mesoscopic or even macroscopic quantities. Thus it is reasonable to seek for an effective description with practically measurable quantities. Similar to Thermodynamic limits, one tries to rescale the system and obtain another description, here for the density of the system. As a consequence the new description will not contain all information about the microscopic behaviour, for instance it does not contain information about asymptotic clustering, which will be explained lateron, and individual trajectories of the Markov process. Such Mesoscopic, i.e. kinetic description, describes instead of microscopic quantities the density of the system via a closed system of equations. Typical for such systems of equations is their non-linear structure and the appearance of convolutions of the density with the potentials involved in the interactions of cells. The analysis of such equations is a topic of applied mathematics and is studied intensively since the last 30 years, c.f. [10, 12, 20].

This kinetic description can give information about the long-time behaviour, invariant and stationary states, asymptotic speed of growth, front wave propagation and several other effects. Its analysis should be realized separately for each model. Let us outline the general approach and motivate the scaling used to derive the kinetic description. In general one suppresses the interactions of cells via a factor $\varepsilon > 0$. In the same way the density of the system is increased. Such attempt will suppress all correlations between the cells within the system and therefore a kinetic description will not contain this information. In the last step we will seek for a limiting description of this system and will arrive in a reduced description of the microscopic model. This reduced description will have not Markovian structure but is still a linear stochastic description involving infinitely many correlation functions. To get a closed equation for the density of the system, remember that all correlations between the cells are suppressed. Thus it is not surprisingly that starting from poissonan statistics, the evolution of the system will preserve this statistics. This property is know as the Chaos preservation principle.

In the following we will first outline a more detailed description of both approaches, introduce all necessary quantities and afterwards state the results for several biological important models of tumour growth, cell division, mortality etc. The last section contains all mathematical details, which are necessary for the analysis of such models.

The aim of this section is to motivate and explain this approach to scientists working in biological or medical research fields. The precise mathematical description will be given and proved afterwards separately.

1.2 One-component models

Let us first outline the necessary structures in the simpler case, where we consider only one type of cells. Since the cells are distinguished only by their positions, we will denote their positions by $x_1, \dots, x_n, \dots \in \mathbb{R}^d$ or more simple as a collection of positions $\gamma = \{x_1, \dots, x_n, \dots\} \subset \mathbb{R}^d$. In reality it is clear, that each organism has only a finite but very large number of cells. For such finite microscopic systems the existence of a Markov process is known. Moreover in [2], [3] asymptotic properties and conditions for explosions respectively non-explosion can be stated. Nevertheless it is still not understood how to derive rigorously, i.e. in the sense of convergence of the corresponding densities, the mesoscopic description. In contrast to infinite systems, i.e. $\gamma \subset \mathbb{R}^d$ contains infinitely many points, behave from the analytical point of view quite different. Here for many models it is already known that the density of the rescaled system will converge to the solution of the kinetic description. In this work we will mainly focus on infinite systems having in mind, that in the kinetic description the initial density should in addition be chosen to be integrable, and hence represents a system consisting only of finitely many cells.

Similar to limits from thermodynamics, some effects like asymptotic clustering or pattern formation can be captured simpler in the limit of infinite particles. Simulations suggest and for some dynamical systems it can be shown, that finite systems with a large number of cells behave in their interior like infinite systems. Finite systems can describe the growth of the system, whereas infinite systems capture the properties of the interior behaviour of cell patterns and their properties. Since we deal with a very large number of cells ($\approx 10^{10}$) it is justified to allow the cell number to be even infinity, so we will use a description which includes both finite and infinite systems.

In this case we have to assume, that locally the number of cells is still finite, i.e. for every bounded volume $\Lambda \subset \mathbb{R}^d$ the number of cells within Λ is finite: $|\gamma \cap \Lambda| < \infty$. This assumption implies, that the local density of the system (also other observables) are locally finite and thus can be measured/observed on each finite volume. Altogether our phase space (configuration space) is

$$\Gamma = \{\gamma \subset \mathbb{R}^d : \quad |\gamma \cap \Lambda| < \infty, \quad \forall \Lambda \subset \mathbb{R}^d \text{ bounded}\}.$$

Clearly the space of finite configurations

$$\Gamma_0 = \{\eta \subset \mathbb{R}^d : \quad |\eta| < \infty\}$$

is a subspace of Γ, i.e. $\Gamma_0 \subset \Gamma$. Heuristically, starting from some configuration $\gamma \in \Gamma$ we would like to describe a Markov process $X_t^\gamma \in \Gamma$ starting at γ, which incorporates all microscopic phenomena we would like to describe. For finite configurations such problem is well understood, c.f. [11] and references therein. The (Markov) dynamics is described via elementary events as birth, death and jumps of cells. A cell located at $x \in \mathbb{R}^d$ can die, i.e. the configuration changes as $\gamma \to \gamma\backslash x$, a cell can jump from x to $y \in \mathbb{R}^d$, i.e. $\gamma \to \gamma\backslash x \cup y$ and finally a new cell at location $y \in \mathbb{R}^d$ may appear, i.e. $\gamma \to \gamma \cup y$. All such events have certain intensities, which will depend on the positions x, y and on the configuration of cells γ. The probability of the new location $y \in \mathbb{R}^d$ is usually described via a probability distribution.

Mathematically a Markov process X_t^γ starting from a configuration of cells $\gamma \in \Gamma$ can be described completely in terms of the corresponding Markov generator L, c.f [18], which acts on functions F called observables. Therefore in order to describe the model it is enough to write down the expression for this Markov operator. For our models all terms contained in the operator have a simple interpretation, e.g. the general form of such generator is simply

$$\begin{aligned}(LF)(\gamma) = &\sum_{x\in\gamma} d(x,\gamma\backslash x)(F(\gamma\backslash x) - F(\gamma)) + \int_{\mathbb{R}^d} b(y,\gamma)(F(\gamma \cup y) - F(\gamma))dy \\ &+ \sum_{x\in\gamma}\int_{\mathbb{R}^d} c(x,y,\gamma\backslash x)(F(\gamma\backslash x \cup y) - F(\gamma))dy. \qquad (1)\end{aligned}$$

Here $0 \le d(x,\gamma\backslash x)$ is the intensity of death of a cell $x \in \mathbb{R}^d$ depending on all other cells $\gamma\backslash x$, $0 \le b(y,\gamma)$ is the intensity, that a cell is born at position $y \in \mathbb{R}^d$ and $0 \le c(x,y,\gamma\backslash x)$ is the intensity that a cell jumps from position x to the new position $y \in \mathbb{R}^d$. Let us stress, that since we will deal with infinite systems the study of the operator L is extremely hard and was carried out only for a few models, e.g. [16]. In the framework of cell biology typically new cells can only be born due to proliferation and hence we can specify the intensity $b(x,\gamma)$ to be of the form

$$b(x,\gamma) = \sum_{y\in\gamma} b_0(x,y,\gamma\backslash x).$$

This means that each cell at $y \in \gamma$ may split and therefore create a new cell at location $x \in \mathbb{R}^d$. The intensity of such events is given by $b_0(x,y,\gamma\backslash x)$. Put in other words, if $\gamma \in \Gamma_0$ then heuristically

$$\mathbb{P}(X_{t+h}^\gamma = X_t^\gamma \cup \{x\} | X_t^\gamma) = b(x,\gamma)h + o(h)$$

as $h \to 0$. Similar statements hold for $d(x,\gamma\backslash x)$ and $c(x,y,\gamma\backslash x)$. In the case of $|\gamma| = \infty$ such description can be interpreted only heuristically, since in each interval $[t, t+h]$, $h > 0$ infinitely many microscopic events will take place. Hence the notion of first time of a change of a system state if not meaningful, whereas in finite systems an explicit construction of the Markov process deeply uses this fact.

Within this framework we will study distributions $\mathbb{P}(X_t^\gamma)^{-1} =: \mu_t$ called states of the system, instead of the process itself. From cell-biological point of view it is not necessary and realistic to know all positions of cells, but one can observe and model statistics respectively their distribution μ_t, which is probability measure on Γ. One simple example is the poissonian statistics. There the probability of finding n-cells within the volume $\Lambda \subset \mathbb{R}^d$ is given by

$$\mathbb{P}_n(\Lambda) = \frac{1}{n!}\Big(\int\limits_\Lambda \rho(x)dx\Big)^n \exp\Big(-\int\limits_\Lambda \rho(x)dx\Big),$$

where $0 \le \rho$ is locally integrable and describes the cell-density. Let us denote the Poisson measure on Γ by π_ρ and the collection of all probability measures on Γ by $\mathscr{P}$. The Poisson measure plays the role of a chaotic, i.e. free state, where all cells are not correlated. Starting from a state $\mu \in \mathscr{P}$, the description of the microscopic evolution will consist of describing the evolution of states $t \longmapsto \mu_t \in \mathscr{P}$. Compared to the description via a process X_t^μ the evolution of statistics μ_t is connected via the equality

$$\int\limits_\Gamma F(\gamma)\mu_t(d\gamma) = \int\limits_\Omega F(X_t^\mu)d\mathbb{P}, \quad F : \Gamma \longrightarrow \mathbb{R}$$

where Ω is the probability space and $\mathbb{P}$ the probability measure for the process X_t^μ starting with initial distribution $\mu\mathscr{P}$. The study of the evolution μ_t can be done via studying its moments, which are functions of arbitrary large number of variables. The definition of this functions, if they exist, is given as follows, c.f. [13]

$$\int\limits_\Gamma \sum_{\{x_1,\dots,x_n\}\subset\gamma} f^{(n)}(x_1,\dots,x_n)d\mu(\gamma)$$
$$= \frac{1}{n!}\int\limits_{(\mathbb{R}^d)^n} f^{(n)}(x_1,\dots,x_n)k^{(n)}(x_1,\dots,x_n)dx_1\dots dx_n \quad (2)$$

for symmetric functions $f^{(n)}$, which are measurable and have compact support. The left-hand side is the mean of the observable $f^{(n)}$, i.e. we sum over all possible n-point configurations $\{x_1,\dots,x_n\} \subset \gamma$ and afterwards integrate over all possible configurations γ. We assume that this mean can be represented via a density $k^{(n)}$ and the factor $\frac{1}{n!}$ is a combinatorial factor describing the number of all possible choices to order the positions $x_1,\dots,x_n$. Let us denote the collection of all correlation functions by $(k^{(n)})_{n=0}^\infty = k$, where $0 \le k = k(\eta)$ is a function of finite configurations $\eta \subset \mathbb{R}^d$ ($|\eta| < \infty$).

Example 1.

- In the case $n = 0$ one has

$$1 = \mu(\Gamma) = \int\limits_\Gamma d\mu(\gamma) = k^{(0)}$$

- For $n=1$ take a Borel set $A \subset \mathbb{R}^d$ and
$$f^{(1)}(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}.$$
Then we obtain
$$\int_\Gamma |A \cap \gamma| d\mu(\gamma) = \int_\Gamma \sum_{x\in\gamma} f^{(1)}(x) d\mu(\gamma) = \int_A k^{(1)}(x)dx$$
and the left-hand side is the expected number of particles within the volume A, whereas the right-hand side is a measure in A. Therefore $k^{(1)}$ is the particle density of the system.

- The same procedure with
$$f^{(2)}(x,y) = \begin{cases} 1, & x,y \in A \\ 0, & \textit{otherwise} \end{cases}$$
leads to
$$\frac{1}{2}\int_\Gamma |\gamma \cap A|^2 d\mu(\gamma) - \frac{1}{2}\int_\Gamma |\gamma \cap A| d\mu(\gamma)$$
$$= \int_\Gamma \binom{|\gamma \cap A|}{2} d\mu(\gamma) = \frac{1}{2}\int_A\int_A k^{(2)}(x,y)dxdy$$
and we see that
$$\int_A\int_A k^{(2)}(x,y)dxdy$$
is the Variance of the cell number operator with kernel $0 \leq k^{(2)}(x,y)$. Similarly $k^{(n)}$ describe higher order moments of the system.

- The correlation functions for the Poisson measure π_ρ are
$$k^{(n)}(x_1,\dots,x_n) = \rho(x_1)\cdots\rho(x_n).$$

The evolution of states $t \longmapsto \mu_t$ is determined as the solution to the Fokker-Planck equation for measures
$$\frac{\partial}{\partial t}\int_\Gamma F(\gamma)d\mu_t(\gamma) = \int_\Gamma (LF)(\gamma)d\mu_t(\gamma) \tag{3}$$
with initial distribution $\mu_t|_{t=0} = \mu_0$. Let us assume that for each state μ_t the correlation function of arbitrary order $n \in \mathbb{N}$ exists, then the evolution μ_t can be described by the collection of all such correlation functions $k_t = (k_t^{(n)})_{n=0}^\infty$. Similar to equation (3), this collection will satisfy the Fokker–Planck hierarchy
$$\frac{\partial k_t}{\partial t} = L^\Delta k_t \tag{4}$$

or written in components

$$\frac{\partial k_t^{(n)}}{\partial t}(x_1, \ldots, x_n) = (L^{\Delta} k_t)^{(n)}(x_1, \ldots, x_n), \quad n \in \mathbb{N}_0,$$

where the operator L^{Δ} acts on the whole vector $k_t = (k_t^{(n)})_{n=0}^{\infty}$, i.e. on each correlation function and can be seen as an infinite matrix. Therefore above equation is a vector equation with the matrix operator L^{Δ}. We have therefore transformed the equation for the evolution of states μ_t to an equation for its moments $(k_t^{(n)})_{n=0}^{\infty}$. As a consequence the system of equations for $(k_t^{(n)})_{n=0}^{\infty}$ will be coupled, hence it is not possible to obtain directly a closed equation for $k_t^{(n)}$, where only $k_t^{(n)}$ enters. Attempts to derive from such system a closed equation are known as moment closure techniques. In our approach scaling of the system yields a closed equation for the density of the system. Let us show for a special choice of L how to derive this equation for $k_t^{(1)}$. The general case, will be postponed to the second part of the article.

As a simple example let us look at a free branching process, where each cell has a random exponentially distributed lifetime with parameter $m > 0$ and can proliferate with intensity $\lambda > 0$, i.e the time to create a new cell is also exponentially distributed with parameter $\lambda > 0$. The position of the new cell born from $x \in \gamma$ will be distributed due to the probability distribution $a(x - y)dy$, where $y \in \mathbb{R}^d$ is the position of the new cell. The heuristic Markov generator will have the form

$$\begin{aligned}(LF)(\gamma) &= m \sum_{x \in \gamma} (F(\gamma \backslash x) - F(\gamma)) \\ &\quad + \lambda \sum_{x \in \gamma} \int_{\mathbb{R}^d} a(x - y)(F(\gamma \cup y) - F(\gamma)) dy. \end{aligned} \tag{5}$$

Let us take in (3) the special choice $F(\gamma) = \sum_{x \in \gamma} \varphi(x)$ with a measurable, bounded function φ with compact support. For this choice the left-hand side of (3) will become, c.f. example 2,

$$\frac{\partial}{\partial t} \int_{\Gamma} \sum_{x \in \gamma} \varphi(x) d\mu_t(\gamma) = \frac{\partial}{\partial t} \int_{\mathbb{R}^d} \varphi(x) k_t^{(1)}(x) dx.$$

A short combinatorial computation shows the equality

$$(LF)(\gamma) = -m \sum_{x \in \gamma} \varphi(x) + \lambda \sum_{x \in \gamma} \int_{\mathbb{R}^d} a(x - y) \varphi(y) dy$$

and thus

$$\begin{aligned}\int_{\Gamma} (LF)(\gamma) d\mu_t(\gamma) &= -m \int_{\Gamma} \sum_{x \in \gamma} \varphi(x) d\mu_t(\gamma) + \lambda \int_{\Gamma} \sum_{x \in \gamma} (a * \varphi)(x) d\mu_t(\gamma) \\ &= -m \int_{\mathbb{R}^d} \varphi(x) k_t^{(1)}(x) dx + \lambda \int_{\mathbb{R}^d} (a * \varphi)(x) k^{(1)}(x) dx.\end{aligned}$$

For the second integral we use Fubini and $a(x-y) = a(y-x)$ to get

$$\int_{\mathbb{R}^d} (a * \varphi)(x) k^{(1)}(x) dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x-y)\varphi(y) k^{(1)}(x) dy dx$$
$$= \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} a(y-x) k^{(1)}(x) dx \Big) \varphi(y) dy.$$

Altogether this gives

$$\int_{\mathbb{R}^d} \varphi(x) \frac{\partial k_t^{(1)}}{\partial t}(x) dx = \int_{\mathbb{R}^d} \varphi(x) \left(-m k_t^{(1)}(x) + \lambda (a * k_t^{(1)})(x) \right) dx$$

for each function φ and thus

$$\frac{\partial k_t^{(1)}}{\partial t}(x) = -m k_t^{(1)} + \lambda (a * k_t)(x).$$

Note, that this is a closed equation, in general it not the case, e.g. the equation for $k_t^{(2)}$

$$\frac{\partial k_t^{(2)}}{\partial t}(x,y) = -2m k_t^{(2)}(x,y) + \lambda \int_{\mathbb{R}^d} a(x-z) k_t^{(2)}(x,z) dz$$
$$+ \lambda \int_{\mathbb{R}^d} a(y-z) k_t^{(2)}(z,y) dz + a(x-y) \left(k_t^{(1)}(x) + k_t^{(1)}(y) \right)$$

does depend on the functions of order 1 and 2.

Let us now turn to scaling and outline the general approach. The first step is to scale the intensities of the interaction of the system, usually one dumps the potentials by a factor $\varepsilon > 0$, e.g. for (5) this means $a \to \varepsilon a$. In general let us assume we scaled the intensities in a proper way, i.e. have expressions $d_\varepsilon, b_\varepsilon$ and c_ε within expression (1). The exact scaling will be carried out for each model separately. To get a limit, we have also accelerate birth by putting a factor $\frac{1}{\varepsilon}$ in front of it, so in the case of (5) this will mean that L is not changed, which is a direct consequence of the independence of the stochastic evolution of each cell. Finally the resulting generator has the form

$$(L_\varepsilon F)(\gamma) = \sum_{x \in \gamma} d_\varepsilon(x, \gamma \backslash x)(F(\gamma \backslash x) - F(\gamma)) + \frac{1}{\varepsilon} \int_{\mathbb{R}^d} b_\varepsilon(y, \gamma)(F(\gamma \cup y) - F(\gamma)) dy$$
$$+ \sum_{x \in \gamma} \int_{\mathbb{R}^d} c_\varepsilon(x, y, \gamma \backslash x)(F(\gamma \backslash x \cup y) - F(\gamma)) dy.$$

The second step is to increase the density of the system, i.e. we consider initial conditions $k_{0,\varepsilon}^{(n)}$, which satisfy

$$\varepsilon^n k_{0,\varepsilon}^{(n)} \to r_0^{(n)}, \quad \varepsilon \to 0$$

with a symmetric function $r_0^{(n)}$ and $n \in \mathbb{N}_0$. Clearly, this implies that the initial condition $k_{0,\varepsilon}^{(n)}$ has a singularity at $\varepsilon > 0$, which can be interpreted as a growth of the initial densities in $\varepsilon > 0$. The functions $r_0^{(n)}$ are a subject of choice for concrete models. The important case is

$$r_0^{(n)}(x_1, \ldots, x_n) = \rho_0(x_1) \cdots \rho_0(x_n), \tag{6}$$

which represents the poissonian statistics. Denote by L_ε^Δ the operator corresponding to L_ε and by $k_{t,\varepsilon}$ the collection of correlation functions, defined as the solutions of the equation

$$\frac{\partial k_{t,\varepsilon}}{\partial t} = L_\varepsilon^\Delta k_{t,\varepsilon}, \quad k_{t,\varepsilon}|_{t=0} = k_{0,\varepsilon}. \tag{7}$$

In the final step we will seek for a limiting correlation function, describing the scaled system, i.e. we want

$$\varepsilon^n k_{t,\varepsilon}^{(n)} \to r_t, \quad \varepsilon \to 0 \tag{8}$$

for each $n \in \mathbb{N}_0$. Again the collection $r_t = (r_t^{(n)})_{n=0}^\infty$ will satisfy some system of equations similar to (4), i.e.

$$\frac{\partial r_t}{\partial t} = L_V^\Delta r_t, \quad r_t|_{t=0} = r_0.$$

This limiting description is known as the Vlasov hierarchy containing less information as the original model, but is simpler to analyse. Starting from initial function r_0 as he correlation function of the Poisson measure π_{ρ_0}, c.f. (6), one finds that the solution r_t will be of the form

$$r_t^{(n)}(x_1, \ldots, x_n) = \rho_t(x_1) \cdots \rho_t(x_n)$$

and so r_t is again the correlation function of a Poisson measure with the new density ρ_t. This density is determined by the mesoscopic equation, which we will also call kinetic description,

$$\frac{\partial \rho_t}{\partial t}(x) = v(\rho_t)(x), \quad \rho_t|_{t=0} = \rho_0 \tag{9}$$

and this property is known as the Chaos preservation principle. All previous steps can be computed for many models explicitly, which will be realized later for each model. The function ρ_t is the approximate density of this system, i.e. plays the same role as $k_t^{(1)}$, whereas it is determined in general by a non-linear and non-local equation. For the special case (5) the equation for ρ_t is the same as for $k_t^{(1)}$, which again is the consequence of the independence of each cell. Given a microscopic model through its (formal) Markov generator L, we will say that (9) is the kinetic description of the microscopic model. This description is produced by taking the formal limit within (7). Without further investigation it is not known, whether also the corresponding solutions converge, i.e. (8) holds. We will say that the kinetic description (9) corresponds to the microscopic model if (8) holds. In particular this implies that for given initial condition ρ_0 and ρ_t the solution to (9) one has

$$\lim_{\varepsilon \to 0} \varepsilon k^{(1)}_{t,\varepsilon} = \rho_t.$$

The precise notion of the convergence might depend on the particular model and shall be checked for each model separately. In many cases one knows that ρ_t will be bounded and hence the limit is uniformly in all spatial variables.

1.3 Clusterization and pattern formation

From an intuitive point of view many scientists understand under the terminus of clusterization that with increased probability we will observe cells aggregating in some bounded regions. For mathematical analysis such understanding has to be reformulated in terms of mathematical objects. In applications biologists often observe the density of the system, and find regions with peaks and at the same time regions with rather small density. Such phenomena is also often called clusterization. In this work we will call such phenomena pattern formation. One example for pattern formation would be the density, for simplicity one-dimensional,

$$\rho(x) = \begin{cases} a, & x \in [2k, 2k+1) \\ b, & x \in [2k+1, 2(k+1)) \end{cases},$$

where $0 < a < b$ and $k \in \mathbb{Z}$. Such density is periodic, and if b is compared to a much larger we will observe macroscopically in the regions $[2k+1, 2(k+1))$ an aggregation of cells or molecules. One could think about the density for the description of periodic crystal structures, whereas we do not relate such density to any specific model, since we have not it derived from any microscopic model.

The notion of clusters will be used in this work to relate to higher correlations of an interacting cell system, whereas pattern formation is connected only to the first correlation function, i.e. the density of the system. Having in mind that the sequence of correlation functions $k^{(n)}$ describe the densities of the moments of a state of the system, i.e. a probability measure on Γ, we would like to fix in the next step a reference measure, which shall be regarded as completely uncorrelated. In Physics it is know that for a completely uncorrelated system the correlation functions will have product structure, i.e.

$$k^{(n)}(x_1, \dots, x_n) = \rho(x_1) \cdots \rho(x_n), \quad x_1, \dots, x_n \in \mathbb{R}^d$$

for each $n \in \mathbb{N}$. Here $0 \le \rho$ is the density. Therefore we regard the Poisson measure π_ρ as the reference measure to measure correlations and clusterization of the system. One important special case is the choice $\rho(x) = z$, where $z > 0$ is constant. In such case the system is distributed uniformly in all $\mathbb{R}^d$ and due to the product structure all cells are independent. Such cases were already analysed in physics for the free gas. We will call a systems non-clustering, if its correlation functions satisfy

$$k^{(n)}(x_1, \dots, x_n) \le \rho(x_1) \cdots \rho(x_n) C, \quad x_1, \dots, x_n \in \mathbb{R}^d \tag{10}$$

for all $n \in \mathbb{N}_0$ and some constant $C > 0$. In the case, where

$$k^{(n)}(x_1, \ldots, x_n) \leq \rho(x_1) \cdots \rho(x_n) n!^{\delta}, \quad x_1, \ldots, x_n \in \mathbb{R}^d \tag{11}$$

for all $n \in \mathbb{N}_0$ and some $\delta > 0$, we will say that the system admits clusterization. Note that, this does not mean that the system will be really clustering. In general one should also have a bound from below. We will say the system is clustering, if the following bounds hold for all $n \in \mathbb{N}_0$, $x_1, \ldots, x_n \in \mathbb{R}^d$

$$\rho_0(x_1) \cdots \rho_0(x_n)(n!)^{\delta_0} C_0 \leq k^{(n)}(x_1, \ldots, x_n) \leq \rho_1(x_1) \cdots \rho_1(x_n)(n!)^{\delta_1} C_1,$$

where ρ_0, ρ_1 are non-negative locally integrable functions, $C_0, C_1, \delta_0, \delta_1 > 0$ are constants. The evolution of a system will be clustering if for each fixed time $t > 0$, the correlation functions $k_t^{(n)}$ admits above estimations.

Let us now turn to the interpretation of this conditions. In the case of (10), we observe that the moments of the system are bounded from above by the moments of the Poisson measure. For instance the probability density of finding n cells at positions $x_1, \ldots, x_n$ is given by $k^{(n)}(x_1, \ldots, x_n)$. In the case of (11) this density is fast growing with respect to n and hence it is more likely to find configurations which consist of a higher number of cells. Therefore we see that in contrast to pattern formation, here we incorporate also the microscopic description of the system via its configurations. In the next section we will see, that a free branching process will always be clustering. In order to prevent clusterization it is therefore necessary to introduce microscopic interactions, which will regulate the system. Two examples are given by either increasing the death of cells, in such a way that in dense regions cells will have an increased intensity to kill each other, or dumping down the intensity for the branching of cells, which means that in dense regions cells will have only a small chance to proliferate.

2 Results

2.1 One-component models

Free branching process

The first model we start with is a toy model in the sense that mathematically all corresponding equations can be solved explicitly. This model consists of the two elementary events birth and death of a cell. First of all each cell have an exponentially distributed lifetime with parameter $m > 0$, so the time each cell will survive is given by an exponentially distributed random variable and the mean lifetime is $\frac{1}{m}$. After the death of a cell located at position $x \in \mathbb{R}^d$ the configuration of all cells changes $\gamma \to \gamma \backslash x$. Written in terms of the heuristic Markov generator this part has the form

$$(L_d F)(\gamma) = \sum_{x \in \gamma} m(F(\gamma \backslash x) - F(\gamma)).$$

Moreover, each cell located at position $x \in \gamma$ can divide into two new cells located at the positions y_1, y_2. Thus the configuration changes in the following way $\gamma \to \gamma \backslash x \cup y_1 \cup y_2$. The probability of finding cells in the volume element $\mathrm{d}y_1\mathrm{d}y_2$, is given by

$$a(x - y_1, x - y_2)\mathrm{d}y_1\mathrm{d}y_2,$$

and the intensity of the event of cell-division is given by $\lambda > 0$. We assume that $0 \leq a$ is a probability density, hence is normalized to 1 and assume that this kernel is symmetric in both arguments, so

$$a(x, -y) = a(x, y), \qquad a(-x, y) = a(x, y), \quad x, y \in \mathbb{R}^d.$$

In terms of the heuristic Markov generator this leads to

$$(L_b F)(\gamma) = \lambda \sum_{x\in\gamma} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x - y_1, x - y_2)(F(\gamma \backslash x \cup y_1 \cup y_2) - F(\gamma))\mathrm{d}y_1\mathrm{d}y_2.$$

Incorporating both effects independently of each other in one process, the overall Markov generator will have the form $L = L_d + L_b$. Clearly this description shows, that each cell is independent of all other cells. Thus this description really reflects the effects of free proliferation of cells within some region. If the kernel a is a product of two probability distributions, i.e. $a(x, y) = b(x)c(y)$, then the positions y_1 and y_2 will be distributed independent of each other. In some applications the positions y_1, y_2 are not independent of each other, i.e. choosing position y_1 influences the position of y_2. In the special case, where the position y_2 would be determined completely by the position y_1 one would take e.g. $a(x, y) = b(x)\delta(x + y)$ with a non-negative integrable function b, which is normalized to 1. Therefore the position y_2 is given by $y_2 = x+(x-y_1) = 2x-y_1$, meaning that cells prefer to proliferate in opposite directions, such that the distance $|y_1 - y_2|$ is maximal.

Here we will investigate the general case and analyse some properties of the system. The first observation shows, that if the birth kernel a is such that new cells may appear arbitrary close to the mother cell located at position x, then the dynamics will admit asymptotic clustering.

Theorem 2.1. *Assume that a is continuous such that $a(0) > 0$, then starting from poissonian statistics, i.e. correlation functions $k(\eta) = C^{|\eta|}$, the evolution of correlation functions k_t will satisfy for each η such that all points are sufficiently close to each other*

$$k_t^{(n)}(x_1, \dots, x_n) \geq \beta^n e^{-(m-\lambda)nt} n!$$

for some constant $\beta > 0$ depending on λ, a and C. Moreover there exists $C(t) > 0$ non-decreasing such that

$$k_t^{(n)}(x_1, \dots, x_n) \leq C(t)^n n!$$

for all $n \in \mathbb{N}_0$ and $x_1, \dots, x_n \in \mathbb{R}^d$.

Above estimate implies due to the presence of the factor $n!$, that independent of β, m and λ with high probability many cells can be observed in a small region, which reflects the effect of clustering. The second estimate shows, that factorial growth of correlation functions k_t is the worst case we can observe. This estimate remains true without any conditions on the birth kernel

a. The next Theorem formulates the result concerning the kinetic description of this model.

Theorem 2.2. *For each initial distribution of cells $\rho_0(x) \geq 0$, which is essentially bounded, there exist a unique solution $\rho_t(x) \geq 0$ to the kinetic equation*

$$\frac{\partial \rho_t}{\partial t}(x) = -(m+\lambda)\rho_t(x) + \lambda \int_{\mathbb{R}^d} b(x-y)\rho_t(y)\mathrm{d}y \tag{12}$$

with initial condition $\rho_t|_{t=0} = \rho_0$. Such solution is also essentially bounded and corresponds to the rescaled system, i.e. the Vlasov hierarchy. The function $0 \leq b$ is given by

$$b(x) = \int_{\mathbb{R}^d} a(x,y)\mathrm{d}y + \int_{\mathbb{R}^d} a(y,x)\mathrm{d}y.$$

The absence of non-linearities is due to the absence of interactions of cells. For $m > \lambda$ all cells will die, whereas for $m < \lambda$ the number of cells will grow exponentially. In the critical case $m = \lambda$ the total number of cells is conserved and the equation describes a random walk in continuous time. The general solution to (12) is given by

$$\rho_t(x) = \rho_0(x)e^{-(m+\lambda)t} + e^{-(m+\lambda)t}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}(b^{*n} * \rho_0)(x).$$

The same Mesoscopic equation and the same results about asymptotic clustering of the system can be achieved, if we simplify the birth by setting

$$a(x,y) = \delta(x)b(y),$$

which means that each cell will create a new cell located at position $y \in \mathbb{R}^d$ without disappearing from the system. Mathematically such situation is due to less computational work simpler to analyse. Results concerning invariant states, existence of a Markov process etc. can be found in [15]. In the following we will always restrict ourselves to this case, called Contact model. Its heuristic Markov generator has the form

$$\begin{aligned}(L_{CM}F)(\gamma) = m &\sum_{x\in\gamma}(F(\gamma\backslash x) - F(\gamma)) \\ &+ \lambda\sum_{x\in\gamma}\int_{\mathbb{R}^d} a(x-y)(F(\gamma\cup y) - F(\gamma))\mathrm{d}y.\end{aligned} \tag{13}$$

Spatial logistic model

As already discussed in the Contact model all cells are independent of each other. For a wide class of biologically relevant models such behaviour is not adequate, so one has to introduce additional microscopic mechanisms, which regularize the overall dynamics in such a way that all correlation functions become sub-poissonian. This can be achieved if one includes either additional density dependent mortality or one introduces density dependent birth in such a way, that in regions of high density cells will have a small probability to proliferate. Here we will state some results about the model with additional density dependent mortality.

Let us start with with the usual Contact model, so L_{CM} given in (13) and introduce additional death. Each cell $x \in \gamma$ may cause death of another cell $y \in \gamma \backslash x$ with rate $\lambda^- a^-(x-y)$. The overall rate of death caused by the cell x is simply $\lambda^- \sum\limits_{y\in\gamma^-\backslash x} a^-(x-y)$ and describes some sort of competition of cells for resources within the body. Therefore the complete heuristic Markov generator will have the form

$$(LF)(\gamma) = (L_{CM}F)(\gamma) + \lambda^- \sum_{x\in\gamma}\sum_{y\in\gamma\backslash x} a^-(x-y)(F(\gamma\backslash x) - F(\gamma)). \qquad (14)$$

This model was analysed in several articles, c.f. [5, 7]. It is known, c.f. [21] that if $m > 0$ is arbitrary small and there is $\theta > 0$ such that $a^- - \theta a$ is a stable potential, then there exists an evolution of states, such that its correlation functions satisfy $k_t^{(n)} \leq C^n$ for some constant $C > 0$. Moreover, it can be shown, that in the regime of high mortality the only invariant state is the one representing the empty configuration. Namely if

$$\lambda a \leq \lambda^- a^-$$

then the unique invariant distribution is $\mu_{inv} = \delta_{\{\emptyset\}}$, i.e. $k_{inv}^{(n)} = 0$ for $n \geq 1$ and $k_{inv}^{(0)} = 1$. Here we will only summarize the result for the kinetic description, [6]

Theorem 2.3. *Assume $a, a^- \geq 0$ are symmetric, integrable and normalized to 1. Then for each initial measurable density $\rho_0(x) \leq C$ for a.a. $x \in \mathbb{R}^d$ there exists a unique solution ρ_t to the kinetic equation*

$$\frac{\partial \rho_t}{\partial t}(x) = -m\rho_t(x) - \lambda^-\rho_t(x)(a^- * \rho_t)(x) + \lambda(a * \rho_t)(x) \qquad (15)$$

with $\rho_t|_{t=0} = \rho_0$. Moreover the function $k_t(\eta) = e_\lambda(\rho_t;\eta)$ is a solution to the Vlasov hierarchy

$$\frac{\partial r_t}{\partial t}(\eta) = L_V^{\Delta} r_t(\eta), \quad r_t|_{t=0} = r_0$$

with $r_0(\eta) = e_\lambda(\rho_0;\eta)$. This solution ρ_t will again be bounded by the same constant C, i.e. $\rho_t(x) \leq C$ for a.a. $x \in \mathbb{R}^d$. Moreover equation (15) is the kinetic description, and if $\lambda^- a^- - \lambda a$ is stable, then also (8) holds.

Clearly there are two stationary solutions to (15) given by 0 and $\dfrac{\lambda - m}{\lambda^-}$. Such solutions are biologically relevant if they are positive, so $m \leq \lambda$. Let us now assume, that a^- is strongly localized. Then we can approximate the convolution by a multiplication, which leads to $a^- * \rho_t \approx \rho_t$. In this case the kinetic equation simplifies to

$$\frac{\partial \rho_t}{\partial t}(x) = -m\rho_t(x) - \lambda^-\rho_t(x)^2 + \lambda(a * \rho_t)(x).$$

This equation was analysed in several articles in the one-dimensional case. A function $\rho_t(x)$ is called a traveling wave solution with monotone profile and speed c, if $\rho_t(x) = \psi(x - ct)$ for some monotone function $\phi \in C^1(\mathbb{R})$. E.g. in [20] it was shown that if a is exponentially integrable, i.e. there exist $\alpha > 0$ such that

$$\int_{\mathbb{R}} e^{-\alpha y} a(y) \mathrm{d}y < \infty$$

then there exists $c_* > 0$ such that for each $c \geq c_*$ there exist a traveling wave solution with monotone profile and speed c. A For each $c < c_*$ there exist no periodic traveling wave solution of speed c. The constant c_* is called spreading speed. For the time-inhomogeneous case $\lambda = \lambda(t)$ in [12] a similar result was shown and a formula for c_* has been derived. In contrast if a do not satisfy the exponential integrability condition, then the speed of propagation will be not constant, c.f. [10]. Therefore modelling complex cell systems one has also to distinguish between different classes of kernels a^-, a. For example taking for a a gaussian distribution, one gets a constant spreading speed, whereas taking a as the Cauchy distribution one gets an accelerated spreading speed.

Branching with fecundity

Instead of density dependent mortality here we will summarize the case of density dependent birth. So each cell have again an exponentially distributed lifetime with parameter $m > 0$ and each cell at position $x \in \gamma$ can create a new cell with intensity $e^{-E(x,\gamma\backslash x)}$. The relative energy $E(x, \gamma\backslash x)$ is given by

$$E(x, \gamma\backslash x) = \sum_{y \in \gamma\backslash x} \varphi(x - y) \geq 0.$$

The potential $\varphi \geq 0$ is assumed to be symmetric and integrable. In dense regions around a cell x the energy will be large and thus the exponential $e^{-E(x,\gamma\backslash x)}$ will dump the intensity of creating a new cell at some position. Such kind of self-regulation can be interpreted as a lack of energy, material or resources for the cell at position x. If now x creates a new cell, then again the probability of finding the new cell within the region $\mathrm{d}y$ is given by $a(x - y)\mathrm{d}y$. The generator is given for functions $F : \Gamma \longrightarrow \mathbb{R}$ by

$$\begin{aligned}(LF)(\gamma) &= m \sum_{x \in \gamma} (F(\gamma\backslash x) - F(\gamma)) \\ &\quad + \lambda \sum_{x \in \gamma} e^{-E(x,\gamma\backslash x)} \int_{\mathbb{R}^d} a(x - y)(F(\gamma \cup y) - F(\gamma))\mathrm{d}y. \qquad (16)\end{aligned}$$

Such model was discussed in [8] and it was shown, that under some conditions on the potentials a and φ such self-regulation will prevent asymptotic clustering of the system. More precisely, if there is a constant $\theta > 0$ such that for a.a. $x \in \mathbb{R}^d$ the conditions

$$a(x) \leq \theta \varphi(x) e^{-\varphi(x)}$$

$$\lambda \left(1 + \frac{\theta}{eC}\right) < \frac{m}{2} \exp\left(-C \int_{\mathbb{R}^d} (1 - e^{-\varphi(x)})\mathrm{d}x\right)$$

hold, then there is $0 < C' < C$ such that for $k_0(\eta) \le C'^{|\eta|}$ there exist a unique classical solution k_t to (4) such that $k_t(\eta) \le C^{|\eta|}$. Here the dispersion kernel a should be dominated by the interaction kernel suppressing the intensity of birth and in the second condition one assumes that the constant mortality is high enough. Pattern formations might still appear, such effects are of mesoscopic nature and thus should by studied within the kinetic description. So let us state the general result for the mesoscopic limit.

Theorem 2.4. *Assume that*

$$a(x) \le \theta\varphi(x)e^{-\varphi(x)}$$
$$2e^{C\langle\varphi\rangle}\lambda\left(1 + \frac{\theta}{eC}\right) < m$$
$$\lambda\left(1 + \frac{\theta\langle\varphi\rangle}{e}\right) < m,$$

where $\langle\varphi\rangle = \int_{\mathbb{R}^d} \varphi(x)\mathrm{d}x$ *denotes the mean of the potential* φ*. Then there is* $\alpha_0 \in (0,1)$ *such that for all* $\alpha \in (\alpha_0, 1)$ *and each initial condition* $0 \le \rho_0 \le \alpha C$ *there exists a unique solution* $0 \le \rho_t \le \alpha C$ *to the kinetic equation*

$$\frac{\partial \rho_t}{\partial t}(x) = -m\rho_t(x) + \lambda\left(a * \rho_t e^{-\varphi*\rho_t}\right)(x) \tag{17}$$

and the function $r_t = e_\lambda(\rho_t)$ *solves the Vlasov-hierarchy.*

The property $\rho_t \le \alpha C$ means, that the density of the system will be bounded and so no explosions of the cell population may appear. The main difference to the Contact model is the presence of the additional term $e^{-\varphi*\rho_t}$ which suppresses the growth of the density. The first condition states that the interaction should dominate the proliferation The other two conditions require high mortality and are sufficient to prevent the growth of the density of the system. Without these two conditions we expect that the density will grow exponentially, but still will not admit clusterization.

Contact model with motion

Last we would like to draw the attention to another self-regulation mechanism. The usual Contact model described by the heuristic Markov generator L_{CM}, as mentioned before, consists of asymptotic clusters. To avoid this effect, let us assume that each cell have the additional possibility to move within the system. Similar to previous model let us assume that there are two main contributions to the intensity of the motion. On the one hand-side a cell at position $x \in \gamma$ will try to move outside a dense area of cells and on the other hand-side the destination point will be chosen in such way, that it is less dense. All in one cells will try to jump from dense areas to less dense areas. Such heuristic description can be summarized in the following Markov generator

$$(LF)(\gamma) = (L_{CM}F)(\gamma) + \sum_{x\in\gamma} e^{E_\phi(x,\gamma\backslash x)} \int_{\mathbb{R}^d} e^{-E_\psi(y,\gamma)} c(x-y)(F(\gamma\backslash x \cup y) - F(\gamma))\mathrm{d}y. \tag{18}$$

As before, the energies have the form $E_\phi(x,\gamma\backslash x) = \sum\limits_{y\in\gamma\backslash x}\phi(x-y)$ and $E_\psi(y,\gamma) = \sum\limits_{z\in\gamma}\psi(y-z)$, where the potentials $\phi,\psi \geq 0$ are symmetric and integrable. The probability of finding the new cell within $\mathrm{d}y$ is approximately

$$\frac{1}{N}e^{-E_\psi(y,\gamma)}c(x-y)\mathrm{d}y$$

with a normalization constant $N = N(\gamma)$ and a probability distribution $0 \leq c(x) = c(-x)$. To this time such model was never analysed in this generality and therefore it is not clear how the microscopic behaviour will look like. Nevertheless, simulations suggest that such mechanism can lead to less asymptotic clustering of the evolution, but due to the motion of the system, started from a compactly supported density, will spread out faster then in the Contact model. We also expect that the local density ρ_t within the kinetic description will be growing at most sub-exponential. Within this work we derive the kinetic description for this model. Questions concerned about front wave propagation and bounds on the density should be studied in detail afterwards.

Theorem 2.5. *The kinetic description corresponding to the microscopic description of the Contact model in the presence of density dependent jumps is given by a density $\rho_t \geq 0$, which solves the Mesoscopic equation*

$$\begin{aligned}\frac{\partial\rho_t}{\partial t}(x) &= -m\rho_t(x) + \lambda(a*\rho_t)(x)\\ &\quad + \left(c*\left(\rho_t e^{\phi*\rho_t}\right)\right)(x)e^{-(\psi*\rho_t)(x)} - e^{(\phi*\rho_t)(x)}(c*e^{-\psi*\rho_t})(x)\rho_t(x).\end{aligned}$$

Already here, we can observe how complicated the mesoscopic description might become. Of course one could simplify the situation by only investigating the case, where only one of the potentials ϕ,ψ is non-vanishing. So let us assume $\psi = 0$. Then the equation becomes

$$\frac{\partial\rho_t}{\partial t}(x) = -m\rho_t(x) + \lambda(a^+*\rho_t)(x) + \left(c*\left(\rho_t e^{\phi*\rho_t}\right)\right)(x) - \langle c\rangle e^{(\phi*\rho_t)(x)}\rho_t(x).$$

The first two terms describe the free proliferation, whereas the last two terms describe the impact of motion on ρ_t. The total number of particles is not affected by this two terms, i.e. $\frac{\partial}{\partial t}\langle\rho_t\rangle = (\lambda - m)\langle\rho_t\rangle$. The local number of cells, $\int\limits_\Lambda \rho_t(x)\mathrm{d}x$, within some volume $\Lambda \subset \mathbb{R}^d$ might have a drastically different behaviour.

2.2 Two-type models

In contrast to previous models here we will present some results about multi-type models. In reality cells have different tasks and hence should be described by different microscopic interactions. In contrast to previous modelling here we will consider two type of configurations $\gamma^+ = \{x_1,\dots,x_n,\dots\}$ and $\gamma^- = \{y_1,\dots,y_n,\dots\}$. Both should be locally finite and distinct, so $\gamma^+\cap\gamma^- = \emptyset$. The collection of all such configurations will be denoted by Γ^2. Not only the elementary events birth, death and jumping of cells can be treated, we now have the possibility of switching cell-type, i.e. a +-cell becomes a −-cell and vice versa. More important, the densities for all events might also depend on the cells of other type, so that e.g. +-cells are being affected by −-cells etc. Within this framework the kinetic description will be a coupled system of two equations, which describe the rescaled density ρ^+ for +-cells and the rescaled density ρ^- for −-cells.

Let us now outline how to extend previous considerations to this case. A state of the system is a probability distribution, i.e. measure $\mu \in \mathscr{P}(\Gamma^2)$, on the two-component phase space Γ. For the given μ, the corresponding correlation functions $k^{(n,m)}(x_1, \ldots, x_n; y_1, \ldots, y_m)$, if they exist, are defined via the equation

$$\int_{\Gamma^2} \sum_{\{x_1,\ldots,x_n\}\subset\gamma^+} \sum_{\{y_1,\ldots,y_m\}\subset\gamma^-} f^{(n,m)}(x_1,\ldots,x_n;y_1,\ldots y_m)d\mu(\gamma^+,\gamma^-)$$
$$= \frac{1}{n!m!} \int_{(\mathbb{R}^d)^n} \int_{(\mathbb{R}^d)^m} f^{(n,m)}(x_1,\ldots,x_n;y_1,\ldots,y_m)$$
$$\times k^{(n,m)}(x_1,\ldots,x_n;y_1,\ldots,y_m)d^nxd^my$$

for all symmetric functions $f^{(n,m)}$ which are integrable with compact support. Again $k^{(n,m)}$ describe the moments of the state μ of the system and in the special case $n = 1 = m$ the function $k^{(1,1)}$ is the density of the system, whereas $k^{(n,0)}$ and $k^{(0,m)}$ correspond to the boundary distributions.

As before the correlation functional k_t will satisfy the equation

$$\frac{\partial k_t}{\partial t}(\eta^+,\eta^-) = (L^\Delta k_t)(\eta^+,\eta^-), \tag{19}$$

which has to be studied for a rigorous mathematical analysis.

Similar to the one-component case, the kinetic scaling starts with dumping the potentials by multipliying them by a factor $\varepsilon > 0$. Therefore we get a scaled version of the equation (19), i.e. L^Δ_ε instead of L^Δ. Let us assume for the initial conditions $k^{(n,m)}_{0,\varepsilon}$

$$\varepsilon^{n+m}k^{(n,m)}_{0,\varepsilon} \to r^{(n,m)}_0, \quad \varepsilon \to 0$$

with a symmetric function $r^{(n,m)}_0$ and $n, m \in \mathbb{N}_0$. The important case is to take

$$r^{(n,m)}_0(x_1,\ldots,x_n;y_1,\ldots,y_m) = \rho^+_0(x_1)\cdots\rho^+_0(x_n)\rho^-_0(y_1)\cdots\rho^-_0(y_m). \tag{20}$$

Denote by $k^{(n,m)}_{t,\varepsilon}$ the solutions to equation (19) with L^Δ_ε instead of L^Δ and assume that this solutions preserve the order of singularity, namely

$$\varepsilon^{n+m}k^{(n,m)}_{t,\varepsilon} \to r_t, \quad \varepsilon \to 0 \tag{21}$$

for each $n, m \in \mathbb{N}_0$. This is equivalent to investigate the Cauchy problem for the operators $L^\Delta_{\varepsilon,ren} = R_\varepsilon L^\Delta_\varepsilon R_{\varepsilon^{-1}}$, where

$$(R_\varepsilon k)(\eta^+,\eta^-) = \varepsilon^{|\eta^+|+|\eta^-|}k(\eta^+,\eta^-)$$

and seek for a limit $L^{\Delta}_{\varepsilon,ren} \to L^{\Delta}_{V}$. Using the initial condition r_0 as given in (20), the solution to

$$\frac{\partial r_t}{\partial t} = L^{\Delta}_{V} r_t, \quad r_t|_{t=0} = r_0$$

will again have the form

$$r_t^{(n,m)}(x_1,\ldots,x_n;y_1,\ldots,y_m) = \rho_t^+(x_1)\cdots\rho_t^+(x_n)\rho_t^-(y_1)\cdots\rho_t^-(y_m)$$

and ρ_t^+,ρ_t^- is determined by the kinetic equations

$$\begin{cases} \dfrac{\partial \rho_t^+}{\partial t} = v^+(\rho_t^+,\rho_t^-), \\ \dfrac{\partial \rho_t^-}{\partial t} = v^-(\rho_t^+,\rho_t^-). \end{cases} \tag{22}$$

Then similarly to the one-component case the solutions ρ_t^+ and ρ_t^- to (22) will be called kinetic description of the microscopic model. If in addition (21) holds, then we will say that the kinetic description corresponds to the microscopic model. In such case one has

$$\lim_{\varepsilon\to 0} k_{t,\varepsilon}^{(1,1)}(x,y) = \rho_t^+(x)\rho_t^-(y),$$

where ρ_t^+,ρ_t^- are the solutions to (22) with initial condition ρ_0^+ and ρ_0^-. Let us explain the details and state the results for several important models in the last part of this section. Since these models were not investigated mathematically, we will give only some simple preliminary results and state the kinetic description. Its analysis and properties of the description should be analyzed for each model separately.

Necrosis model

Looking at a cell system, with free branching and constant mortality $m > 0$, i.e the Contact model, one possible extension to more realistic situations is to modify the death of cells. After the death of a cell, it triggers several biological mechanisms which effect surrounding cells. If the number of deaths will exceed some critical value, then the surrounding cells will have an increased intensity of death. Such effects will cause cascades of dying cells infecting neighbouring cells. To model this effect we will introduce to types of cells. The +-cells will be the usual cells with constant mortality and free proliferation, i.e. the generator is similar to the generator L_{CM} from the Contact model. The $-$-cells will represent the dead cells, which exceeded the critical value and therefore will cause death of +-cells. These dead cells will disappear due to some exponentially distributed time with parameter $m_1 > 0$. The affect of $-$-cells on +-cells will be described similar to the spatial logistic model, c.f. (14). To summarize this explanation we will write down the form of the heuristic Markov generator, i.e.

$$(LF)(\gamma^+,\gamma^-) = (AF)(\gamma^+,\gamma^-) + (BF)(\gamma^+,\gamma^-) + (VF)(\gamma^+,\gamma^-). \tag{23}$$

The first operator A is similar to the Contact model for the normal cells and has the form

$$(AF)(\gamma^+,\gamma^-) = m_0 \sum_{x\in\gamma^+} (F(\gamma^+\backslash x, \gamma^- \cup x) - F(\gamma^+,\gamma^-)) \\ + \lambda^+ \sum_{x\in\gamma^+} \int_{\mathbb{R}^d} a^+(x-y)(F(\gamma^+ \cup y, \gamma^-) - F(\gamma^+,\gamma^-))\mathrm{d}y.$$

The operator B describes the evolution of $-$ cells, which can only disappear from our system, so it is simply

$$(BF)(\gamma^+,\gamma^-) = m_1 \sum_{x\in\gamma^-} (F(\gamma^+, \gamma^-\backslash x) - F(\gamma^+,\gamma^-)).$$

The last part describes the interaction of both types and is assumed to be of the form

$$(VF)(\gamma^+,\gamma^-) = \lambda^- \sum_{x\in\gamma^+} \Big(\sum_{y\in\gamma^-} \varphi(x-y)\Big)(F(\gamma^+\backslash x, \gamma^- \cup x) - F(\gamma^+,\gamma^-)).$$

The potentials $a^+, \varphi \geq 0$ are assumed to be symmetric, integrable, normalized to 1, and the constants $m_0, m_1, \lambda^+, \lambda^-$ are strictly positive. Ignoring the effects caused by changing the types $+$ to $-$ and vice versa, the overall evolution should be similar to the dynamics of the spatial logistic model with constant mortality $m_0 + m_1$, dispersion $\lambda^+ a^+$ and competition kernel $a^- = \varphi$. Effects caused by changing the type may cause waves of dying cells and by this regulate the local density, which will prevent the explosion of the local number of cells. A rigorous mathematical analysis and simulations are the first steps for a better understanding of this system.

Finally let us give the kinetic description of this model.

Theorem 2.6. *Let $\rho_0 \geq 0$ be essentially bounded and ρ_t a non-negative solution to the system of mesoscopic equations*

$$\frac{\partial \rho_t^+}{\partial t}(x) = -(m_0 + \lambda^-(\varphi * \rho_t^-)(x))\rho_t^+(x) + \lambda^+(a^+ * \rho_t^+)(x) \tag{24}$$

$$\frac{\partial \rho_t^-}{\partial t}(x) = -m_1\rho_t^-(x) + \lambda^- \rho_t^+(x)(\varphi * \rho_t^-)(x) + m_0\rho_t^+(x). \tag{25}$$

Then $r_t(\eta^+,\eta^-) = e_\lambda(\rho_t^+;\eta^+)e_\lambda(\rho_t^-;\eta^-)$ is the solution with initial condition $r_0(\eta^+,\eta^-) = e_\lambda(\rho_0^+;\eta^+)e_\lambda(\rho_0^-;\eta^-)$ corresponding to the scaled Vlasov hierarchy.

Go-and-grow models

Here we will assume that tumour cells have two possible states. On the one side the cells can be in a proliferating state, which we call $-$-state. This state is responsible for the growth of the tumour. In the second state, called $+$-state, a cell will be moving and so contribute to additional spreading of the tumour, where the length of the distance should be large compared with the spreading size of the proliferation. We have the freedom to take several different types of interactions and intensities for proliferation, movement, and changing the type of state. Let us first summarize briefly all common effects and afterwards give an extended description for each choice of intensities.

In principal all proliferating cells have their own development and will spread within the system due to either the Contact model or the Contact model with fecundity. Moreover, they will have the possibility to change their type to moving cells by random. Such switching can be either spontaneously or triggered by surrounding cells in dense areas. This moving cell will start to randomly hop inside the tumour, essentially with high probability this jumps will be far compared to the distance of proliferation. After a certain time this moving cell will reach a substantially less dense region and will start to proliferate again. Such microscopic dynamics may cause the creation of new tumour patters where the distance to the old pattern is large compared to proliferation length.

A medical difficulty is to observe such moving cells, therefore a treatment of a tumour is essentially restricted to the treatment of proliferating cells. One goal is to determine the front wave propagation, derive reasonable extremal statistics, and consequently predict the size and possible locations of a significantly wider amount of tumour cells. We expect that this kind of insights will lead to a better understanding of the microscopic structure of the tumours and hence to new therapeutical treatments of tumour and cancer.

In the following we will give 4 examples with concrete types of intensities and derive their kinetic description. The moving cells will always evolve as a free jumping process, meaning each moving cell will independent of all other cells randomly hop within the system. In addition each moving cell will have a density independent death of parameter $d \geq 0$. The heuristic Markov generator for the moving cells is simply

$$(L_{hop}F)(\gamma^+,\gamma^-) = d\sum_{x\in\gamma^+}(F(\gamma^+\backslash x,\gamma^-) - F(\gamma^+,\gamma^-)$$
$$+\sum_{x\in\gamma^+}\int_{\mathbb{R}^d} c(x-y)(F(\gamma^+\backslash x\cup y,\gamma^-) - F(\gamma^+,\gamma^-))\mathrm{d}y.$$

Each model will either have different rates at which the cells will change their state or the type of proliferation is varying.

First model

Let us assume that the proliferating cells will be described by the Contact model, c.f. (13) and that within dense areas the proliferating cells have an increased intensity to change their state to moving cells. For simplicity we assume first, that cells within the moving state will stay an exponential distributed time with parameter $q > 0$ in this state and afterwards start to proliferate again.

Changing the state is the described by the heuristic Markov generator

$$(VF)(\gamma^+,\gamma^-) = q\sum_{x\in\gamma^+}(F(\gamma^+\backslash x,\gamma^-\cup y)-F(\gamma^+,\gamma^-)) \\ +\sum_{x\in\gamma^-}\Big(p+\sum_{y\in\gamma^-\backslash x}\varphi(x-y)\Big)(F(\gamma^+\cup x,\gamma^-\backslash x)-F(\gamma^+,\gamma^-)).$$

Here $p,q>0$ are the intensities to change the type independent of all other cells and $0\le\varphi\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ is a symmetric potential. The overall dynamics is a superposition of all three type of dynamics and has the form $L=L_{CM}+L_{hop}+V$.

The kinetic description for this model is given by:

$$\frac{\partial\rho_t^+}{\partial t}(x) = -(\langle c\rangle+d+q)\rho_t^+(x)+(c*\rho_t^+)(x)+p\rho_t^-(x)+\rho_t^-(x)(\varphi*\rho_t^-)(x)$$
$$\frac{\partial\rho_t^-}{\partial t}(x) = -(m+p)\rho_t^-(x)-\rho_t^-(x)(\varphi*\rho_t^-)(x)+\lambda(a*\rho_t^-)(x)+q\rho_t^+(x).$$

Despite the presence of motion, this example is similar to previous one. Here the spreading speed should be due to the motion increased, whereas in the previous model the spreading speed is constant for exponentially integrable dispersion kernels. The local cell number may be dumped by the motion, but the overall particle number will still grow asymptotically as $e^{(\lambda-m-d)t}\rho_0$, with ρ_0 the initial distribution of cells.

Second model

Let us include density dependent changes from moving to proliferating cells, so the moving cell will have a small probability to change its type if it is still in a dense area of proliferating cells. Such changes could be achieved by the following change of the operator V

$$(VF)(\gamma^+,\gamma^-) = q\sum_{x\in\gamma^+}\exp\Big(-\sum_{y\in\gamma^-}\psi(x-y)\Big)(F(\gamma^+\backslash x,\gamma^-\cup y)-F(\gamma^+,\gamma^-)) \\ +\sum_{x\in\gamma^-}\Big(p+\sum_{y\in\gamma^-\backslash x}\varphi(x-y)\Big)(F(\gamma^+\cup x,\gamma^-\backslash x)-F(\gamma^+,\gamma^-)).$$

Here p,q,φ are the same as before and $0\le\psi\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ is a symmetric, non-negative potential. This model will lead to the following pair of equations describing the local densities ρ_t^+,ρ_t^-

$$\frac{\partial\rho_t^+}{\partial t}(x) = -\,(\langle c\rangle+d+qe^{-(\psi*\rho_t^-)(x)})\rho_t^+(x)+(c*\rho_t^+)(x) \\ +p\rho_t^-(x)+\rho_t^-(x)(\varphi*\rho_t^-)(x)$$
$$\frac{\partial\rho_t^-}{\partial t}(x) = -\,(m+p)\rho_t^-(x)-\rho_t^-(x)(\varphi*\rho_t^-)(x)+\lambda(a*\rho_t^-)(x) \\ +q\rho_t^+(x)e^{-(\psi*\rho_t^-)(x)}.$$

Here the cells will stay a not exponentially distributed lifetime in the moving state. With high probability they will move until they reach an area with less proliferating cells and start to proliferate again. Thus we expect, that the motion outside of a pattern is higher and therefore the speed of growth of the boundary of the tumour is increased compared to previous model.

Third model

Let us assume constant intensities $p, q > 0$ for changing from proliferation to motion and vice versa, i.e. $\varphi = \psi = 0$ from the previous model, so

$$(VF)(\gamma^+, \gamma^-) = q \sum_{x\in\gamma^+} (F(\gamma^+\backslash x, \gamma^- \cup x) - F(\gamma^+, \gamma^-)) \\ + p \sum_{x\in\gamma^-} (F(\gamma^+ \cup x, \gamma^-\backslash x) - F(\gamma^+, \gamma^-)). \tag{26}$$

Instead, we introduce additional density dependent death of proliferating cells, so they are self-regulating themselves, c.f. spatial logistic model. The generator for the $-$ cells is given in such case by

$$\begin{aligned}(L_-F)(\gamma) = &\sum_{x\in\gamma_-} m(F(\gamma^+, \gamma^-\backslash x) - F(\gamma)) \\ &+ \sum_{x\in\gamma^-} \sum_{y\in\gamma^-\backslash x} a^-(x-y)(F(\gamma^+, \gamma^-\backslash x) - F(\gamma)) \\ &+ \sum_{x\in\gamma^+} \int_{\mathbb{R}^d} a^+(x-y)(F(\gamma^+, \gamma^- \cup y) - F(\gamma))\mathrm{d}y\end{aligned}$$

and the overall generator by $L = L_- + L_{hop} + V$. A proliferating cell will have an increased rate for death and will start moving according to an exponentially distributed time with parameter $p > 0$. This cell will continue to move for an another exponentially distributed time with parameter $q > 0$ and afterwards start to proliferate again. Such behaviour will cause a diffusion like movement of the cells where the speed of growth of the patterns should be less then in the previous models. Instead, here the local regulation mechanism will bound the local density in time. Altogether this will lead to the following kinetic description for the local densities ρ_t^+, ρ_t^-

$$\begin{aligned}\frac{\partial \rho_t^+}{\partial t}(x) &= -(\langle c\rangle + q + d)\rho_t^+(x) + (c * \rho_t^+)(x) + p\rho_t^-(x) \\ \frac{\partial \rho_t^-}{\partial t}(x) &= -(m+p)\rho_t^-(x) - \rho_t^-(x)(a^- * \rho_t^-)(x) + \lambda(a * \rho_t^-)(x) + q\rho_t^+(x).\end{aligned}$$

Fourth model

Instead of looking at density dependent mortality for self-regulation of the proliferating cells, we could also take density dependent birth, i.e. branching with fecundity, c.f. [8]. Here the generator is given by $L = L_- + L_{hop} + V$, where L_- is given in (16) and V in (26). Using the same notations we will get the kinetic description

$$\frac{\partial \rho_t^+}{\partial t}(x) = -(\langle c \rangle + q + d)\rho_t^+(x) + (c * \rho_t^+)(x) + p\rho_t^-(x) \tag{27}$$

$$\frac{\partial \rho_t^-}{\partial t}(x) = -(m+p)\rho_t^-(x) + \lambda\left(a * \rho_t^- e^{-\varphi * \rho_t^-}\right)(x) + q\rho_t^+(x). \tag{28}$$

3 General Markov evolutions on configuration spaces

In this section we are going to summarize all necessary definitions and results, so that we can prove the given statements of previous section. First we briefly outline our approach for one-component systems and afterwards point out the steps for a natural extension to two-component systems. The last part deals with the mesoscopic scaling, here all machinery needed to derive the kinetic description for a wide class of models is introduced.

3.1 One-component models

The phase space of the evolutions is described by locally finite configurations $\gamma \in \Gamma$, i.e.

$$\Gamma = \{\gamma \subset \mathbb{R}^d : \ |\gamma \cap K| < \infty \ \ \forall K \subset \mathbb{R}^d \text{ bounded}\}.$$

The topology on Γ is defined as the smallest, such that all maps

$$\Gamma \ni \gamma \longmapsto \langle f, \gamma \rangle = \sum_{x \in \gamma} f(x), \quad f \in C_c(\mathbb{R}^d)$$

are continuous and Γ equipped with this topology has the structure of a polish space, c.f. [14], [1]. Here $C_c(\mathbb{R}^d)$ is the space of all continuous functions f on $\mathbb{R}^d$ with compact support. Denote by $\mathscr{B}(\Gamma)$ the corresponding Borel $\sigma-$algebra and remember that the space of all probability measurses on Γ, i.e. states of the system, is denoted by $\mathscr{P}$. The Poisson measure $\pi_\rho \in \mathscr{P}$ on Γ is defined via the Laplace transform, c.f. [1]

$$\widehat{\pi}(f) = \exp\Bigg(\int_{\mathbb{R}^d} (e^{f(x)} - 1)\rho(x)\mathrm{d}x\Bigg), \quad f \in C_c(\mathbb{R}^d),$$

where $0 \le \rho \in L^1_{loc}(\mathbb{R}^d)$. It is also possible to construct π_ρ directly using the projective structure of Γ. Since we are not going to use this construction, we will refer to [1]. The space of finite configurations $\eta \in \Gamma_0$ is

$$\Gamma_0 = \{\eta \subset \mathbb{R}^d : \ |\eta| < \infty\} = \bigsqcup_{n \in \mathbb{N}} \Gamma_0^{(n)} \tag{29}$$

with $\Gamma_0^{(n)} = \{\eta \subset \mathbb{R}^d : \ |\eta| = n\}$. Also this space can be equipped with a natural topology and the Borel $\sigma-$algebra is denoted by $\mathscr{B}(\Gamma_0)$, c.f. [13]. Denote the bijective symmetrization map by

$$\mathrm{sym}_n : \widetilde{(\mathbb{R}^d)^n} \to \Gamma_0^{(n)}, \quad (x_1, \dots, x_n) \longmapsto \{x_1, \dots, x_n\}$$

with $(x_1, \dots, x_n) \in \widetilde{(\mathbb{R}^d)^n}$ if and only if $x_j \neq x_j$ for all $j \neq k$. The Lebesgue-poisson measure on Γ_0 is defined by

$$\lambda = \delta_{\{\emptyset\}} + \sum_{n=1}^{\infty} \frac{\mathrm{d}x^{(n)}}{n!},$$

where $\mathrm{d}x^{(n)}$ is the image measure of the Lebesgue measure $\mathrm{d}x^{\otimes n}$ on $(\mathbb{R}^d)^n$ under the symmetrization map sym_n. Functions on Γ_0, will be written by $G, k : \Gamma_0 \to \mathbb{R}$, whereas functions on Γ are denoted by $F : \Gamma \to \mathbb{R}$. From (29) we conclude that each function k respectively $G : \Gamma_0 \longrightarrow \mathbb{R}$ has a decomposition to a sequence of symmetric functions $k = (k^{(n)})_{n=0}^{\infty}$ respectively $G = (G^{(n)})_{n=0}^{\infty}$. There is a combinatorial operator similar to Fourier transform translating functions $G : \Gamma_0 \longrightarrow \mathbb{R}$ to functions $F : \Gamma \longrightarrow \mathbb{R}$. This Transformation is called K-transform, see [13], and is defined by

$$(KG)(\gamma) = \sum_{\eta \Subset \gamma} G(\eta). \tag{30}$$

Here the symbol $\Subset$ means, that the summation is taken only about all finite configurations $\eta \subset \gamma$. The inverse map K^{-1} has the form

$$(K^{-1}G)(\eta) = \sum_{\xi \subset \eta} (-1)^{|\eta \backslash \xi|} G(\xi).$$

Expression (30) is well-defined for instance for bounded functions G having bounded support, i.e. there is $\Lambda \subset \mathbb{R}^d$ compact, $N \in \mathbb{N}$ and $C > 0$ such that $|G(\eta)| \leq C$, and for all $\eta \in \Gamma_0$ with $|\eta| > N$ or $\eta \not\subset \Lambda$ one has $G(\eta) = 0$. In such case KG is a cylindrical function on Γ, for details see [13].

Next introduce a convolution for measurable functions $G, H : \Gamma_0 \longrightarrow \mathbb{R}$ via

$$(G \star H)(\eta) = \sum_{\xi \subset \eta} \sum_{\zeta \subset \xi} G(\xi) H(\eta \backslash \xi \cup \zeta). \tag{31}$$

This convolution will satisfy a similar property to the Fourier transform of functions, namely

$$(KG)(KH) = K(G \star H), \tag{32}$$

provided $G, H \in L^1(\Gamma_0, d\lambda)$. This transformation allows us to associate to each probability measure $\mu \in \mathscr{P}(\Gamma)$ with finite local moments, i.e.

$$\int_{\Gamma} |\gamma \cap \Lambda|^n \mu(\mathrm{d}\gamma) < \infty$$

for all compacts $\Lambda \subset \mathbb{R}^d$, a locally finite measure ρ_μ on Γ_0 via an extension of the relation

$$\rho_\mu(A) = \int_{\Gamma} (K 1_A)(\gamma) \mu(\mathrm{d}\gamma), \quad A \in \mathscr{B}(\Gamma_0).$$

Let us assume, that ρ_μ is absolutely continuous with respect to the Lebesgue-Poisson measure λ. then the Radon-Nikodym derivative $k_\mu = \dfrac{\mathrm{d}\rho_\mu}{\mathrm{d}\lambda}$ is the correlation function defined in (2) (corresponding to the measure μ). Conversely given a function $k : \Gamma_0 \longrightarrow \mathbb{R}$ the following inverse statement for the construction of a measure $\mu \in \mathscr{P}$ from k holds. The proof can be found in [13].

Theorem 3.1. *Assume that k is positive definite in the sense that*

$$\int_{\Gamma_0} G(\eta)k(\eta)\mathrm{d}\lambda(\eta) \geq 0 \tag{33}$$

for all G bounded with bounded support, such that $KG \geq 0$. Then there exists a probability measure μ on Γ with correlation function k.

The Lebesgue-Poisson exponential $e_\lambda(f;\eta) := \prod_{x\in\eta} f(x)$ satisfy the combinatorial formula $Ke_\lambda(f) = e_\lambda(f+1)$, i.e.

$$\sum_{\xi\subset\eta} e_\lambda(\rho;\xi) = e_\lambda(\rho+1;\eta).$$

The following equality will be useful for several computations

$$\int_{\Gamma_0} e_\lambda(\rho;\eta)\mathrm{d}\lambda(\eta) = \exp\Big(\int_{\mathbb{R}^d} \rho(x)\mathrm{d}x\Big).$$

Let us take $f \in C_c(\mathbb{R}^d)$ and compute on the one-hand-side

$$\int_{\Gamma} e^{\langle f,\gamma\rangle}\mathrm{d}\pi_\rho(\gamma) = \exp\Big(\int_{\mathbb{R}^d} (e^{f(x)}-1)\rho(x)\mathrm{d}x\Big) = \int_{\Gamma_0} e_\lambda(e^f-1)e_\lambda(\rho)\mathrm{d}\lambda$$
$$= \int_{\Gamma_0} Ke_\lambda(e^f)e_\lambda(\rho)\mathrm{d}\lambda$$

thus

$$\int_{\Gamma} e^{\langle f,\gamma\rangle}\mathrm{d}\pi_\rho(\gamma) = \int_{\Gamma_0} Ke_\lambda(e^f)e_\lambda(\rho)\mathrm{d}\lambda,$$

which shows that the correlation measure for π_ρ is given by $e_\lambda(\rho)\mathrm{d}\lambda$. Finally we will explain the approach to describe statistical dynamics on this spaces, i.e. the approach to analyse the evolution $t \longmapsto \mu_t$. So let us start with a heuristic Markov generator L, e.g. (1) or (13). In the general framework of Markov processes one would study the evolution of observables, i.e. solutions to the equation

$$\frac{\partial F_t}{\partial t} = LF_t, \quad F_t|_{t=0} = F_0.$$

Its solution can give the possibility to construct under certain conditions a Markov process $(X_t^\gamma)_{t\geq 0}$ such that

$$F_t(\gamma) = \mathbb{E}^\gamma(F_0(X_t)).$$

Alternatively we can try to investigate the equation for measures μ_t, c.f. (3). But since we are dealing with infinite configurations, both approaches are very difficult and it was possible to realize them only in a few examples, c.f. [16]. Instead one tries to rewrite the equation using the $K-$transform to an equation for functions on Γ_0 and investigate this equation. This approach should be interpreted as a change of variables, so we define the operator $\widehat{L} = K^{-1}LK$, which acts now on functions $G : \Gamma_0 \longrightarrow \mathbb{R}$ and try to solve the Cauchy problem

$$\frac{\partial G_t}{\partial t} = \widehat{L}G_t, \quad G_t|_{t=0} = G_0.$$

In this article we will investigate this equation in one of the following Banach spaces

$$\mathbb{B}_\alpha = L^1(\Gamma_0, e^{\alpha|\cdot|}\mathrm{d}\lambda)$$

with $\alpha \in \mathbb{R}$ and the norm given by

$$\|G\|_\alpha = \int\limits_{\Gamma_0} |G(\eta)|e^{\alpha|\eta|}\mathrm{d}\lambda(\eta) = \sum_{n=0}^{\infty} \frac{e^{\alpha n}}{n!} \int\limits_{(\mathbb{R}^d)^n} |G^{(n)}(x_1, \ldots, x_n)|\mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

An evolution $t \longmapsto G_t \in \mathbb{B}_\alpha$ determines a dual evolution $t \longmapsto k_t^D$ by

$$\int\limits_{\Gamma_0} G_t(\eta)k_0(\eta)\mathrm{d}\lambda(\eta) = \int\limits_{\Gamma_0} G_0(\eta)k_t^D(\eta)\mathrm{d}\lambda(\eta)$$

and since $G_t \in \mathbb{B}_\alpha$ this dual evolution will obey the Ruelle bound

$$|k_t^D(\eta)| \le Ce^{\alpha|\eta|}, \quad \eta \in \Gamma_0$$

and hence be sub-poissonian. As already mentioned such an evolution describes a system, which is not asymptotically clustering, but still could include some pattern formation. It is also possible to study the equation for k_t directly, therefore using duality it is possible to compute the expression for L^Δ directly via

$$\int\limits_{\Gamma_0} (\widehat{L}G)(\eta)k(\eta)\mathrm{d}\lambda(\eta) = \int\limits_{\Gamma_0} G(\eta)(L^\Delta k)(\eta)\mathrm{d}\lambda(\eta)$$

for each function G bounded with bounded support and k locally integrable. One special case was computed already for the first and second correlation functions. Finally one would seek for a solution to

$$\frac{\partial k_t}{\partial t} = L^\Delta k_t, \quad k_t|_{t=0} = k_0$$

and construct if possible the evolution of states $t \longmapsto \mu_t$. This sketch has to be realized for each model separately, like all operators $L, \widehat{L}, L^\Delta$ have to be defined on a proper set of functions, which is large enough to determine the evolution of states. Note that even if we have solved the equation (4) it is not clear, whether the evolution $t \longmapsto k_t$ really determines an evolution of states $t \longmapsto \mu_t$ and therefore is of biological interest. Such task has to be carefully proved and was realized for several important models, c.f. [15]. The main problem is that the evolution has to be positive definite, c.f. (33).

3.2 Two-component models

Let us now outline the major differences of two-component models. Afterwards it will be clear how to extend all considerations to models with any number of components $n \in \mathbb{N}$. First of all let us denote by $+$ respectively $-$ the types of cells and by γ^+ and γ^- their (locally finite) configurations. Since no cells of different type can be located at the same position we will assume $\gamma^+ \cap \gamma^- = \emptyset$, therefore

$$\Gamma^2 = \{(\gamma^+, \gamma^-) : \quad \gamma^+, \gamma^- \in \Gamma, \quad \gamma^+ \cap \gamma^- = \emptyset\}.$$

Similarly the space of finite configurations Γ_0^2 and the topologies on these spaces are defined. Since for each $\xi \in \Gamma_0$ the set

$$\{\eta \in \Gamma_0 : \quad \eta \cap \xi \neq \emptyset\}$$

is a set of measure zero with respect to λ we can define the Lebesgue Poisson measure λ^2 on Γ_0^2 as the product measure $\lambda \otimes \lambda$ and calculate as in the one-component case. Similarly the Poisson measure will be the product measure of two copies of π. The K-transform is a composition of two K-transforms for each type of cells, i.e.

$$(KG)(\gamma^+, \gamma^-) = \sum_{\eta^+ \Subset \gamma^+} \sum_{\eta^- \Subset \gamma^-} G(\eta^+, \eta^-)$$

and K^{-1} is just

$$(K^{-1}F)(\eta^+, \eta^-) = \sum_{\xi^+ \subset \eta^+} \sum_{\xi^- \subset \eta^-} (-1)^{|\eta^+ \setminus \xi^+|} (-1)^{|\eta^- \setminus \xi^-|} F(\xi^+, \xi^-).$$

The Lebesgue-Poisson exponential will be the product of the Lebesgue-Poisson exponentials for each type of cells and the correlation functions become a double indexed vector, i.e. $k^{(n,m)}(x_1, \ldots, x_n; y_1, \ldots y_m)$. The heuristic Markov generator L now acts on functions $F : \Gamma^2 \longrightarrow \mathbb{R}$ and L^Δ on collections of correlation functions $k = (k^{(n,m)})_{n,m=0}^\infty$.

3.3 Mesoscopic scaling

As before the approach to derive the kinetic description, respectively the mesoscopic equation, can be described within three steps. In the first step one rescales the potentials, and thus the generator L. The outcome is a new system with smaller interactions, with generator denoted by L_ε. In the second step we will choose some admissible class of initial states $k_{0,\varepsilon} = (k_{0,\varepsilon}^{(n)})_{n=0}^\infty$ such that

$$\varepsilon^{|\eta|} k_{0,\varepsilon}(\eta) \to r_0(\eta), \quad \varepsilon \to 0$$

for each $\eta \in \Gamma_0$. Finally let $k_{t,\varepsilon}$ be the solution of

$$\frac{\partial k_{t,\varepsilon}}{\partial t}(\eta) = L_\varepsilon^\Delta k_{t,\varepsilon}(\eta), \tag{34}$$

where L_ε^Δ is the adjoint operator to $\widehat{L}_\varepsilon = K^{-1} L_\varepsilon K$. We will seek for a limit

$$\varepsilon^{|\eta|} k_{t,\varepsilon}(\eta) \to r_t(\eta), \quad \varepsilon \to 0$$

for each $\eta \in \Gamma_0$ and t. Such limit implies that

$$\varepsilon^n k_{t,\varepsilon}^{(n)}(x_1, \dots, x_n) \to \rho_t(x_1) \cdots \rho_t(x_n), \quad \varepsilon \to 0 \tag{35}$$

if $r_0^{(n)}(x_1, \dots, x_n) = \rho_0(x_1) \cdots \rho_n(x_n)$ for all $n \in \mathbb{N}$. This is equivalent to solve the equations

$$\frac{\partial k_{t,\varepsilon}^{ren}}{\partial t} = L_{\varepsilon,ren}^{\Delta} k_{t,\varepsilon}^{ren}, \quad k_{t,\varepsilon}^{ren} = R_\varepsilon k_{0,\varepsilon}$$

and seek for the limits

$$\lim_{\varepsilon \to 0} k_{t,\varepsilon}^{ren} = r_t, \tag{36}$$

where r_t solves the equation

$$\frac{\partial r_t}{\partial t} = L_V^{\Delta} r_t, \quad r_t|_{t=0} = r_0. \tag{37}$$

Here

$$L_{\varepsilon,ren}^{\Delta} = R_\varepsilon L_\varepsilon^{\Delta} R_\varepsilon^{-1} \to L_V^{\Delta} \tag{38}$$

and $(R_\varepsilon k)(\eta) = \varepsilon^{|\eta|} k(\eta)$. Summarizing this approach, we first rescale the system and arrive at an expression for the operator $L_{\varepsilon,ren}^{\Delta}$. Form this one computes the expression for L_V^{Δ}. Finally putting $r_0 = e_\lambda(\rho_0)$ in equation (37) one deduces the kinetic description

$$\frac{\partial \rho_t}{\partial t} = v(\rho_t), \quad \rho_t|_{t=0} = \rho_0. \tag{39}$$

The analysis of (36) is quite hard and needs several technical tool and such problem should be solved for each model separately. Nevertheless it is important, since it relates the mesoscopic evolution as the limiting evolution of the microscopic evolution. This means for instance, that starting with $r_0(x_1, \dots, x_n) = \rho_0(x_1) \cdots \rho_0(x_n)$, and denoting by ρ_t the solution to (39), we get for all $n \in \mathbb{N}$ (35). The precise notion of convergence should be chosen adequately to the model. In this work we will focus on (38) and compute equations (37) and (39) for several models. However, convergence of equations (38) does not imply convergence of solutions, i.e. (36), thus it is important to determine conditions which imply $k_{t,\varepsilon}^{ren} \to r_t$. If such convergence happens to be false in some case, then we know that this kinetic description, also if it is well analysed, will not describe the original model and hence has no biological significance.

4 One-component systems

Within this section we will prove the results stated in the previous section and derive for many possible individual based interactions their related operators on quasi-observables, correlation functions and the kinetic description. The main technical tools introduced in the last section will be applied for each model directly.

We will work in scales of Banach spaces defined for $\alpha \in \mathbb{R}$ by $\mathbb{B}_\alpha = L^1(\Gamma_0, e^{\alpha|\cdot|}\mathrm{d}\lambda)$, i.e. equivalence classes of measurable functions $G : \Gamma_0 \longrightarrow \mathbb{R}$ satisfying

$$\|G\|_\alpha = \int_{\Gamma_0} |G(\eta)|e^{\alpha|\eta|}\mathrm{d}\lambda(\eta) = \sum_{n=0}^{\infty} \frac{e^{\alpha n}}{n!} \int_{(\mathbb{R}^d)^n} |G^{(n)}(x_1, \dots, x_n)|\mathrm{d}x_1 \dots \mathrm{d}x_n < \infty. \tag{40}$$

The dual space is given by $\mathbb{B}_\alpha^* = L^\infty(\Gamma_0, e^{-\alpha|\cdot|}\mathrm{d}\lambda)$, so measurable functions $k : \Gamma_0 \longrightarrow \mathbb{R}$ such that

$$\|k\|_\alpha = \operatorname*{ess\,sup}_{\eta \in \Gamma_0} |k(\eta)|e^{-\alpha|\eta|} < \infty. \tag{41}$$

The duality pairing is simply

$$\langle G, k\rangle = \int_{\Gamma_0} G(\eta)k(\eta)\mathrm{d}\lambda(\eta) \tag{42}$$

and satisfies $|\langle G, k\rangle| \le \|G\|_\alpha \|k\|_\alpha$. Let $\widehat{L} \in L(\mathbb{B}_\alpha, \mathbb{B}_{\alpha'})$ for each $\alpha' < \alpha$ and L^Δ the dual operator with respect to (42). Then

$$\|\widehat{L}\|_{\alpha\alpha'} = \|L^\Delta\|_{\alpha'\alpha} \tag{43}$$

where the norms are determined by (40) and (41). Consequently for several aspects it is enough to analyse only the operator $\widehat{L}$. It is possible to assign to each $\widehat{L}$ a measurable function $M_\alpha : \Gamma_0 \longrightarrow \mathbb{R}_+$ such that

$$\|\widehat{L}G\|_\alpha = \int_{\Gamma_0} |\widehat{L}G(\eta)|e^{\alpha|\eta|}\mathrm{d}\lambda(\eta) \le \int_{\Gamma_0} |G(\eta)|M_\alpha(\eta)e^{\alpha|\eta|}\mathrm{d}\lambda(\eta) = \|M_\alpha G\|_\alpha.$$

The operator $(\widehat{L}, D(M_\alpha))$ is well-defined on

$$D(M_\alpha) = \{G \in \mathbb{B}_\alpha : \ M_\alpha \cdot G \in \mathbb{B}_\alpha\} \tag{44}$$

and if $M_\alpha(\eta) \le P_\alpha(|\eta|)e^{\delta|\eta|}$ with some polynomial P_α and $\delta > 0$, then the estimate

$$|\eta|^k e^{-\delta|\eta|} \le \left(\frac{k}{e\delta}\right)^k \tag{45}$$

implies $\widehat{L} \in L(\mathbb{B}_\alpha, \mathbb{B}_{\alpha'})$ for all $\alpha - \alpha' > \delta$. From (43) the same estimate is valid for L^Δ. Practically we have only to determine the expression for M_α and analyse its growth. Concerning the construction of microscopic dynamics via semigroups on the scale of Banach spaces $\mathbb{B}_\alpha$ one would compute the function $D_\alpha : \Gamma_0 \longrightarrow \mathbb{R}_+$ given by

$$\int_{\Gamma_0} (\widehat{L}G)(\eta)e^{\alpha|\eta|}\mathrm{d}\lambda(\eta) = \int_{\Gamma_0} G(\eta)D_\alpha(\eta)e^{\alpha|\eta|}\mathrm{d}\lambda(\eta).$$

By (42) it means $(L^{\Delta} e_{\lambda}(e^{\alpha}))(\eta) = D_{\alpha}(\eta)$, and analyse its properties. In many cases both functions M_{α} and D_{α} have a simple relation, but M_{α} is not unique. Similarly define $D_{\Delta}(M_{\alpha}) \subset \mathbb{B}^*_{\alpha}$ as the set of all $k \in \mathbb{B}^*_{\alpha}$ such that $M_{\alpha} k \in \mathbb{B}^*_{\alpha}$. Then L^{Δ} is well-defined on $D_{\Delta}(M_{\alpha})$.

Within the mesoscopic scaling we will consider the rescaled operators $\widehat{L}_{\varepsilon,ren}$ and $L^{\Delta}_{\varepsilon,ren}$. Denote by N_{α} the function determined by

$$\int_{\Gamma_0} |\widehat{L}_{\varepsilon,ren} G(\eta)| e^{\alpha|\eta|} d\lambda(\eta) \leq \int_{\Gamma_0} N_{\alpha}(\eta) |G(\eta)| e^{\alpha|\eta|} d\lambda(\eta).$$

Note, that in general such function does not need to exist, it will be necessary to show that for all reasonable models under some general assumptions we can find the function N_{α}. As before the operators $(\widehat{L}_{\varepsilon,ren}, D(N_{\alpha}))$ and $(L^{\Delta}_{\varepsilon,ren}, D_{\Delta}(N_{\alpha}))$ are well-defined. The limiting operator $\widehat{L}_V$ given by $\widehat{L}_{\varepsilon,ren} \to \widehat{L}_V$ as $\varepsilon \to 0$ determines another function N^V_{α} via

$$\int_{\Gamma_0} |\widehat{L}_V G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta) \leq \int_{\Gamma_0} N^V_{\alpha}(\eta) |G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta).$$

So we define $(\widehat{L}_V, D(N^V_{\alpha}))$ and $(L^{\Delta}_V, D_{\Delta}(N^V_{\alpha}))$.

Theorem 4.1. *For all subsequent interactions, the following holds.*

1. *For any $G \in D(N_{\alpha}) \cap D(N^V_{\alpha})$ the convergence*

$$\widehat{L}_{\varepsilon,ren} G \to \widehat{L}_V G, \quad \varepsilon \to 0$$

holds. If in addition there is a polynomial P and $\delta > 0$ such that $N_{\alpha}(\eta), N^V_{\alpha}(\eta) \leq P(|\eta|) e^{\delta|\eta|}$, then $\widehat{L}, \widehat{L}_{\varepsilon,ren}$ and $\widehat{L}_V$ act as bounded linear operators in $L(\mathbb{B}_{\alpha}, \mathbb{B}_{\alpha'})$ for any $\alpha - \alpha' > \delta$ and

$$\|\widehat{L}_{\varepsilon,ren} - \widehat{L}_V\| \alpha\alpha' \to 0, \quad \varepsilon \to 0.$$

2. *For any $k \in D_{\Delta}(N_{\alpha}) \cap D_{\Delta}(N^V_{\alpha})$ the convergence*

$$L^{\Delta}_{\varepsilon,ren} k \to L^{\Delta}_V k, \quad \varepsilon \to 0$$

holds. And if in addition there is a polynomial P and $\delta > 0$ such that $N_{\alpha}(\eta), N^V_{\alpha}(\eta) \leq P(|\eta|) e^{\delta|\eta|}$, then $L^{\Delta}, L^{\Delta}_{\varepsilon,ren}$ and L^{Δ}_V act as bounded linear operators in $L(\mathbb{B}^*_{\alpha'}, \mathbb{B}^*_{\alpha})$ for any $\alpha - \alpha' > \delta$ and

$$\|L^{\Delta}_{\varepsilon,ren} - L^{\Delta}_V\| \alpha'\alpha \to 0, \quad \varepsilon \to 0.$$

3. *If ρ_t is a solution to the corresponding kinetic description determined by L^{Δ}_V, c.f.* (39)*, then $e_{\lambda}(\rho_t)$ is a solution to the Cauchy problem associated to L^{Δ}_V, i.e. solves the Cauchy problem* (37).

In the following denote by $E(x, \gamma \backslash x) = \sum_{y \in \gamma \backslash x} \phi(x-y)$ the relative energy of the cell x with respect to the rest of the configuration $\gamma \backslash x$. Here $0 \leq \phi \in L^1(\mathbb{R}^d)$ is assumed to be symmetric. For infinite configurations such sum will be infinite in general, but e.g. for the Poisson measure it is possible to define $E(x, \gamma \backslash x)$ for almost all $\gamma \in \Gamma$, such that this sum is convergent.

We will use also the following well-known result.

Lemma 4.2. *Let $H : \mathbb{R}^d \times \Gamma_0 \longrightarrow \mathbb{R}$ and $G : \Gamma_0 \times \Gamma_0 \times \Gamma_0 \longrightarrow \mathbb{R}$ be measurable, then the following formulas hold, provided one side of the corresponding equality exists*

$$\int_{\Gamma_0} \sum_{x\in\eta} H(x,\eta) d\lambda(\eta) = \int_{\Gamma_0} \int_{\mathbb{R}^d} H(x, \eta \cup x) dx d\lambda(\eta)$$

and

$$\int_{\Gamma_0} \sum_{\xi\subset\eta} H(\xi, \eta\backslash\xi, \eta) d\lambda(\eta) = \int_{\Gamma_0} \int_{\Gamma_0} H(\xi, \eta, \eta \cup \xi) d\lambda(\xi) d\lambda(\eta).$$

4.1 Death dynamics

Let us investigate here the dynamics of the microscopic event death.

Example 2 (constant mortality)**.** The Markov generator has here the form

$$(LF)(\gamma) = \sum_{x\in\gamma} m(x)(F(\gamma\backslash x) - F(\gamma)),$$

where $0 \leq m \in L^\infty_{loc}(\mathbb{R}^d)$. Each cell located in position $x \in \mathbb{R}^d$ has an exponential distributed liftetime with parameter $m(x)$. In the case when $m(x) = 0$, the cell will not die due to this mechanism. The operator $\widehat{L}$ on quasi-observables has the form

$$(\widehat{L}G)(\eta) = -\sum_{x\in\eta} m(x)G(\eta)$$

and likewise L^Δ is given by the same expression. Moreover we see that it is possible to take $M_\alpha(\eta) = N_\alpha(\eta) = N^V_\alpha(\eta) = \sum_{x\in\eta} m(x)$. Since here no scaling is necessary we obtain $\widehat{L}_V = \widehat{L}$ and $L^\Delta_V = L_V$. Consequently the kinetic description is simply

$$\frac{\partial \rho_t}{\partial t}(x) = -m(x)\rho_t(x).$$

Example 3 (quadratic mortality)**.** The Markov generator is given by

$$(LF)(\gamma) = \sum_{x\in\gamma} E(x, \gamma\backslash x)(F(\gamma\backslash x) - F(\gamma)).$$

Here each cell located at position $x \in \mathbb{R}^d$ may die, where the intensity of death is given by the intensity $\sum_{y\in\gamma\backslash x} \phi(x - y)$, i.e. the death of the cell is caused by interaction with another cell located at position $y \in \gamma\backslash x$. The case where $y = x$ is already included in the constant mortality $m = m(x)$. The operator for quasi-observables is now given by

$$(\widehat{L}G)(\eta) = -\sum_{x\in\eta} E(x, \eta\backslash x)G(\eta) - \sum_{x\in\eta} E(x, \eta\backslash x)G(\eta\backslash x)$$

and the operator on correlation functions by

$$(L^{\Delta}k)(\eta) = -\sum_{x\in\eta} E(x,\eta\backslash x)k(\eta) - \sum_{x\in\eta}\int_{\mathbb{R}^d} \phi(x-y)k(\eta\cup y)dy.$$

Similarly we can choose $M_\alpha(\eta) = N_\alpha(\eta) = \sum_{x\in\eta} E(x,\eta\backslash x) + \langle a\rangle e^{\alpha}|\eta|$. Within the scaling and after renormalization we arrive at new operators, where only the multiplicative part will be multiplied by $\varepsilon > 0$. Hence after limit transition $\varepsilon \to 0$ we obtain the operators for the Vlasov hierarchy given by

$$(\widehat{L}_V G)(\eta) = -\sum_{x\in\eta}\sum_{y\in\eta\backslash x} a^-(x-y)G(\eta\backslash x)$$

and likewise

$$(L_V^{\Delta}k)(\eta) = -\sum_{x\in\eta}\int_{\mathbb{R}^d} a^-(x-y)k(\eta\cup y)\mathrm{d}y$$

so that $N_\alpha^V(\eta) = e^{\alpha}\langle a^-\rangle|\eta|$. Finally the kinetic description is given by

$$\frac{\partial\rho_t}{\partial t}(x) = -\rho_t(x)(a^- * \rho_t)(x).$$

Example 4. Let us look at the stronger death intensity described by the Markov generator

$$(LF)(\gamma) = \sum_{x\in\gamma} e^{E(x,\gamma\backslash x)}(F(\gamma\backslash x) - F(\gamma)),$$

Here each particle located at position $x \in \mathbb{R}^d$ may die, whereas the intensity of such microscopic event is given by $e^{E(x,\gamma\backslash x)}$, in the case of $E(x,\gamma\backslash x) = \infty$ one can think of immediate death. The corresponding operator on quasi-observables is given by

$$(\widehat{L}G)(\eta) = -\sum_{\xi\subset\eta} G(\xi)\sum_{x\in\xi} e^{E(x,\xi\backslash x)} e_\lambda(e^{\phi(x-\cdot)} - 1;\eta\backslash\xi)$$

and on correlation functions by

$$(L^{\Delta}k)(\eta) = -\sum_{x\in\eta} e^{E(x,\eta\backslash x)}\int_{\Gamma_0} e_\lambda(e^{\phi(x-\cdot)} - 1;\xi)k(\eta\cup\xi)\mathrm{d}\lambda(\xi).$$

We can choose the function $M_\alpha(\eta) = \beta_1(\alpha)\sum_{x\in\eta} e^{E(x,\eta\backslash x)}$, where

$$\beta_1(\alpha) = \exp\Big(e^{\alpha}\int_{\mathbb{R}^d}(e^{\phi(x)} - 1)\mathrm{d}x\Big).$$

For the mesoscopic scaling let us rescale the potential as $\phi \to \varepsilon\phi$ and after renormalization we arrive at

$$(\widehat{L}_{\varepsilon,ren}G)(\eta) = -\sum_{\xi\subset\eta} G(\xi)\sum_{x\in\xi} e^{\varepsilon E(x,\xi\backslash x)} e_\lambda\left(\frac{e^{\varepsilon\phi(x-\cdot)} - 1}{\varepsilon};\eta\backslash\xi\right)$$

and

$$(L^{\Delta}_{\varepsilon,ren}k)(\eta) = -\sum_{x\in\eta} e^{\varepsilon E(x,\eta\setminus x)} \int\limits_{\Gamma_0} e_\lambda\left(\frac{e^{\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\xi\right) k(\eta\cup\xi)\mathrm{d}\lambda(\xi).$$

After limit transition $\varepsilon \to 0$ we arrive at

$$(\widehat{L}_V G)(\eta) = -\sum_{\xi\subset\eta} G(\xi) \sum_{x\in\xi} e_\lambda(\phi(x-\cdot);\eta\setminus\xi)$$

and

$$(L^{\Delta}_V k)(\eta) = -\sum_{x\in\eta}\int\limits_{\Gamma_0} e_\lambda(\phi(x-\cdot);\xi)k(\eta\cup\xi)\mathrm{d}\lambda(\xi).$$

Here we can take $N_\alpha(\eta) = \beta(\alpha)\sum\limits_{x\in\eta} e^{E(x,\eta\setminus x)}$, where we have to assume that

$$\beta(\alpha) = \sup_{\varepsilon\in(0,1]} \exp\left(\frac{e^\alpha}{\varepsilon}\int\limits_{\mathbb{R}^d} |e^{\varepsilon\phi(x)}-1|\mathrm{d}x\right) < \infty.$$

Finally, $N^V_\alpha(\eta) = \exp(e^\alpha\langle\phi\rangle)|\eta|$ and for the kinetic description

$$\frac{\partial\rho_t}{\partial t}(x) = -\rho_t(x)e^{(\phi*\rho_t)(x)}.$$

4.2 Birth dynamics

Here we will describe the microscopic event responsible for the appearance of new cells.

Example 5 (Sourgailis birth)**.** The most simple form of birth, is where in each region $\Lambda\subset\mathbb{R}^d$ the intensity that a new cell appear in Λ is given by $\int\limits_\Lambda z(x)\mathrm{d}x$, where $0\le z\in L^1_{loc}(\mathbb{R}^d)$ is the intensity. Each such event is independent of the other and describes thus free growth of the system. In this case the Markov generator is given by

$$(LF)(\gamma) = \int\limits_{\mathbb{R}^d} z(x)(F(\gamma\cup x)-F(\gamma))\mathrm{d}x$$

and on quasi-observables by

$$(\widehat{L}G)(\eta) = \int\limits_{\mathbb{R}^d} z(x)G(\eta\cup x)\mathrm{d}x.$$

For correlation functions the adjoint operator is given by

$$(L^\Delta k)(\eta) = \sum_{x\in\eta} z(x)k(\eta\setminus x).$$

Take $M_\alpha(\eta) = N_\alpha(\eta) = N^V_\alpha(\eta) = e^{-\alpha}\sum\limits_{x\in\eta} z(x)$. Since scaling will not affect this operators, we immediately arrive at the kinetic description given by

$$\frac{\partial\rho_t}{\partial t}(x) = z(x).$$

Example 6 (Gibbs-type birth)**.** Let us assume that L is of the form

$$(LF)(\gamma) = z \int_{\mathbb{R}^d} e^{-E(x,\gamma)}(F(\gamma \cup x) - F(\gamma))\mathrm{d}x,$$

where $z > 0$. The creation of cells in some volume $\Lambda \subset \mathbb{R}^d$ is given by the intensity $\int_\Lambda ze^{-E(x,\gamma)}dx \leq z|\Lambda|$, where $|\Lambda|$ denotes the Lebesgue volume of Λ. The operator for quasi-observables is given by

$$(\widehat{L}G)(\eta) = z \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} e_\lambda(e^{-\phi(x-\cdot)} - 1; \eta \setminus \xi) e^{-E(x,\xi)} G(\xi \cup x)\mathrm{d}x$$

and for correlation functions by

$$(L^\Delta k)(\eta) = z \sum_{x \in \eta} e^{-E(x,\eta \setminus x)} \int_{\Gamma_0} e_\lambda(e^{-\phi(x-\cdot)} - 1; \xi) k(\eta \setminus x \cup \xi) d\lambda(\xi).$$

Here we can take $M_\alpha(\eta) = \beta(\alpha) \sum_{x \in \eta} e^{-E(x,\eta \setminus x)}$, where

$$\beta(\alpha) = \exp\Big(e^\alpha \int_{\mathbb{R}^d} |1 - e^{-\phi(x)}| \mathrm{d}x \Big)$$

and $N_\alpha(\eta) = \exp(e^\alpha \langle \phi \rangle) \sum_{x \in \eta} e^{-E(x,\eta \setminus x)}$. After scaling and renormalization we will arrive at

$$(\widehat{L}_\varepsilon G)(\eta) = z \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} e_\lambda \left(\frac{e^{-\varepsilon\phi(x-\cdot)} - 1}{\varepsilon}; \eta \setminus \xi \right) e^{-\varepsilon E(x,\xi)} G(\xi \cup x)\mathrm{d}x$$

which tends in the limit $\varepsilon \to 0$ to

$$(L_V G)(\eta) = z \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} e_\lambda \left(\phi(x - \cdot); \eta \setminus \xi \right) G(\xi \cup x)\mathrm{d}x.$$

In the same way we obtain

$$(L_\varepsilon^\Delta k)(\eta) = z \sum_{x \in \eta} e^{-\varepsilon E(x,\eta \setminus x)} \int_{\Gamma_0} e_\lambda \left(\frac{e^{-\varepsilon\phi(x-\cdot)} - 1}{\varepsilon}; \xi \right) k(\eta \setminus x \cup \xi)\mathrm{d}\lambda(\xi)$$

and hence when $\varepsilon \to 0$

$$(L_V^\Delta k)(\eta) = z \sum_{x \in \eta} \int_{\Gamma_0} e_\lambda(\phi(x - \cdot); \xi) k(\eta \setminus x \cup \xi)\mathrm{d}\lambda(\xi).$$

The function N_α^V can be chosen as $N_\alpha^V(\eta) = z \exp(e^\alpha \langle \phi \rangle) e^{-\alpha} |\eta|$. Therefore the kinetic description is given by

$$\frac{\partial \rho_t}{\partial t}(x) = z e^{(\phi * \rho_t)(x)}.$$

Example 7 (free branching)**.** In the simplest way free branching is described by

$$(LF)(\gamma) = \sum_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} a^+(x-y)(F(\eta\cup y) - F(\eta))\mathrm{d}y,$$

where $0 \le a^+ \in L^1(\mathbb{R}^d)$ is symmetric. Here each cell located at position $x \in \gamma$ may create a new cell located at position $y \in \mathbb{R}^d$. The intensity of such event is given by $\langle a^+\rangle = \int_{\mathbb{R}^d} a^+(z)\mathrm{d}z$ and the new particle is distributed according to the probability measure $\frac{1}{\langle a^+\rangle} a^+(x-y)\mathrm{d}y$. On the level of quasi-observables this effect is described via

$$(\widehat{L}G)(\eta) = \sum_{x\in\eta} \int_{\mathbb{R}^d} a^+(x-y)G(\eta\backslash x\cup y)\mathrm{d}y + \sum_{x\in\eta}\int_{\mathbb{R}^d} a^+(x-y)G(\eta\cup y)\mathrm{d}y.$$

Likewise on correlation functions it is given by

$$(L^\triangle k)(\eta) = \sum_{x\in\eta}\int_{\mathbb{R}^d} a^+(x-y)k(\eta\backslash x\cup y)\mathrm{d}y + \sum_{x\in\eta}\sum_{y\in\eta\backslash x} a^+(x-y)k(\eta\backslash x).$$

It is sufficient to take $M_\alpha(\eta) = N_\alpha(\eta) = e^{-\alpha}\sum\limits_{x\in\eta}\sum\limits_{y\in\eta\backslash x} a^+(x-y) + \langle a^+\rangle|\eta|$. Here we can explicitly compute the correlation functions, which will be done later on. After scaling and renormalization we observe that only the second summands will be multiplied by $\varepsilon > 0$. Hence in the limit we arrive at

$$(\widehat{L}_V G)(\eta) = \sum_{x\in\eta}\int_{\mathbb{R}^d} a^+(x-y)G(\eta\backslash x\cup y)\mathrm{d}y$$

and likewise for correlation functions $L^\triangle k$ is given by the same expression, namely we can chose $N_\alpha^V(\eta) = \langle a^+\rangle|\eta|$. For the kinetic description we obtain

$$\frac{\partial \rho_t}{\partial t}(x) = (a^+ * \rho_t)(x).$$

Similarly we can also consider the case, where each cell at $x\in\gamma$ may split into two new cells at positions y_1, y_2. The intensity of such transition would be, for simplicity, again constant $\langle a^+\rangle$. The probability distribution is given by $\frac{1}{\langle a^+\rangle}a^+(x-y_1, x-y_2)\mathrm{d}y_1\mathrm{d}y_2$, where $0\le a^+ \in L^1(\mathbb{R}^d\times\mathbb{R}^d)$ is symmetric in both variables. The Markov generator is of the form

$$(LF)(\gamma) = \sum_{x\in\gamma}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a^+(x-y_1, x-y_2)(F(\gamma\backslash x\cup y_1\cup y_2) - F(\gamma))\mathrm{d}y_1\mathrm{d}y_2.$$

For quasi-observables this yields

$$(\widehat{L}G)(\eta) = \sum_{x\in\eta}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a^+(x-y_1, x-y_2)G(\eta\backslash x\cup y_1\cup y_2)\mathrm{d}y_1\mathrm{d}y_2$$
$$+ \sum_{x\in\eta}\int_{\mathbb{R}^d} b(x-y)(G(\eta\backslash x\cup y) - G(\eta))\mathrm{d}y + \langle a^+\rangle|\eta|G(\eta),$$

where $b(x) = \int_{\mathbb{R}^d} a(x,y)\mathrm{d}y + \int_{\mathbb{R}^d} a(y,x)\mathrm{d}y$. Likewise we can compute the adjoint operator, which is given by

$$(L^{\triangle}k)(\eta) = \sum_{y_1\in\eta}\sum_{y_2\in\eta\backslash y_1}\int_{\mathbb{R}^d} a^+(x-y_1, x-y_2)k(\eta\cup x\backslash y_1\backslash y_2)dx$$
$$+\sum_{x\in\eta}\int_{\mathbb{R}^d} b(x-y)(k(\eta\backslash x\cup y) - k(\eta))dy + \langle a^+\rangle k(\eta).$$

Similarly we can choose

$$M_\alpha(\eta) = N_\alpha(\eta) = e^{-\alpha}\sum_{y_1\in\eta}\sum_{y_2\in\eta\backslash y_1}\int_{\mathbb{R}^d} a(x-y_1, x-y_2)\mathrm{d}x + 3\langle a^+\rangle|\eta|.$$

Within the scaling we have to multiply a^+ by ε and afterwards rescale the operators. Effectively it will consist only of multiplying the first term by ε, and after limit transition we arrive at

$$(\widehat{L}_V G)(\eta) = \sum_{x\in\eta}\int_{\mathbb{R}^d} b(x-y)(G(\eta\backslash x\cup y) - G(\eta))\mathrm{d}y + \langle a^+\rangle|\eta|G(\eta)$$

and

$$(L_V^{\triangle}k)(\eta) = \sum_{x\in\eta}\int_{\mathbb{R}^d} b(x-y)(k(\eta\backslash x\cup y) - k(\eta))\mathrm{d}y + \langle a^+\rangle|\eta|k(\eta),$$

so that $N_\alpha^V(\eta) = 3\langle a^+\rangle|\eta|$. Therefore the kinetic description is simply given by

$$\frac{\partial\rho_t}{\partial t}(x) = -\langle a^+\rangle\rho_t(x) + (b*\rho_t)(x) = \big((b*\rho_t)(x) - \langle b^+\rangle\rho_t(x)\big) + \langle a^+\rangle\rho_t(x).$$

Note that the solution is increasing, e.g. if ρ_0 is integrable, then the solution ρ_t will be integrable as well and satisfy

$$\frac{\partial}{\partial t}\langle\rho_t\rangle = \langle a^+\rangle\langle\rho_t\rangle,$$

which yields $\langle\rho_t\rangle = e^{\langle a^+\rangle t}\langle\rho_0\rangle$.

Example 8 (establishment)**.** Let us take a look on the birth dynamics with establishment. Here each cell located at position $x\in\gamma$ will have a dumped probability to produce a new cell at position $y\in\mathbb{R}^d$, if there are many cells around y. The Markov generator is given by

$$(LF)(\gamma) = \sum_{x\in\gamma}\int_{\mathbb{R}^d} e^{-E(y,\gamma)}a^+(x-y)(F(\gamma\cup y) - F(\gamma))dy,$$

where $0\le a^+\in L^1(\mathbb{R}^d)$ is symmetric. Each cell at position $x\in\gamma$ will create a new cell at position y, but the intensity of this effect is dumped by the relative energy in the exponential. Calculations yield the following form for the generator on quasi-observables

$$
\begin{aligned}
&(\widehat{L}G)(\eta) \\
&= \sum_{\xi\subset\eta}\sum_{x\in\xi}\int_{\mathbb{R}^d} G(\xi\cup y)e_\lambda(e^{-\phi(y-\cdot)}-1;\eta\backslash\xi)e^{-E(y,\xi)}a^+(x-y)\mathrm{d}y \\
&\quad+\sum_{\xi\subset\eta}\sum_{x\in\eta\backslash\xi}\int_{\mathbb{R}^d} G(\xi\cup y)e^{-E(y,\xi)}e_\lambda(e^{-\phi(y-\cdot)}-1;\eta\backslash\xi\backslash x)a^+(x-y)e^{-\phi(x-y)}\mathrm{d}y.
\end{aligned}
$$

Likewise we obtain for L^Δ on correlation functions

$$
\begin{aligned}
&(L^\Delta k)(\eta) \\
&= \sum_{x\in\eta}\sum_{y\in\eta\backslash x} e^{-E(x,\eta\backslash x)}a^+(x-y)\int_{\Gamma_0} e_\lambda(e^{-\phi(x-\cdot)}-1;\xi)k(\eta\cup\xi\backslash x)\mathrm{d}\lambda(\xi) \\
&\quad+\sum_{x\in\eta} e^{-E(x,\eta\backslash x)}\int_{\Gamma_0}\int_{\mathbb{R}^d} e_\lambda(e^{-\phi(x-\cdot)}-1;\xi)a^+(x-y)e^{-\phi(x-y)}k(\eta\cup\xi\backslash x\cup y)\mathrm{d}y\mathrm{d}\lambda(\xi).
\end{aligned}
$$

Hence M_α is given by

$$
\begin{aligned}
M_\alpha(\eta) = e^{-\alpha}\beta(\alpha)&\sum_{x\in\eta}\sum_{y\in\eta\backslash x} a^+(x-y)e^{-E(x,\eta\backslash x)} \\
&+\beta(\alpha)\langle a^+e^{-\phi}\rangle\sum_{x\in\eta} e^{-E(x,\eta\backslash x)}
\end{aligned}
$$

with $\beta(\alpha) = \exp\Big(e^\alpha \int_{\mathbb{R}^d} |1-e^{-\phi(x)}|\mathrm{d}x\Big)$. Rescaling the interactions, i.e. $a^+ \to \varepsilon a^+$, $\phi\to\varepsilon\phi$, putting $L \to \frac{1}{\varepsilon}L_\varepsilon$ and rescaling both operators we arrive at

$$
\begin{aligned}
&(\widehat{L}_{\varepsilon,ren}G)(\eta) \\
&= \varepsilon\sum_{\xi\subset\eta}\sum_{x\in\xi}\int_{\mathbb{R}^d} G(\xi\cup y)e_\lambda\left(\frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon};\eta\backslash\xi\right)e^{-\varepsilon E(y,\xi)}a^+(x-y)dy \\
&\quad+\sum_{\xi\subset\eta}\sum_{x\in\eta\backslash\xi}\int_{\mathbb{R}^d} G(\xi\cup y)e^{-\varepsilon E(y,\xi)}e_\lambda\left(\frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon};\eta\backslash\xi\backslash x\right)a^+(x-y)e^{-\varepsilon\phi(x-y)}dy.
\end{aligned}
$$

and

$$
\begin{aligned}
&(L^\Delta_{\varepsilon,ren}k)(\eta) \\
&= \varepsilon\sum_{x\in\eta}\sum_{y\in\eta\backslash x} e^{-\varepsilon E(x,\eta\backslash x)}a^+(x-y)\int_{\Gamma_0} e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\xi\right)k(\eta\cup\xi\backslash x)\mathrm{d}\lambda(\xi) \\
&\quad+\sum_{x\in\eta} e^{-\varepsilon E(x,\eta\backslash x)}\int_{\Gamma_0}\int_{\mathbb{R}^d} e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\xi\right)a^+(x-y)e^{-\varepsilon\phi(x-y)} \\
&\qquad\qquad\times k(\eta\cup\xi\backslash x\cup y)dy\mathrm{d}\lambda(\xi),
\end{aligned}
$$

which shows

$$N_\alpha(\eta) = e^{-\alpha}\exp(e^\alpha\langle\phi\rangle)\sum_{x\in\eta}\sum_{y\in\eta\backslash x} a^+(x-y) + \exp(e^\alpha\langle\phi\rangle)\langle a^+\rangle|\eta|.$$

The limiting operators as $\varepsilon\to 0$ are given by

$$(\widehat{L}_V G)(\eta) = \sum_{\xi\subset\eta}\sum_{x\in\eta\backslash\xi}\int_{\mathbb{R}^d} G(\xi\cup y)e_\lambda(-\phi(y-\cdot);\eta\backslash\xi\backslash x)a^+(x-y)\mathrm{d}y.$$

and

$$(L_V^\Delta k)(\eta) = \sum_{x\in\eta}\int_{\Gamma_0}\int_{\mathbb{R}^d} e_\lambda(-\phi(x-\cdot);\xi)a^+(x-y)k(\eta\cup\xi\backslash x\cup y)\mathrm{d}y\mathrm{d}\lambda(\xi)$$

and so $N_\alpha^V(\eta) = \langle a^+\rangle\exp(e^\alpha\langle\phi\rangle)|\eta|$. Therefore the kinetic description is given by

$$\frac{\partial\rho_t}{\partial t}(x) = (a^+ * \rho_t)(x)e^{-(\phi*\rho_t)(x)}.$$

Example 9. (fecundity) Let us take a look on the birth dynamics with fecundity. Here each cell at $x\in\gamma$ will produce new cells according to the distribution $a^+(x-y)dy$, whereas the intensity is dumped by a factor $e^{-E(x,\gamma\backslash x)}$. The Markov generator is given by

$$(LF)(\gamma) = \sum_{x\in\gamma} e^{-E(x,\gamma\backslash x)}\int_{\mathbb{R}^d} a^+(x-y)(F(\gamma\cup y)-F(\gamma))dy,$$

where $0\le a^+\in L^1(\mathbb{R}^d)$ is symmetric. Calculations yield the following form for the generator on quasi-observables

$$\begin{aligned}(\widehat{L}G)(\eta) = &\sum_{\xi\subset\eta}\sum_{x\in\eta\backslash\xi} e^{-E(x,\xi)}e_\lambda(e^{-\phi(x-\cdot)}-1;\eta\backslash\xi\backslash x)\int_{\mathbb{R}^d} G(\xi\cup y)a^+(x-y)\mathrm{d}y\\ &+\sum_{\xi\subset\eta}\sum_{x\in\xi} e^{-E(x,\xi\backslash x)}e_\lambda(e^{-\phi(x-\cdot)}-1;\eta\backslash\xi)\int_{\mathbb{R}^d} G(\xi\cup y)a^+(x-y)\mathrm{d}y.\end{aligned}$$

Likewise we obtain for L^Δ on correlation functions

$$\begin{aligned}&(L^\Delta k)(\eta)\\ &=\sum_{x\in\eta}\int_{\Gamma_0}\int_{\mathbb{R}^d} e^{-E(y,\xi\backslash x)}a^+(x-y)e_\lambda(e^{-\phi(y-\cdot)}-1;\xi)k(\eta\cup\xi\backslash x\cup y)\mathrm{d}y\mathrm{d}\lambda(\xi)\\ &+\sum_{x\in\eta}\sum_{y\in\eta\backslash x} e^{-E(x,\eta\backslash x)}e^{\phi(x-y)}a^+(x-y)\int_{\Gamma_0} e_\lambda(e^{-\phi(x-\cdot)}-1;\eta)k(\eta\cup\xi\backslash y)\mathrm{d}\lambda(\xi)\end{aligned}$$

and hence

$$M_\alpha(\eta) = e^{-\alpha}\beta(\alpha)\sum_{x\in\eta}\sum_{y\in\eta\backslash x} a^+(x-y)e^{\phi(x-y)}e^{-E(x,\eta\backslash x)} + \beta(\alpha)\langle a^+e^\phi\rangle|\eta|.$$

Rescaling the interactions, i.e. $a^+ \to \varepsilon a^+$, $\phi \to \varepsilon\phi$, putting $L \to \dfrac{1}{\varepsilon}L_\varepsilon$ and rescaling both operators we arrive at

$$
\begin{aligned}
&(\widehat{L}_{\varepsilon,ren}G)(\eta)\\
&= \sum_{\xi\subset\eta}\sum_{x\in\eta\setminus\xi} e^{-\varepsilon E(x,\xi)} e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\eta\setminus\xi\setminus x\right)\int_{\mathbb{R}^d} G(\xi\cup y)a^+(x-y)\mathrm{d}y\\
&\quad + \varepsilon\sum_{\xi\subset\eta}\sum_{x\in\xi} e^{-\varepsilon E(x,\xi\setminus x)} e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\eta\setminus\xi\right)\int_{\mathbb{R}^d} G(\xi\cup y)a^+(x-y)\mathrm{d}y.
\end{aligned}
$$

Likewise we obtain for L^Δ on correlation functions

$$
\begin{aligned}
&(L^\Delta_{\varepsilon,ren}k)(\eta)\\
&= \sum_{x\in\eta}\int_{\Gamma_0}\int_{\mathbb{R}^d} e^{-\varepsilon E(y,\xi\setminus x)}a^+(x-y)e_\lambda\left(\frac{e^{-\varepsilon\phi(y-\cdot)}-1}{\varepsilon};\xi\right)k(\eta\cup\xi\setminus x\cup y)\mathrm{d}y\mathrm{d}\lambda(\xi)\\
&\quad +\varepsilon\sum_{x\in\eta}\sum_{y\in\eta\setminus x} e^{-\varepsilon E(x,\eta\setminus x)}e^{\varepsilon\phi(x-y)}a^+(x-y)\\
&\qquad\qquad \times\int_{\Gamma_0} e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\eta\right)k(\eta\cup\xi\setminus y)\mathrm{d}\lambda(\xi)
\end{aligned}
$$

so

$$
N_\alpha(\eta) = e^{-\alpha}\exp(e^\alpha\langle\phi\rangle)\sum_{x\in\eta}\sum_{y\in\eta\setminus x} a^+(x-y)e^{\phi(x-y)} + \langle a^+\rangle\exp(e^\alpha\langle\phi\rangle)|\eta|.
$$

The limiting operators as $\varepsilon\to 0$ are given by

$$
(\widehat{L}_V G)(\eta) = \sum_{\xi\subset\eta}\sum_{x\in\eta\setminus\xi} e_\lambda(-\phi(x-\cdot);\eta\setminus\xi\setminus x)\int_{\mathbb{R}^d} G(\xi\cup y)a^+(x-y)\mathrm{d}y
$$

and

$$
(L^\Delta_V k)(\eta) = \sum_{x\in\eta}\int_{\Gamma_0}\int_{\mathbb{R}^d} a^+(x-y)e_\lambda(-\phi(y-\cdot);\xi)k(\eta\cup\xi\setminus x\cup y)\mathrm{d}y\mathrm{d}\lambda(\xi)
$$

Therefore the kinetic description is given by

$$
\frac{\partial\rho_t}{\partial t}(x) = (a^+ * (e^{-\phi*\rho_t}\rho_t))(x).
$$

4.3 Moving cells

Here let us describe possible microscopic events, which lead to a motion of cells. The first model describes the most simple possibility.

Example 10 (free jumps)**.** The Markov generator of a system of free jumping cells is given by

$$
(LF)(\gamma) = \sum_{x\in\gamma}\int_{\mathbb{R}^d} c(x-y)(F(\gamma\setminus x\cup y) - F(\gamma))\mathrm{d}y,
$$

where $0 \le c \in L^1(\mathbb{R}^d)$ is symmetric. Here each cell jumps independently of the others according to a jump rate $\langle c \rangle$ and a probability distribution $\frac{1}{\langle c \rangle} c(x-y)\mathrm{d}y$, where $x \in \gamma$. The mechanism can be described on quasi-observables via

$$(\widehat{L}G)(\eta) = \sum_{x\in\eta} \int_{\mathbb{R}^d} c(x-y)(G(\eta\backslash x \cup y) - G(\eta))\mathrm{d}y$$

and L^Δ is given by the same formula. Since after scaling nothing is changed we obtain immediately for the kinetic description

$$\frac{\partial \rho_t}{\partial t}(x) = (c * \rho_t)(x) - \rho_t(x) = \int_{\mathbb{R}^d} c(x-y)(\rho_t(y) - \rho_t(x))\mathrm{d}x,$$

which is the equation describing a random walk in continuous time. Then functions M_α, N_α and N_α^V can be chosen as $\langle c \rangle |\eta|$.

Another possibility of describing the free motion of particles is given by the next example.

Example 11 (free diffusion)**.** Let the Markov generator be given by

$$(LF)(\gamma) = \sum_{x\in\Gamma} (\Delta_x F)(\gamma).$$

Here each cell undergoes a free diffusion independent of all other cells. Rewriting this operators to quasi-observables we arrive at

$$(\widehat{L}G)(\eta) = \sum_{x\in\eta} (\Delta_x G)(\eta)$$

and likewise the expression for L^Δ is given by the same formula. Since scaling will not change the operators we arrive at the kinetic description

$$\frac{\partial \rho_t}{\partial t}(x) = \Delta \rho_t(x).$$

Example 12 (jumps with additive intensity)**.** The Markov generator for jumping cells, with density dependent intensity is given by

$$(LF)(\gamma) = \sum_{x\in\gamma} \sum_{y\in\gamma\backslash x} b(x-y) \int_{\mathbb{R}^d} c(x-z)(F(\gamma\backslash x \cup z) - F(\gamma))\mathrm{d}z,$$

where $0 \le c, b \in L^1(\mathbb{R}^d)$ are symmetric. Each cell at $x \in \gamma$ will jump with intensity $\langle c \rangle \sum\limits_{y\in\gamma\backslash x} b(x-y)$ and the position is determined by the distribution $\frac{1}{\langle c \rangle} c(x-z)\mathrm{d}z$. The description via quasi-observables will give

$$\begin{aligned}(\widehat{L}G)(\eta) = &\sum_{x\in\eta} \sum_{y\in\eta\backslash x} b(x-y) \int_{\mathbb{R}^d} c(x-z)(G(\eta\backslash x\backslash y \cup z) - G(\eta\backslash y))\mathrm{d}z \\ &+ \sum_{x\in\eta} \sum_{y\in\eta\backslash x} b(x-y) \int_{\mathbb{R}^d} c(x-z)(G(\eta\backslash x \cup z) - G(\eta))\mathrm{d}z\end{aligned}$$

and similarly for correlation functions

$$(L^{\Delta}k)(\eta) = \sum_{x\in\eta}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} c(x-y)b(y-z)(k(\eta\backslash x\cup y\cup z) - k(\eta\cup z))\mathrm{d}y\mathrm{d}z$$
$$+\sum_{x\in\eta}\sum_{y\in\eta\backslash x} b(x-y)\int_{\mathbb{R}^d} a(x-z)(k(\eta\backslash x\cup z) - k(\eta))\mathrm{d}z.$$

The rigorous derivation of the kinetic description was already done in [4]. Scaling the potential as $b\to\varepsilon b$ and rescaling, we see that only the last terms in $\widehat{L}$ and L^{Δ} will be multiplied by ε. Hence after limit transition $\varepsilon\to 0$ we arrive at

$$(\widehat{L}_V G)(\eta) = \sum_{x\in\eta}\sum_{y\in\eta\backslash x} b(x-y)\int_{\mathbb{R}^d} c(x-z)(G(\eta\backslash x\backslash y\cup z) - G(\eta\backslash y))\mathrm{d}z$$

and

$$(L_V^{\Delta}k)(\eta) = \sum_{x\in\eta}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} c(x-y)b(y-z)(k(\eta\backslash x\cup y\cup z) - k(\eta\cup z))\mathrm{d}y\mathrm{d}z,$$

which yields the kinetic description

$$\frac{\partial\rho_t}{\partial t}(x) = (c*((b*\rho_t)\cdot\rho_t))(x) - (c*(b*\rho_t))(x)\rho_t(x).$$

Example 13 (density dependent jumps)**.** Let $0\leq\phi,\psi,c\in L^1(\mathbb{R}^d)$ symmetric with $\phi\in L^{\infty}(\mathbb{R}^d)$, set $E_{\phi}(x,\gamma\backslash x) = \sum\limits_{y\in\gamma\backslash x}\phi(x-y)$ and likewise $E_{\psi}(y,\gamma) = \sum\limits_{x\in\gamma}\psi(x-y)\geq 0$. Define the formal Markov generator

$$(LF)(\gamma) = \sum_{x\in\gamma} e^{E_{\phi}(x,\gamma\backslash x)}\int_{\mathbb{R}^d} e^{-E_{\psi}(y,\gamma)}c(x-y)(F(\gamma\backslash x\cup y) - F(\gamma))\mathrm{d}y.$$

Here each cell located at $x\in\gamma$ will have a high intensity to jump, if there are many other cells around and due to $e^{-E_{\psi}(y,\gamma)}$ it will prefer to jump in regions, which have a small density of cells. Let us compute the the operator $\widehat{L}$. For $G\in B_{bs}(\Gamma_0)$, $x\in\gamma$ and $y\notin\gamma$ we obtain

$$\begin{aligned}
&(KG)(\gamma\backslash x\cup y) - (KG)(\gamma)\\
&= \sum_{\eta\Subset\gamma\backslash x\cup y} G(\eta) - \sum_{\eta\Subset\gamma} G(\eta)\\
&= \sum_{\eta\Subset\gamma\backslash x} G(\eta) + \sum_{\eta\Subset\gamma\backslash x} G(\eta\cup y) - \sum_{\eta\Subset\gamma\backslash x} G(\eta) - \sum_{\eta\Subset\gamma\backslash x} G(\eta\cup x)\\
&= \sum_{\eta\Subset\gamma\backslash x}(G(\eta\cup y) - G(\eta\cup x)).
\end{aligned}$$

Therefore using (31) and (32) we get

$$\sum_{x\in\gamma} e^{E_\phi(x,\gamma\setminus x)} \int_{\mathbb{R}^d} e^{-E_\psi(y,\gamma\setminus x)} e^{-\psi(x-y)} c(x-y) \sum_{\eta\Subset\gamma\setminus x} (G(\eta\cup y) - G(\eta\cup x))$$

$$= \sum_{x\in\gamma} \int_{\mathbb{R}^d} e^{-\psi(x-y)} c(x-y) K e_\lambda(f(x,y))(\gamma\setminus x) \sum_{\eta\Subset\gamma\setminus x} (G(\eta\cup y) - G(\eta\cup x)) \mathrm{d}y$$

$$= \sum_{x\in\gamma} \sum_{\eta\Subset\gamma\setminus x} \int_{\mathbb{R}^d} e_\lambda\left(f(x,y)\right) \star (G(\cdot\cup y) - G(\cdot\cup x))(\eta) e^{-\psi(x-y)} c(x-y)\mathrm{d}y$$

$$= \sum_{\eta\Subset\gamma} \sum_{x\in\eta} \int_{\mathbb{R}^d} e_\lambda\left(f(x,y)\right) \star (G(\cdot\cup y) - G(\cdot\cup x))(\eta\setminus x) e^{-\psi(x-y)} c(x-y)\mathrm{d}y$$

with $f(x,y;w) = e^{\phi(x-w)-\psi(y-w)} - 1$. Again using the definition (31) we get

$$(\widehat{L}G)(\eta)$$
$$= \sum_{x\in\eta} \int_{\mathbb{R}^d} e_\lambda\left(f(x,y)\right) \star (G(\cdot\cup y) - G(\cdot\cup x))(\eta\setminus x) e^{-\psi(x-y)} c(x-y)\mathrm{d}y$$

$$= \sum_{\xi\subset\eta} \sum_{x\in\eta} e^{E_\phi(x,\xi\setminus x)} \int_{\mathbb{R}^d} e^{-E_\psi(y,\xi)} c(x-y) e_\lambda\left(f(x,y);\eta\setminus\xi\right) (G(\xi\cup y) - G(\xi\cup x))\mathrm{d}y.$$

This yields the following formula

$$\sum_{\xi\subset\eta} \sum_{x\in\xi} e^{E_\phi(x,\xi\setminus x)} \int_{\mathbb{R}^d} e^{-E_\psi(y,\xi)} c(x-y) e_\lambda\left(f(x,y);\eta\setminus\xi\right) \left(G(\xi\setminus x\cup y) - G(\xi)\right)\mathrm{d}y.$$

Let us show that the function M_α can be chosen by

$$M_\alpha(\eta) = e^{e^\alpha\kappa} \sum_{x\in\eta} e^{-E_\psi(x,\eta\setminus x)} \int_{\mathbb{R}^d} c(x-y) e^{-\phi(x-y)-\psi(x-y)} e^{E_\phi(y,\eta)}\mathrm{d}y$$
$$+ e^{e^\alpha\kappa} \sum_{x\in\eta} e^{E_\phi(x,\eta\setminus x)} \int_{\mathbb{R}^d} c(x-y) e^{-E_\psi(y,\eta)}\mathrm{d}y,$$

where $\kappa = e^{\|\phi\|_{L^\infty}}\langle\phi\rangle$. So let $G\in D(M_\alpha)$ and note that

$$\int_{\mathbb{R}^d} f(x,y;w)\mathrm{d}w = \int_{\mathbb{R}^d} \left(e^{\phi(x-w)-\psi(y-w)} - 1\right)\mathrm{d}w \le e^{\|\phi\|_{L^\infty}}\langle\phi\rangle = \kappa.$$

Now using the formulas from Lemma 4.2 we arrive at

$$e^\alpha \int_{\Gamma_0^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{E_\phi(x,\xi)} e^{-E_\psi(y,\xi\cup x)} c(x-y) e_\lambda\left(|f(x,y)|;\eta\right)$$
$$\times |G(\xi\cup y)| e^{\alpha|\xi|} e^{\alpha|\eta|}\mathrm{d}y\mathrm{d}x\mathrm{d}\lambda(\eta,\xi)$$
$$\le e^\alpha e^{e^\alpha\kappa} \int_{\Gamma_0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\alpha|\xi|} e^{-E_\psi(y,\xi\cup x)} e^{E_\phi(x,\xi)} c(x-y)|G(\xi\cup y)|\mathrm{d}x\mathrm{d}y\mathrm{d}\lambda(\xi)$$

$$= e^{e^{\alpha}\kappa} \int_{\Gamma_0} |G(\xi)| \sum_{y\in\xi} e^{-E_{\psi}(y,\xi\backslash y)} \int_{\mathbb{R}^d} c(x-y) e^{-\psi(x-y)-\phi(x-y)} e^{E_{\phi}(x,\xi)} e^{\alpha|\xi|} \mathrm{d}x \mathrm{d}\lambda(\xi)$$

and for the second part of $\widehat{L}$ at

$$\int_{\Gamma_0^2} \mathrm{d}\lambda^2(\eta,\xi) e^{\alpha|\eta|} e^{\alpha|\xi|} \sum_{x\in\eta} \int_{\mathbb{R}^d} dy c(x-y) e^{E_{\phi}(x,\xi\backslash x)} e^{-E_{\psi}(y,\xi)} e_{\lambda}\left(|f(x,y)|;\eta\right) |G(\xi)|$$

$$\leq e^{e^{\alpha}\kappa} \int_{\Gamma_0} \Big(\sum_{x\in\xi} e^{E_{\phi}(x,\xi\backslash x)} \int_{\mathbb{R}^d} c(x-y) e^{-E_{\psi}(y,\xi)} dy \Big) |G(\xi)| e^{\alpha|\xi|} \mathrm{d}\lambda(\xi).$$

Since $M_{\alpha}(\eta) \leq 2\langle c\rangle e^{e^{\alpha}\kappa} |\eta| e^{\|\phi\|_{L^{\infty}}|\eta|}$ we get for $\alpha - \alpha' > \|\phi\|_{L^{\infty}}$ that $\widehat{L} \in L(\mathbb{B}_{\alpha}, \mathbb{B}_{\alpha'})$ and

$$\|\widehat{L}\|_{\alpha\alpha'} \leq \frac{2\langle c\rangle e^{e^{\alpha'}\kappa}}{e(\alpha - \alpha' - \|\phi\|_{L^{\infty}})}.$$

Turning now to correlation functions, the action of the operator L^{Δ} is given by

$$\sum_{y\in\eta} \int_{\Gamma_0} \mathrm{d}\lambda(\xi) \int_{\mathbb{R}^d} \mathrm{d}x c(x-y) e^{E_{\phi}(x,\eta\backslash y)} e^{-E_{\psi}(y,\eta\cup x\backslash y)} e_{\lambda}\left(f(x,y);\xi\right) k(\eta\cup\xi\backslash y\cup x)$$

$$- \sum_{y\in\eta} \int_{\Gamma_0} \mathrm{d}\lambda(\xi) \int_{\mathbb{R}^d} \mathrm{d}y c(x-y) e^{E_{\phi}(y,\eta\backslash y)} e^{-E_{\psi}(y,\eta)} e_{\lambda}\left(f(x,y);\eta\right) k(\eta\cup\xi)$$

and similarly L^{Δ} is a bounded linear operator in $L(\mathbb{B}^*_{\alpha'}, \mathbb{B}^*_{\alpha})$ for each $\alpha - \alpha' > \|\phi\|_{L^{\infty}}$. In order to see this let $G \in B_{bs}(\Gamma_0)$ and $k \in \mathbb{B}_{\alpha}$ for some $\alpha \geq 0$, then for the first term we get

$$\int_{\Gamma_0} k(\eta) \sum_{\xi\subset\eta} \sum_{x\in\xi} \int_{\mathbb{R}^d} e^{E_{\phi}(x,\xi\backslash x)} e^{-E_{\psi}(y,\xi)} c(x-y) e_{\lambda}\left(f(x,y);\eta\right) G(\xi\backslash x\cup y) \mathrm{d}y \mathrm{d}\lambda(\eta)$$

$$= \int_{\Gamma_0^2} k(\eta\cup\xi) \sum_{x\in\xi} \int_{\mathbb{R}^d} c(x-y) e^{E_{\phi}(x,\xi\backslash x)} e^{-E_{\psi}(y,\xi)} e_{\lambda}\left(f(x,y);\eta\right) G(\xi\backslash x\cup y) \mathrm{d}y \mathrm{d}\lambda(\eta,\xi)$$

$$= \int_{\Gamma_0^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k(\eta\cup\xi\cup x) G(\xi\cup y) c(x-y) e^{E_{\phi}(x,\xi)} e^{-E_{\psi}(y,\xi\cup x\backslash y)} e_{\lambda}\left(f(x,y);\eta\right) \mathrm{d}x \mathrm{d}y \mathrm{d}\lambda(\eta,\xi)$$

$$= \int_{\Gamma_0^2} \sum_{y\in\xi} \int_{\mathbb{R}^d} k(\eta\cup\xi\cup x\backslash y) G(\xi) c(x-y) e^{E_{\phi}(x,\xi\backslash y)} e^{-E_{\psi}(y,\xi\cup x\backslash y)} e_{\lambda}\left(f(x,y);\eta\right) \mathrm{d}x \mathrm{d}\lambda(\eta,\xi).$$

For the second term we have

$$\int_{\Gamma_0} k(\eta) \sum_{\xi\subset\eta} \sum_{x\in\xi} \int_{\mathbb{R}^d} c(x-y) e^{E_{\phi}(x,\xi\backslash x)} e^{-E_{\psi}(y,\xi)} e_{\lambda}\left(f(x,y);\eta\backslash\xi\right) G(\xi) \mathrm{d}y \mathrm{d}\lambda(\eta)$$

$$= \int_{\Gamma_0^2} k(\eta\cup\xi) \sum_{x\in\xi} \int_{\mathbb{R}^d} c(x-y) e^{E_{\phi}(x,\xi\backslash x)} e^{-E_{\psi}(y,\xi)} e_{\lambda}\left(f(x,y);\eta\right) G(\xi) \mathrm{d}y \mathrm{d}\lambda(\eta,\xi).$$

Here we have to rescale the potentials $\phi \to \varepsilon\phi$ and $\psi \to \varepsilon\psi$. Since we are interested in the limit $\varepsilon \to 0$, we will restrict the range of ε to $(0,1]$. The rescaled operator will have the

$$\sum_{\xi\subset\eta}\sum_{x\in\xi} e^{\varepsilon E_\phi(x,\xi\backslash x)} \int_{\mathbb{R}^d} e^{-\varepsilon E_\psi(y,\xi)} c(x-y) \times e_\lambda\left(e^{\varepsilon\phi(x-\cdot)-\varepsilon\psi(y-\cdot)} - 1; \eta\backslash\xi\right)(G(\xi\backslash x\cup y) - G(\xi))\mathrm{d}y.$$

In order to get the normalized expression we have again to consider the composition $\widehat{L}_\varepsilon^{ren} = R_{\varepsilon^{-1}}\widehat{L}_\varepsilon R_\varepsilon$, this leads to the following expression for $\widehat{L}_\varepsilon^{ren}$

$$\sum_{\xi\subset\eta}\sum_{x\in\xi} e^{\varepsilon E_\phi(x,\xi\backslash x)} \int_{\mathbb{R}^d} e^{-\varepsilon E_\psi(y,\xi)} c(x-y) e_\lambda\left(f_\varepsilon(x,y);\eta\backslash\xi\right)(G(\xi\backslash x\cup y) - G(\xi))dy$$

with $f_\varepsilon(x,y;w) = \dfrac{e^{\varepsilon\phi(x-w)-\varepsilon\psi(y-w)} - 1}{\varepsilon}$. For each fixed $\eta\in\Gamma_0$ this expression converges to

$$(\widehat{L}_V G)(\eta) = \sum_{\xi\subset\eta}\sum_{x\in\xi}\int_{\mathbb{R}^d} c(x-y) e_\lambda\left(\phi(x-\cdot)-\psi(y-\cdot);\eta\backslash\xi\right)(G(\xi\backslash x\cup y) - G(\xi))\mathrm{d}y.$$

Lemma 4.3. *For $\widehat{L}_\varepsilon^{ren}$ the corresponding function N_α is given by*

$$N_\alpha(\eta) = e^{e^\alpha\kappa}\sum_{x\in\eta}\int_{\mathbb{R}^d} c(x-y)e^{E_\phi(y,\eta)}\mathrm{d}y + \langle c\rangle e^{e^\alpha\kappa}\sum_{x\in\eta} e^{E_\phi(x,\eta\backslash x)}$$

and $N_\alpha^V(\eta) = 2\langle c\rangle\exp(e^\alpha(\langle\phi\rangle + \langle\psi\rangle))|\eta|$. Moreover for each $\alpha-\alpha' > \|\phi\|_{L^\infty}$ the estimate holds:

$$\|\widehat{L}_{\varepsilon,ren}\|_{\alpha\alpha'} \le \frac{2\langle c\rangle e^{e^{\alpha'}\kappa}}{e(\alpha-\alpha'-\varepsilon\|\phi\|_{L^\infty})} \le \frac{2\langle c\rangle e^{e^{\alpha'}\kappa}}{e(\alpha-\alpha'-\|\phi\|_{L^\infty})},$$
$$\|\widehat{L}_V\|_{\alpha\alpha'} \le \frac{2\langle c\rangle\exp(e^{\alpha'}(\langle\phi\rangle+\langle\psi\rangle))}{e(\alpha-\alpha')}$$

for all $\alpha' < \alpha$.

Proof. For this purpose we have first to estimate $f_\varepsilon(x,y)$ by

$$|f_\varepsilon(x,y;w)| \le e^{\|\phi\|_{L^\infty}}\phi(x-w)$$

for almost all $w\in\mathbb{R}^d$ and afterwards to use

$$N_\alpha(\eta) \le 2\langle ce^{e^\alpha\kappa}\rangle|\eta|e^{\|\phi\|_{L^\infty}|\eta|}.$$ □

Clearly we have $D(M_\alpha)\subset D(N_\alpha)\subset D(N_\alpha^V)$.

Theorem 4.4. *Let $G\in D(N_\alpha)$ such that $|\eta|^2 e^{\|\phi\|_{L^\infty}|\eta|}G\in\mathbb{B}_\alpha$, then*

$$\widehat{L}_\varepsilon^{ren}G \to \widehat{L}_V G, \quad \varepsilon\to 0$$

in $\mathbb{B}_\alpha$. In addition the renormalized operator for quasi-observables converges in the uniform operator topology of $L(\mathbb{B}_\alpha,\mathbb{B}_{\alpha'})$, i.e. the following holds

$$\|\widehat{L}_V - \widehat{L}_\varepsilon^{ren}\|_{\alpha\alpha'} \to 0, \quad \varepsilon\to 0.$$

Proof. Fix $\alpha' < \alpha$ and $G \in D(M_{\alpha'})$ such that $|\eta|^2 e^{\|\phi\|_{L^\infty}|\eta|} G \in \mathbb{B}_{\alpha'}$. Let us divide $\widehat{L}_V$ and $\widehat{L}_\varepsilon^{ren}$ in two parts according to $G(\xi \backslash x \cup y)$ and $G(\xi)$ and investigate their differences separately. Starting with the term containing $G(\xi \backslash x \cup y)$ we obtain for the difference

$$
\begin{aligned}
& e^{\alpha'} \int_{\Gamma_0^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{d}\lambda(\eta,\xi) \mathrm{d}x \mathrm{d}y e^{\alpha'|\eta|} e^{\alpha'|\xi|} c(x-y) |G(\xi \cup y)| \\
& \quad \times \left| e^{\varepsilon E_\phi(x,\xi)} e^{-\varepsilon E_\psi(y,\xi\cup x)} e_\lambda\left(f_\varepsilon(x,y);\eta\right) - e_\lambda(\phi(x-\cdot) - \psi(y-\cdot);\eta) \right| \\
&= \int_{\Gamma_0^2} \mathrm{d}\lambda(\eta,\xi) e^{\alpha'|\eta|} e^{\alpha'|\xi|} |G(\xi)| \sum_{y\in\xi} \int_{\mathbb{R}^d} \mathrm{d}x c(x-y) \\
& \quad \times \left| e^{\varepsilon E_\phi(x,\xi\backslash y)} e^{-\varepsilon E_\psi(y,\xi\backslash y\cup x)} e_\lambda\left(f_\varepsilon(x,y);\eta\right) - e_\lambda(\phi(x-\cdot) - \psi(y-\cdot);\eta) \right|.
\end{aligned}
$$

Using $|f_\varepsilon(x,y;w)| \leq e^{\|\phi\|_{L^\infty}} \phi(x-w)$,

$$
|f_\varepsilon(x,y;w) - (\phi(x-w) - \psi(y-w))| \leq e^r |\phi(x-w) - \psi(y-w)|
$$

where $r = \|\phi\|_{L^\infty} + \|\psi\|_{L^\infty}$ and

$$
|e^{\varepsilon E_\phi(x,\xi\backslash y)} - 1| \leq \varepsilon E_\phi(x,\xi\backslash y) e^{\|\phi\|_{L^\infty}|\xi\backslash y|} \leq \varepsilon \|\phi\|_{L^\infty} |\xi| e^{\|\phi\|_{L^\infty}|\xi|}
$$

the modulus in the integral can be estimated by

$$
\begin{aligned}
& \left| e^{\varepsilon E_\phi(x,\xi\backslash y)} e^{-\varepsilon E_\psi(y,\xi\backslash y\cup x)} e_\lambda\left(f_\varepsilon(x,y);\eta\right) - e_\lambda(\phi(x-\cdot) - \psi(y-\cdot);\eta) \right| \\
&\leq |e^{\varepsilon E_\phi(x,\xi\backslash y)} - 1| \left| e^{-\varepsilon E_\psi(y,\xi\backslash y\cup x)} e_\lambda\left(f_\varepsilon(x,y);\eta\right) \right| \\
&\quad + |e^{-\varepsilon E_\psi(y,\xi\backslash y\cup x)} - 1| \, |e_\lambda\left(|f_\varepsilon(x,y)|;\eta\right)| \\
&\quad + |e_\lambda\left(|f_\varepsilon(x,y)|;\eta\right) - e_\lambda(\phi(x-\cdot) - \psi(y-\cdot);\eta)| \\
&\leq \varepsilon \|\phi\|_{L^\infty} |\xi| e^{\varepsilon\|\phi\|_{L^\infty}|\xi|} e_\lambda\left(e^{\|\phi\|_{L^\infty}} \phi(x-\cdot)|;\eta\right) \\
&\quad + \varepsilon (E_\psi(y,\xi\backslash y) + \psi(x-y)) e_\lambda\left(e^{\|\phi\|_{L^\infty}} \phi(x-\cdot)|;\eta\right) \\
&\quad + \varepsilon e^r \sum_{w\in\eta} |\phi(x-w) - \psi(y-w)| e_\lambda(e^r|\phi(x-\cdot) - \psi(y-\cdot)|;\eta\backslash w).
\end{aligned}
$$

Invoking this in previous estimations one obtains with some generic constant $C > 0$ independent of α', α, and ε

$$
\begin{aligned}
& \leq \varepsilon C e^{e^{\alpha'+r}\kappa} \int_{\Gamma_0} e^{\alpha'|\xi|} |G(\xi)| \sum_{y\in\xi} \int_{\mathbb{R}^d} c(x-y) \mathrm{d}x |\xi| e^{\varepsilon\|\phi\|_{L^\infty}|\xi|} \mathrm{d}\lambda(\xi) \\
& \quad + \varepsilon C e^{e^{\alpha'+r}\kappa} \int_{\Gamma_0} e^{\alpha'|\xi|} |G(\xi)| \sum_{y\in\xi} (\langle c\rangle + \langle c\psi\rangle) E_\psi(y,\xi\backslash y) \\
& \quad + \varepsilon C e^{\alpha'} e^{e^{\alpha'+r}\kappa} \int_{\Gamma_0} e^{\alpha'|\xi|} |G(\xi)| |\xi| \mathrm{d}\lambda(\xi) \\
& \leq \varepsilon C e^{e^{\alpha'+r}\kappa} \int_{\Gamma_0} |G(\xi)| \left(|\xi|^2 e^{\varepsilon\|\phi\|_{L^\infty}|\xi|} e^{\alpha'} + 1 + |\xi| + |\xi|^2 \right) e^{\alpha'|\xi|} \mathrm{d}\lambda(\xi)
\end{aligned}
$$

This shows the first part. For the second part take $0 < \varepsilon < \frac{\alpha - \alpha'}{\|\phi\|_{L^\infty}}$, then $\mathbb{B}_\alpha \subset D(N_{\alpha'})$ and for $G \in \mathbb{B}_\alpha$ above integral is bounded by

$$\leq \varepsilon C \left(\frac{4}{e^2(\alpha - \alpha' - \|\phi\|_{L^\infty})^2} + \frac{4}{e^2(\alpha - \alpha')^2} + \frac{1}{e(\alpha - \alpha')} + 1 \right) \|G\|_\alpha e^{e^{\alpha' + r}\kappa}$$

which shows the assertion for the terms containing $G(\xi \cup y \backslash x)$. Similarly the differences including $G(\xi)$ can be estimated. □

For correlation functions the rescaled operator has the form

$$\begin{aligned}
&(L_\varepsilon^\Delta k)(\eta) \\
&= \sum_{y \in \eta} \int_{\Gamma_0} \mathrm{d}\lambda(\xi) \int_{\mathbb{R}^d} \mathrm{d}x c(x-y) e^{\varepsilon E_\phi(x, \eta \backslash y)} e^{-\varepsilon E_\psi(y, \eta \cup x \backslash y)} \\
&\qquad \times e_\lambda \left(e^{\varepsilon \phi(x - \cdot) - \varepsilon \psi(y - \cdot)} - 1; \xi \right) k(\eta \cup \xi \cup x \backslash y) \\
&\quad - \sum_{y \in \eta} \int_{\Gamma_0} \mathrm{d}\lambda(\xi) \int_{\mathbb{R}^d} \mathrm{d}x c(x-y) e^{\varepsilon E_\phi(y, \eta \backslash y)} e^{-\varepsilon E_\psi(x, \eta)} \\
&\qquad \times e_\lambda \left(e^{\varepsilon \phi(y - \cdot) - \varepsilon \psi(x - \cdot)} - 1; \xi \right) k(\eta \cup \xi).
\end{aligned}$$

Again computing the renormalized operator one gets similarly to the case for quasi-observables

$$\begin{aligned}
&(L_\varepsilon^{\Delta, ren} k)(\eta) \\
&= \sum_{y \in \eta} \int_{\Gamma_0} \mathrm{d}\lambda(\xi) \int_{\mathbb{R}^d} dx c(x-y) e^{\varepsilon E_\phi(x, \eta \backslash y)} e^{-\varepsilon E_\psi(y, \eta \cup x \backslash y)} e_\lambda \left(f_\varepsilon(x,y); \xi \right) k(\eta \cup \xi \cup x \backslash y) \\
&\quad - \sum_{y \in \eta} \int_{\Gamma_0} \mathrm{d}\lambda(\xi) \int_{\mathbb{R}^d} dx c(x-y) e^{\varepsilon E_\phi(y, \eta \backslash y)} e^{-\varepsilon E_\psi(x, \eta)} e_\lambda \left(f_\varepsilon(x,y); \xi \right) k(\eta \cup \xi)
\end{aligned}$$

and for each fixed $\eta \in \Gamma_0$ this operator converges to

$$\begin{aligned}
(L_V^\Delta k)(\eta) &= \sum_{y \in \eta} \int_{\Gamma_0} \mathrm{d}\lambda(\xi) \int_{\mathbb{R}^d} dx c(x-y) e_\lambda \left(\phi(x - \cdot) - \psi(y - \cdot); \xi \right) k(\eta \cup \xi \backslash y \cup x) \\
&\quad - \sum_{y \in \eta} \int_{\Gamma_0} \mathrm{d}\lambda(\xi) \int_{\mathbb{R}^d} dx c(x-y) e_\lambda \left(\phi(y - \cdot) - \psi(x - \cdot); \xi \right) k(\eta \cup \xi).
\end{aligned}$$

Lemma 4.5. *The renormalized operator on correlation functions converges in the uniform operator topology of $L(\mathbb{B}_{\alpha'}^*, \mathbb{B}_\alpha^*)$ to L_V^Δ, i.e. for all $\alpha' < \alpha$*

$$\|L_V^\Delta - L_\varepsilon^{\Delta, ren}\|_{\alpha'\alpha} \to 0, \quad \varepsilon \to 0.$$

Since $L_\varepsilon^{\Delta, ren}$ is dual to $\widehat{L}_\varepsilon^{ren}$ with respect to (42) the assertion follows from (45). Finally let us derive the kinetic description for this model. Therefore we have to compute $L_V^\Delta e_\lambda(\rho)$ for a function $0 \leq \rho \in L^\infty(\mathbb{R}^d)$. This expression is given by

$$
\begin{aligned}
&(L_V^{\Delta} e_\lambda)(\eta) \\
&= \sum_{y\in\eta} e_\lambda(\rho;\eta\backslash y) \int_{\Gamma_0} \mathrm{d}\lambda(\xi) \int_{\mathbb{R}^d} \mathrm{d}x c(x-y)\rho(x) e_\lambda(\phi(x-\cdot)-\psi(y-\cdot);\xi) e_\lambda(\rho;\xi) \\
&\quad - \sum_{y\in\eta} e_\lambda(\rho;\eta\backslash y)\rho(y) \int_{\Gamma_0} \mathrm{d}\lambda(\xi) \int_{\mathbb{R}^d} \mathrm{d}x c(x-y) e_\lambda(\phi(y-\cdot)-\psi(x-\cdot);\xi) e_\lambda(\rho;\xi) \\
&= \sum_{y\in\eta} e_\lambda(\rho;\eta\backslash y) \int_{\mathbb{R}^d} \mathrm{d}x c(x-y)\rho(x) \exp\Big(\int_{\mathbb{R}^d} \mathrm{d}w \,(\phi(x-w)-\psi(y-w))\,\rho(w)\Big) \\
&\quad - \sum_{y\in\eta} e_\lambda(\rho;\eta\backslash y)\rho(y) \int_{\mathbb{R}^d} \mathrm{d}x c(x-y) \exp\Big(\int_{\mathbb{R}^d} \mathrm{d}w \,(\phi(y-w)-\psi(x-w))\,\rho(w)\Big) \\
&= \sum_{y\in\eta} e_\lambda(\rho;\eta\backslash y) \left(e^{-(\psi*\rho)(y)}(c*\rho\cdot e^{\phi*\rho})(y) - \rho(y) e^{(\phi*\rho)(y)}(c*e^{-(\psi*\rho)})(y)\right)
\end{aligned}
$$

and since

$$\frac{\partial}{\partial t} e_\lambda(\rho_t;\eta) = \sum_{x\in\eta} e_\lambda(\rho_t;\eta\backslash y)\frac{\partial\rho_t}{\partial t}(y)$$

we obtain for the contribution of the jumps to the mesoscopic equation the terms

$$\left(c*\left(\rho_t e^{\phi*\rho_t}\right)\right)(y) e^{-(\psi*\rho_t)(y)} - e^{(\phi*\rho_t)(y)}(c*e^{-\psi*\rho_t})(y)\rho_t(y).$$

4.4 Free branching process

Let us recap shortly the description of the free branching process. Here the heuristic Markov generator is given by

$$
\begin{aligned}
(LF)(\gamma) &= m\sum_{x\in\gamma}(F(\gamma\backslash x) - F(\gamma)) \\
&\quad + \lambda \sum_{x\in\gamma} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x-y_1, x-y_2)(F(\gamma\backslash x\cup y_1\cup y_2) - F(\gamma))\mathrm{d}y_1\mathrm{d}y_2
\end{aligned}
$$

with constant mortality $m > 0$ and intensity of cell-division $\lambda > 0$. The potential $0 \le a \in L^1(\mathbb{R}^d\times\mathbb{R}^d)$ is assumed to by symmetric in both coordinates and the total mass is normalized to 1. This model describes a cell population, where each cell will die with exponential distributed lifetime of parameter $m > 0$ and will divide into two cells after another exponential distributed time of parameter $\lambda > 0$. The position of the new cells is determined by the probability distribution

$$a(x-y_1, x-y_2)\mathrm{d}y_1\mathrm{d}y_2,$$

where $x\in\gamma$ is the position of the mother cell. The generator L is well defined for all functions $F = KG$, where $G \in B_{bs}(\Gamma_0)$, i.e. is bounded and has bounded support, i.e. there exist a compact $\Lambda \subset \mathbb{R}^d$ and $N \in \mathbb{N}$ such that G is bounded and for any $\eta \in \Gamma_0$ with $|\eta| > N$ or $\eta \not\subset \Lambda$ on has $G(\eta) = 0$. Following the general approach of section 3, we are first going to calculate the operators $\widehat{L}$ for quasi-observables G and L^{Δ} for correlation functions k.

Theorem 4.6. *For $G \in B_{bs}(\Gamma_0)$ the operator $\widehat{L} = \widehat{L}_V + \widehat{B}$ is given by*

$$(\widehat{L}_V G)(\eta) = -(m+\lambda)|\eta|G(\eta) + \lambda \sum_{x\in\eta} \int_{\mathbb{R}^d} b(x-y)G(\eta\backslash x \cup y)\mathrm{d}y \tag{46}$$

with $\widehat{B}$ given by

$$(\widehat{B}G)(\eta) = \lambda \sum_{x\in\eta} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x-y_1, x-y_2)G(\eta\backslash x \cup y_1 \cup y_2)\mathrm{d}y. \tag{47}$$

Here $0 \le b$ describes the effective proliferation and is given by

$$b(x) = \int_{\mathbb{R}^d} a(x,y)\mathrm{d}y + \int_{\mathbb{R}^d} a(y,x)\mathrm{d}y.$$

The function $M_\alpha = M_\alpha^V + M_\alpha^B$ is given by $M_\alpha^V(\eta) = (m+3\lambda)|\eta|$ and

$$M_\alpha^B(\eta) = \lambda e^{-\alpha} \sum_{y_1\in\eta} \sum_{y_2\in\eta\backslash y_1} \int_{\mathbb{R}^d} a(x-y_1, x-y_2)\mathrm{d}x$$

If in addition the expression

$$\theta = \min\left\{ \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} a(x-y,x)dx, \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} a(x,x-y)dx \right\} \tag{48}$$

is finite, then $\widehat{L}$ acts as a bounded operator from $\mathbb{B}_\alpha$ to $\mathbb{B}_{\alpha'}$ for each $\alpha' < \alpha$. In this case the estimate

$$\|\widehat{L}\|_{\alpha\alpha'} \le \|\widehat{L}_V\|_{\alpha\alpha'} + \|\widehat{B}\|_{\alpha\alpha'} \le \frac{m+3\lambda}{e(\alpha-\alpha')} + \frac{4\lambda\theta e^{-\alpha'}}{e^2(\alpha-\alpha')^2}. \tag{49}$$

holds.

Proof. Using the $K-$transform we obtain for $x \in \gamma$

$$(KG)(\gamma\backslash x) - (KG)(\gamma) = -\sum_{\eta\Subset\gamma\backslash x} G(\eta\cup x)$$

and therefore for the first part

$$m\sum_{x\in\gamma}((KG)(\gamma\backslash x) - (KG)(\gamma)) = -m\sum_{x\in\gamma}\sum_{\eta\Subset\gamma\backslash x} G(\eta\cup x)$$
$$= -m\sum_{\eta\Subset\gamma}\sum_{x\in\eta} G(\eta) = -mK(|\cdot|G)(\gamma).$$

Applying the inverse $K-$transform we arrive at the expression $-m|\eta|G(\eta)$ reflecting the natural death of each cell. For the cell-division we first note that for $x \in \gamma$ and $y_1, y_1 \notin \gamma$

$$\begin{aligned}&(KG)(\gamma \backslash x \cup y_1 \cup y_1) - (KG)(\gamma)\\ &= \sum_{\eta \Subset \gamma \backslash x} \left(G(\eta \cup y_1) + G(\eta \cup y_2) + G(\eta \cup y_1 \cup y_2) - G(\eta \cup x)\right).\end{aligned}$$

Therefore the birth-part is given by

$$\begin{aligned}&\sum_{x \in \eta} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x - y_1, x - y_2)\\ &\qquad \times \left(G(\eta \backslash x \cup y_1) + G(\eta \backslash x \cup y_2) + G(\eta \backslash x \cup y_1 \cup y_2) - G(\eta)\right) \mathrm{d}y_1 \mathrm{d}y_2.\end{aligned}$$

In the first two terms of the second part the integration over y_1 respectively y_2 can be carried out, which gives together with the substitution $y_1, y_2 \to y$

$$\begin{aligned}&\lambda \sum_{x \in \eta} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x - y_1, x - y_2) \left(G(\eta \backslash x \cup y_1) + G(\eta \backslash x \cup y_2)\right) \mathrm{d}y_1 \mathrm{d}y_2\\ &= \lambda \sum_{x \in \eta} \int_{\mathbb{R}^d} b(x - y) G(\eta \backslash x \cup y) \mathrm{d}y.\end{aligned}$$

Altogether we obtain formulas (46) and (47). Let us now compute M_α, so let $G \in D(M_\alpha)$ defined in (44), then

$$\int_{\Gamma_0} |\widehat{L}G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta) \leq \int_{\Gamma_0} |\widehat{L}_V G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta) + \int_{\Gamma_0} |\widehat{B}G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta)$$

and for the $\widehat{L}_V$ we get

$$\begin{aligned}&\int_{\Gamma_0} |\widehat{L}_V G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta)\\ &\leq \int_{\Gamma_0} (m + \lambda)|\eta||G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta) + \lambda \int_{\Gamma_0} \sum_{x \in \eta} \int_{\mathbb{R}^d} b(x - y) |G(\eta \backslash x \cup y)| e^{\alpha|\eta|} \mathrm{d}y \mathrm{d}\lambda(\eta)\\ &= \int_{\Gamma_0} (m + \lambda)|\eta||G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta) + \lambda e^{\alpha} \int_{\Gamma_0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b(x - y) |G(\eta \cup y)| e^{\alpha|\eta|} \mathrm{d}y \mathrm{d}x \mathrm{d}\lambda(\eta)\\ &= \int_{\Gamma_0} (m + \lambda)|\eta||G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta) + \lambda \int_{\Gamma_0} \sum_{y \in \eta} \int_{\mathbb{R}^d} b(x - y) |G(\eta)| e^{\alpha|\eta|} \mathrm{d}x \mathrm{d}\lambda(\eta)\\ &= \int_{\Gamma_0} (m + \lambda)|\eta||G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta) + 2\lambda \int_{\Gamma_0} |\eta||G(\eta)| e^{\alpha|\eta|} \mathrm{d}\lambda(\eta)\end{aligned}$$

and hence $M_\alpha^V(\eta) = (m + 3\lambda)|\eta|$. For the second part we get

$$\begin{aligned}
&\int\limits_{\Gamma_0} |\widehat{B}G(\eta)|e^{\alpha|\eta|}\mathrm{d}\lambda(\eta) \\
&\leq \lambda \int\limits_{\Gamma_0} \sum_{x\in\eta} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(x-y_1, x-y_2)|G(\eta\backslash x \cup y_1 \cup y_2)|e^{\alpha|\eta|}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\lambda(\eta) \\
&= e^{-\alpha}\lambda \int\limits_{\Gamma_0} \sum_{y_1\in\eta}\sum_{y_2\in\eta\backslash y_1} \int\limits_{\mathbb{R}^d} a(x-y_1, x-y_2)|G(\eta)|e^{\alpha|\eta|}\mathrm{d}x\mathrm{d}\lambda(\eta).
\end{aligned}$$

If (48) holds, then $M_\alpha^B(\eta) \leq \lambda e^{-\alpha}\theta|\eta|^2$, which shows the estimate for the norm of $\|\widehat{L}\|_{\alpha\alpha'}$. □

Let us take a closer look at $\widehat{L}$. This operator is a sum of a particle number preserving part $\widehat{L}_V$ and a upper diagonal part $\widehat{B}$. Rewrite this number preserving part $\widehat{L}_V$ in the form

$$(\widehat{L}_V G)(\eta) = -(m-\lambda)|\eta|G(\eta) + \lambda\sum_{x\in\eta}\int\limits_{\mathbb{R}^d} b(x-y)\left(G(\eta\backslash x\cup y) - G(\eta)\right)\mathrm{d}y.$$

By previous proof we know, that $(\widehat{L}_V, D(\widehat{L}_V))$ is a well-defined linear operator satisfying

$$\|\widehat{L}_V\|_{\alpha\alpha'} \leq \frac{m+3\lambda}{e(\alpha-\alpha')}.$$

Let $G = (G^{(n)})_{n=0}^\infty$ be the decomposition of a measurable function $G : \Gamma_0 \longrightarrow \mathbb{R}$ to its components and set for $n \in \mathbb{N}$

$$\begin{aligned}
&(D_n G^{(n)})(x_1,\dots,x_n) \\
&= -(m-\lambda)nG^{(n)}(x_1,\dots,x_n) \\
&\quad + \lambda\sum_{k=1}^n \int\limits_{\mathbb{R}^d} b(x_k-y)\Big(G^{(n)}(x_1,\dots,\hat{x}_k,y,\dots,x_n) - G^{(n)}(x_1,\dots,x_n)\Big)\,\mathrm{d}y \\
&= -(m-\lambda)nG^{(n)}(x_1,\dots,x_n) + (A_nG)^{(n)}(x_1,\dots,x_n),
\end{aligned}$$

where $\hat{x}_k$ means that integration over the variable x_k should be omitted. For each $n \in \mathbb{N}_0$ the operator $\widehat{L}_V$ is diagonal, i.e. it acts only on $G^{(n)}$. The equation

$$\frac{\partial G_t^{(n)}}{\partial t} = D_n G_t^{(n)}, \quad G_t^{(n)}|_{t=0} = G_0^{(n)}$$

has a solution $G_t^{(n)} = e^{-(m-\lambda)nt}H_t^{(n)}$, were $H_t^{(n)}$ solves

$$\frac{\partial H_t^{(n)}}{\partial t} = A_n H_t^{(n)}, \quad H_t^{(n)}|_{t=0} = G_0^{(n)}.$$

Therefore let us try to understand the meaning of A_n. This part describes a Random walk in continuous time of each cell with intensity 2λ and the probability of a cell located at $x \in \mathbb{R}^d$ to jump in the region $\mathrm{d}y$ is given by

$$\frac{1}{2}b(x-y)\mathrm{d}y.$$

Lemma 4.7. *D_n is a bounded linear operator on $L^1((\mathbb{R}^d)^n)$ and $L^\infty((\mathbb{R}^d)^n)$ and the corresponding semigroup is a positive contraction semigroup. Moreover, if $\lambda \le m$, then $(\widehat{L}_V, D(\widehat{L}_V))$ has an extension to a of a sub-stochastic semigroup on $\mathbb{B}_\alpha$ for each α.*

Proof. The first assertion is a consequence of the Beurling-Deny-Criterion, c.f. [17]. Assume $\lambda \le m$ and consider

$$(\widehat{L}_V G)(\eta) = -(m+\lambda)|\eta|G(\eta) + \sum_{x\in\eta}\int_{\mathbb{R}^d} b(x-y)G(\eta\backslash x \cup y)\mathrm{d}y,$$

the second summand is positive and defined on the same domain as the negative multiplication operator $-(m+\lambda)|\eta|$. Now an application of [19] shows the assertion, provided

$$\int_{\Gamma_0} (\widehat{L}_V G)(\eta) e^{-\alpha|\eta|}\mathrm{d}\lambda(\eta) \le 0$$

for $0 \le G \in D(\widehat{L}_V)$. But this is true, since $\lambda \le m$. □

Note that also for $m < \lambda$ an evolution $t \longmapsto G_t$ can be constructed. Let $G_0 = (G_0^{(n)})_{n\in\mathbb{N}}$ be measurable such that each component $G_0^{(n)}$ is integrable. Then $e^{-(m-\lambda)nt}e^{tA_n}G_0^{(n)} = e^{tD_n}G_0^{(n)}$ is well-defined and the vector $G_t = (e^{tD_n}G_0^{(n)})_{n=0}^\infty$ is the unique component-wise solution to

$$\begin{cases} \dfrac{\partial G_t}{\partial t} = \widehat{L}_V G_t \\ G_t|_{t=0} = G_0 \end{cases}.$$

This solution, if $G_0 \in \mathbb{B}_\alpha$, evolves in the scale of Banach spaces $\mathbb{B}_\alpha$ with $\alpha(t) = \alpha + (m-\lambda)t$, i.e. $G_t \in \mathbb{B}_{\alpha(t)}$, which follows from

$$\begin{aligned} \|G_t\|_{\alpha(t)} &= \sum_{n=0}^\infty \frac{e^{-(m-\lambda)nt}e^{\alpha(t)n}}{n!} \int_{(\mathbb{R}^d)^n} |e^{tA_n}G_0^{(n)}(x_1,\dots,x_n)|\mathrm{d}x_1\dots\mathrm{d}x_n \\ &\le \sum_{n=0}^\infty \frac{e^{\alpha n}}{n!} \int_{(\mathbb{R}^d)^n} |G_0^{(n)}(x_1,\dots,x_n)|\mathrm{d}x_1\dots\mathrm{d}x_n = \|G_0\|_\alpha. \end{aligned}$$

The presence of the perturbation $\widehat{B}$ implies that the solution cannot satisfy $G_t \in \mathbb{B}_{\alpha(t)}$ for $t \ge 0$ and any $\alpha(t)$. Since $\widehat{B}$ sends functions of $n+1$ variables to functions of n variables it is not helpful to discuss a direct solution formula, though it is possible. More precise results will be investigated in terms of correlation functions.

Lemma 4.8. *For $k : \Gamma_0 \longrightarrow \mathbb{R}$ such that $|k(\eta)| \le |\eta|!C^{|\eta|}$ for some constant $C > 0$ the operator L^Δ is given by*

$$L^\Delta = L_V^\Delta + B^\Delta,$$

where L_V^Δ is given by the same expression as $\widehat{L}_V$ and B^Δ by

$$(B^\Delta k)(\eta) = \lambda \sum_{y_1 \in \eta} \sum_{y_2 \in \eta \backslash y_1} \int_{\mathbb{R}^d} a(x-y_1, x-y_2) k(\eta \cup x \backslash y_1 \backslash y_2) dx. \tag{50}$$

Moreover $L_V^\Delta \in L(\mathbb{B}_{\alpha'}^*, \mathbb{B}_\alpha^*)$ and if (48) holds, then $B^\Delta \in L(\mathbb{B}_{\alpha'}^*, \mathbb{B}_\alpha^*)$ with $\|L_V^\Delta\|_{\alpha'\alpha} = \|\widehat{L}_V\|_{\alpha\alpha'}$ and $\|B^\Delta\|_{\alpha'\alpha} = \|\widehat{B}\|_{\alpha\alpha'}$.

Proof. For $G \in B_{bs}(\Gamma_0)$ and k as described above, the operator L^Δ is uniquely determined by the pairing

$$\int_{\Gamma_0} (\widehat{L}G)(\eta) k(\eta) \mathrm{d}\lambda(\eta) = \int_{\Gamma_0} G(\eta)(L^\Delta k)(\eta) \mathrm{d}\lambda(\eta).$$

The negative multiplication part will therefore not change and for the second part we get by the formula from Lemma 4.2

$$\begin{aligned} &\lambda \int_{\Gamma_0} \sum_{x \in \eta} \int_{\mathbb{R}^d} b(x-y) G(\eta \backslash x \cup y) \mathrm{d}y k(\eta) \mathrm{d}\lambda(\eta) \\ &= \lambda \int_{\Gamma_0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b(x-y) G(\eta \cup y) k(\eta \cup x) \mathrm{d}y \mathrm{d}x \mathrm{d}\lambda(\eta) \\ &= \lambda \int_{\Gamma_0} \sum_{y \in \eta} \int_{\mathbb{R}^d} b(x-y) k(\eta \cup x \backslash y) \mathrm{d}x G(\eta) \mathrm{d}\lambda(\eta). \end{aligned}$$

Finally

$$\begin{aligned} &\int_{\Gamma_0} (\widehat{B}G)(\eta) k(\eta) \mathrm{d}\lambda(\eta) \\ &= \lambda \int_{\Gamma_0} \sum_{x \in \eta} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x-y_1, x-y_2) G(\eta \backslash x \cup y_1 \cup y_2) dy_1 dy_2 k(\eta) \mathrm{d}\lambda(\eta) \\ &= \lambda \int_{\Gamma_0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x-y_1, x-y_2) G(\eta \cup y_1 \cup y_2) k(\eta \cup x) \mathrm{d}x \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}\lambda(\eta) \\ &= \lambda \int_{\Gamma_0} \sum_{y_1 \in \eta} \sum_{y_2 \in \eta \backslash y_1} \int_{\mathbb{R}^d} a(x-y_1, x-y_2) k(\eta \cup x \backslash y_1 \backslash y_2) dx G(\eta) \mathrm{d}\lambda(\eta), \end{aligned}$$

proves the assertion. The second part follows from (43). □

Again, the equation for L_V^Δ can be solved explicitly and since B^Δ has now lower diagonal structure the equation

$$\frac{\partial k_t}{\partial t} = L^\Delta k_t$$

has a unique solution given by a recursive formula. More precisely let $k_0 = (k_0^{(n)})_{n=0}^\infty$ be non-negative and measurable such that $k_0^{(n)} \in L^\infty((\mathbb{R}^d)^n)$. Denote by B_n^Δ the operator given by (50) taking functions from $n-1$ variables to functions with n variables. The solution to (4) is given by

$$k_t^{(n+1)} = e^{-(m-\lambda)(n+1)t} e^{tA_{n+1}} k_0^{(n+1)} + \int_0^t e^{-(m-\lambda)(n+1)(t-s)} e^{(t-s)A_{n+1}} B_{n+1}^{\Delta} k_s^{(n)} ds. \tag{51}$$

Theorem 4.9. *For each $k_0 \geq 0$ measurable, such that $k_0^{(n)} \in L^\infty((\mathbb{R}^d)^n)$, there exist a unique solution $k_t \geq 0$, given recursively by formula (51). If θ is finite, then for each initial conditions satisfying $k_0(\eta) \leq |\eta|! C^{|\eta|}$ for some constant $C > 0$, this solution obeys the bound*

$$k_t(\eta) \leq |\eta|!(C+t)^{|\eta|}(1+\theta)^{|\eta|}\kappa(t)^{|\eta|} e^{-(m-\lambda)|\eta|t}$$

with $\kappa(t) = \max\{1, \lambda, \lambda e^{(m-\lambda)t}\}$. If, in addition there is $\delta > 0$ such that $a(x,y) \geq \alpha > 0$ for some $\alpha > 0$ and all $|x|, |y| \leq \delta$, then for each $k_0(\eta) = C^{|\eta|}$ the solution k_t cannot be sub-poissonian, i.e. for any $\eta \in \Gamma_0$ with:

$$\forall x, y \in \eta,\ x \neq y :\ |x-y| < \delta$$

the estimate

$$k_t(\eta) \geq \beta^{|\eta|} e^{-(m-\lambda)|\eta|t} |\eta|! \quad t \geq 1$$

holds, where $\beta = \min\{C, \lambda\alpha, \delta, |B_\delta|\}$ with $\delta = \begin{cases} \frac{1}{\lambda - m}, & \lambda > m \\ 1, & \lambda \leq m \end{cases}$ and $|B_\delta|$ is the Lebesgue volume of the Ball B_δ of radius δ.

Proof. For the bound from above we will proceed by induction on the number of cells $|\eta|$. The first correlation function is given by

$$k_t^{(1)} = e^{-(m-\lambda)t} e^{tA_1} k_0^{(1)}$$

and hence by positivity of $(e^{tA_1})_{t\geq 0}$ and $e^{tA_1} C = C$

$$k_t^{(1)} \leq e^{-(m-\lambda)t} C \leq (C+t)(1+\theta)\kappa(t) e^{-(m-\lambda)t}.$$

For $n \to n+1$ we get with $|\eta| = n+1$

$$\begin{aligned} k_t^{(n+1)} &\leq e^{-(m-\lambda)(n+1)t}(n+1)! C^{n+1} + \int_0^t e^{-(m-\lambda)(n+1)(t-s)} e^{(t-s)A_{n+1}} B_{n+1}^{\Delta} k_s^{(n)} ds \\ &\leq e^{-(m-\lambda)(n+1)t}(n+1)! C^{n+1} \\ &\qquad + (n+1)!\kappa(t)^{n+1}(1+\theta)^{n+1}\left((C+t)^{n+1} - C^{n+1}\right) e^{-(m-\lambda)(n+1)t} \\ &\leq (n+1)!(C+t)^{n+1}(1+\theta)^{n+1}\kappa(t)^{n+1} e^{-(m-\lambda)(n+1)t}. \end{aligned}$$

Here we used the fact that for $s \leq t$ we have $\kappa(s) \leq \kappa(t)$. For the second part let $k_0^{(n)} = C^n$, then $e^{tA_n} k_0 = C^n$ and therefore $k_t^{(1)} = e^{-(m-\lambda)t} C \geq \beta e^{-(m-\lambda)t}$.

For $n \to n+1$ and $t \geq 1$ we obtain

$$\begin{aligned} k_t^{(n+1)} &\geq e^{-(m-\lambda)(n+1)t} C^{n+1} \\ &\qquad + \lambda \int_0^t e^{-(m-\lambda)(n+1)(t-s)} (n+1) n\alpha e^{-(m-\lambda)ns} \beta^n n! \mathrm{d}s |B_\delta| \\ &\geq e^{-(m-\lambda)(n+1)t} \int_0^t e^{(m-\lambda)s} ds \beta^n (n+1)! \alpha\lambda \\ &\geq e^{-(m-\lambda)(n+1)t} \beta^{n+1} (n+1)! \end{aligned}$$

□

This Theorem shows that if the probability distribution for the new cells, has no hard core, i.e. $a(0) > 0$ for continuous distributions, than the system will consist of clusters. Appearance of such clusters are due to the operator B^Δ. The part L_V^Δ contains information about asymptotic behaviour, speed of propagation etc., whereas B^Δ contains information about correlations of the system. Assume for simplicity, that in the cell-division the position of the new cells are independent of each other, then we may write $a(x,y) = c(x)c(y)$ for some symmetric function $0 \leq c \in L^1(\mathbb{R}^d)$ normalized to 1. If for example c is continuous and non-vanishing, then previous assumptions are satisfied and we get the bound

$$\beta^n n! e^{-(m-\lambda)nt} \leq k_t^{(n)}$$

on $\mathbb{R}^d$. Hence the system will be always clustering. The same results were shown for the case $a(x,y) = c(x)\delta(y)$, where each cell creates a new cell and its location is described by the kernel c. In contrast to this model, the old cell will not die. Clearly such models should have the same properties. Previous Theorem justifies the assumption, that it is enough to work with the usual Contact Model $a(x,y) = c(x)\delta(y)$.

Scaling

Following the general scheme of mesoscopic scaling described in previous chapter, we have to scale potentials like $a \longmapsto \varepsilon a$ and accelerate birth by a factor $\frac{1}{\varepsilon}$. Clearly, since the birth only consists of the a-part, this will not change the operator itself, i.e. $L_\varepsilon = L$. First we will look at Quasi-observables. In this case the renormalized operator is given by $\widehat{L}_{\varepsilon,ren} = R_{\varepsilon^{-1}} \widehat{L} R_\varepsilon$, where $R_\varepsilon G(\eta) = \varepsilon^{|\eta|} G(\eta)$. Applying this to this model, one gets $\widehat{L}_{\varepsilon,ren} = \widehat{L}_V + \varepsilon \widehat{B}$. Hence we can realize $\widehat{L}_{\varepsilon,ren}$ on the same domain as $\widehat{L}$.

Lemma 4.10. *For each $G \in D(N_\alpha)$ one has*

$$\widehat{L}_{\varepsilon,ren} G \longrightarrow \widehat{L}_V G, \quad \varepsilon \to 0$$

in the norm $\|\cdot\|_\alpha$ for each $\alpha \in \mathbb{R}$. Moreover if (48) *holds, the operator $\widehat{L}_{\varepsilon,ren}$ converges to $\widehat{L}_V$ in the operator norm of $L(\mathbb{B}_\alpha, \mathbb{B}_{\alpha'})$ for each $\alpha' < \alpha$.*

Proof. Let $G \in D(M_\alpha)$, then $\widehat{L}_{\varepsilon,ren} G - \widehat{L}_V G = \varepsilon \widehat{B} G \in \mathbb{B}_\alpha$, which shows the first assertion. For the second part we know $\widehat{B} \in L(\mathbb{B}_\alpha, \mathbb{B}_{\alpha'})$ and thus, for $G \in \mathbb{B}_\alpha$,

$$\|\widehat{L}_{\varepsilon,ren}G - \widehat{L}_V G\|_{\alpha'} = \varepsilon\|\widehat{B}G\|_\alpha \le \varepsilon\|\widehat{B}\|_{\alpha\alpha'}\|G\|_\alpha. \qquad \square$$

Similarly we get.

Lemma 4.11. *Assume* (48) *holds, then for each* $\alpha' < \alpha$ *the operator* $L^\Delta_{\varepsilon,ren}$ *converges in the operatornorm of* $L(\mathbb{B}^*_{\alpha'}, \mathbb{B}^*_\alpha)$ *to the operator* L^Δ_V.

Proof. Let $k \in \mathbb{B}^*_{\alpha'}$, then

$$\|L^\Delta_{\varepsilon,ren}k - L^\Delta_V k\|_\alpha = \varepsilon\|B^\Delta k\|_\alpha \le \varepsilon\|B^\Delta\|_{\alpha'\alpha}\|k\|_{\alpha'}$$

implies the assertion. $\square$

Hence mesoscopic scaling suppresses the microscopic effects like cell-correlations etc. The resulting model has less information but is simpler to analyse. As already shown $\widehat{L}_V$ or L^Δ_V will lead to evolutions $t \longmapsto G_t$ or $t \longmapsto k_t$, which can preserve the spaces $\mathbb{B}_\alpha$ respectively $\mathbb{B}^*_\alpha$. Finally we will show the chaos preservation property and derive the equations for the local densities of the kinetic description.

Theorem 4.12. *Let* $k_0(\eta) = \prod_{x\in\eta} \rho_0(x)$ *with* $0 \le \rho_0 \in L^\infty(\mathbb{R}^d)$. *Then the unique solution to*

$$\begin{cases} \dfrac{\partial k_t}{\partial t} = L^\Delta_V k_t \\ k_t|_{t=0} = e_\lambda(\rho_0) \end{cases} \tag{52}$$

is given by $k_t(\eta) = \prod_{x\in\eta} \rho_t(x)$, *where* $\rho_t \ge 0$ *is a classical solution to the mesoscopic equation*

$$\begin{cases} \dfrac{\partial \rho_t}{\partial t} = -(m+\lambda)\rho_t + b * \rho_t \\ \rho_t|_{t=0} = \rho_0. \end{cases}$$

Proof. Since for each $k_0 = (k_0^{(n)})_{n=0}^\infty$ such that all $k_0^{(n)}$ are essentially bounded there exists a unique solution, we have only to check that also $k_t(\eta) = \prod_{x\in\eta} \rho_t(x)$ solves (52). Note, that for the given function ρ_0 a unique classical solution for the mesoscopic equation exists on $\mathbb{R}_+$. Computing

$$\frac{\partial k_t}{\partial t}(\eta) = \sum_{x\in\eta} \frac{\partial \rho_t}{\partial t}(x) e_\lambda(\rho_t; \eta\backslash x)$$

and

$$(L^\Delta_V e_\lambda(\rho_t))(\eta) = \sum_{x\in\eta} e_\lambda(\rho_t;\eta\backslash x)\left(-(m+\lambda)\rho_t(x) + \int_{\mathbb{R}^d} b(x-y)\rho_t(y)dy\right)$$

we conclude that k_t given by the formula is a solution. $\square$

In this model all cells are independent of each other, which implies that the equation in the kinetic description will be linear. Non-linearities enter through interactions of cells. So in more realistic models the typical equation will consist of convolutions and powers of ρ_t.

5 Two-component models

The extension to two-component models is straightforward. The Banach spaces $\mathbb{B}_\alpha$ of functions $G : \Gamma_0^2 \longrightarrow \mathbb{R}$ becomes $\mathbb{B}_\alpha = L^1(\Gamma_0^2, e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}d\lambda)$ with $\alpha = (\alpha^+, \alpha^-)$ equipped with the norm

$$\|G\|_\alpha = \int\limits_{\Gamma_0^2} |G(\eta^+, \eta^-)| e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|} \mathrm{d}\lambda(\eta^+, \eta^-)$$

and the dual space of correlation functions $k \in \mathbb{B}_\alpha^* = L^\infty(\Gamma_0^2, e^{-\alpha^+|\eta^+|}e^{-\alpha^-|\eta^-|}\mathrm{d}\lambda)$ with the norm

$$\|k\|_\alpha = \operatorname*{ess\,sup}_{(\eta^+,\eta^-)\in\Gamma_0^2} |k(\eta^+, \eta^-)| e^{-\alpha^+|\eta^+|}e^{-\alpha^-|\eta^-|}.$$

The dual pairing for these spaces is given by

$$\langle G, k\rangle = \int\limits_{\Gamma_0^2} G(\eta^+, \eta^-)k(\eta^+, \eta^-)\mathrm{d}\lambda(\eta^+, \eta^-)$$

and satisfies $|\langle G, k\rangle| \le \|G\|_\alpha \|k\|_\alpha$. For pairs $\alpha' = (\alpha'^+, \alpha'^-)$ and $\alpha = (\alpha^+, \alpha^-)$ we will write $\alpha' < \alpha$ if $\alpha'^+ < \alpha^+$ and $\alpha'^- < \alpha^-$ holds. In such case for an operator $\widehat{L} \in L(\mathbb{B}_\alpha, \mathbb{B}_{\alpha'})$ for all $\alpha' < \alpha$ and its dual operator $L^\Delta \in L(\mathbb{B}_{\alpha'}^*, \mathbb{B}_\alpha^*)$

$$\|\widehat{L}\|_{\alpha\alpha'} = \|L^\Delta\|_{\alpha'\alpha}$$

holds. Also there exists a measurable function $M_\alpha : \Gamma_0^2 \longrightarrow \mathbb{R}_+$ such that

$$\int\limits_{\Gamma_0^2} |\widehat{L}G(\eta^+, \eta^-)| e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}\mathrm{d}\lambda(\eta^+, \eta^-) \le \int\limits_{\Gamma_0^2} M_\alpha(\eta)|G(\eta^+, \eta^-)| e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}\mathrm{d}\lambda(\eta^+, \eta^-),$$

so all previous methods can be applied in this extended two-component setting.

In this section we will derive, similarly as for the one-component case, all necessary formulas to derive the kinetic description. Such list of interactions will be not complete, but should cover most of the interesting models in cell biology. Here we will restrict in many cases to interactions on +-cells. The case of −-cells in the presence of interactions with +-cells can be derived in the same way, simply exchanging all + with − and vice versa.

Define the relative energies $E(x, \gamma^\pm) = \sum\limits_{y\in\gamma^\pm} a(x - y)$ and E_ϕ, E_ψ in the same way with a replaced by ϕ respectively ψ. We will assume that $0 \le a, \phi, \psi \in L^1(\mathbb{R}^d)$ are symmetric.

Example 14. Let us consider first consider the Markov generator

$$(LF)(\gamma) = \sum_{x\in\gamma^+} E(x, \gamma^-)(F(\gamma^+\backslash x, \gamma^-) - F(\gamma)).$$

Each cell at position $x \in \gamma^+$ can die due to the interaction $\sum_{y\in\gamma^-} a(x-y)$ with cells from different type. The operator on quasi-observables is given by

$$(\widehat{L}G)(\eta) = -\sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)G(\eta) - \sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)G(\eta^+, \eta^-\backslash y).$$

The functions M_α and N_α are in such case given by

$$N_\alpha(\eta) = M_\alpha(\eta) = \sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y) + e^{\alpha^-}\langle a\rangle|\eta^+|.$$

After scaling, i.e. $a \to \varepsilon a$ and renormalization, we arrive in the limit to the operator

$$(\widehat{L}_V G)(\eta) = -\sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)G(\eta^+, \eta^-\backslash y),$$

which is defined on $D(N_\alpha^V)$ with $N_\alpha^V(\eta) = e^{-\alpha^-}\langle a\rangle|\eta^+|$. The convergence holds for each $G \in D(N_\alpha)$ in $\mathbb{B}_\alpha$, since only the multiplicative part is multiplied by ε. On the level of correlation functions L^Δ is given by

$$(L^\Delta k)(\eta) = -\sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)k(\eta) - \sum_{x\in\eta^+}\int_{\mathbb{R}^d} a(x-y)k(\eta^+, \eta^- \cup y)\mathrm{d}y$$

and

$$(L_V^\Delta k)(\eta) = -\sum_{x\in\eta^+}\int_{\mathbb{R}^d} a(x-y)k(\eta^+, \eta^- \cup y)\mathrm{d}y.$$

Therefore the kinetic description is given by

$$\frac{\partial\rho_t^+}{\partial t}(x) = -\rho_t^+(x)(a*\rho_t^-)(x), \quad \frac{\partial\rho_t^-}{\partial t}(x) = 0.$$

Example 15. Let us consider here the case, where the interaction is not quadratic in the number of particles, but exponential instead. In such case the Markov generator is given by

$$(LF)(\gamma) = \sum_{x\in\gamma^+} e^{E_\phi(x,\gamma^-)}e^{E_\psi(x,\gamma^+\backslash x)}(F(\gamma^+\backslash x, \gamma^-) - F(\gamma)).$$

The operator on quasi-observables is given by

$$(\widehat{L}G)(\eta) = -\sum_{\xi\subset\eta}\sum_{x\in\xi^+} e^{E_\psi(x,\xi^+\backslash x)}e^{E_\phi(x,\xi^-)} \\ \times e_\lambda(e^{\psi(x-\cdot)}-1;\eta^+\backslash\xi^+)e_\lambda(e^{\phi(x-\cdot)}-1;\eta^-\backslash\xi^-)G(\xi)$$

and on correlation functions by

$$(L^\Delta k)(\eta) = -\sum_{x\in\eta^+} e^{E_\psi(x,\eta^+\backslash x)}e^{E_\phi(x,\eta^-)} \\ \times \int_{\Gamma_0^2} e_\lambda(e^{\psi(x-\cdot)}-1;\xi^+)e_\lambda(e^{\phi(x-\cdot)}-1;\xi^-)k(\eta\cup\xi)\mathrm{d}\lambda^2(\xi),$$

where

$$\beta_\psi(\alpha^+) = \exp\Big(e^{\alpha^+}\int_{\mathbb{R}^d}(e^{\psi(x)}-1)\mathrm{d}x\Big),\quad \beta_\phi(\alpha^-) = \exp\Big(e^{\alpha^-}\int_{\mathbb{R}^d}(e^{\phi(x)}-1)\mathrm{d}x\Big),$$

and

$$M_\alpha(\eta) = \beta_\psi(\alpha^+)\beta_\phi(\alpha^-)\sum_{x\in\eta^+} e^{E_\psi(x,\eta^+\setminus x)}e^{E_\phi(x,\eta^-)}.$$

The rescaled operators $\widehat{L}_{\varepsilon,ren}G$ have for $\eta\in\Gamma_0^2$ the form

$$-\sum_{\xi\subset\eta}\sum_{x\in\xi^+} e^{\varepsilon E_\psi(x,\xi^+\setminus x)}e^{\varepsilon E_\phi(x,\xi^-)} \times e_\lambda\left(\frac{e^{\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\eta^+\setminus\xi^+\right)e_\lambda\left(\frac{e^{\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\eta^-\setminus\xi^-\right)G(\xi)$$

and on correlation functions $L^\Delta_{\varepsilon,ren}k$ is given by

$$-\sum_{x\in\eta^+} e^{\varepsilon E_\psi(x,\eta^+\setminus x)}e^{\varepsilon E_\phi(x,\eta^-)} \times\int_{\Gamma_0^2} e_\lambda\left(\frac{e^{\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\xi^+\right)e_\lambda\left(\frac{e^{\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\xi^-\right)k(\eta\cup\xi)\mathrm{d}\lambda^2(\xi),$$

so

$$\beta^*_\psi(\alpha^+) = \sup_{\varepsilon\in(0,1]}\exp\Big(\frac{e^{\alpha^+}}{\varepsilon}\int_{\mathbb{R}^d}(e^{\varepsilon\psi(x)}-1)\mathrm{d}x\Big),$$

$$\beta^*_\phi(\alpha^+) = \sup_{\varepsilon\in(0,1]}\exp\left(\frac{e^{\alpha^+}}{\varepsilon}\int_{\mathbb{R}^d}(e^{\varepsilon\phi(x)}-1)\mathrm{d}x\right),$$

and

$$N_\alpha(\eta) = \beta^*_\psi(\alpha^+)\beta^*_\phi(\alpha^-)\sum_{x\in\eta^+} e^{E_\psi(x,\eta^+\setminus x)}e^{E_\phi(x,\eta^-)}.$$

Taking the limit $\varepsilon\to 0$ we obtain

$$(\widehat{L}_V G)(\eta) = -\sum_{\xi\subset\eta}\sum_{x\in\xi^-} e_\lambda(\psi(x-\cdot);\eta^+\setminus\xi^+)e_\lambda(\phi(x-\cdot);\eta^-\setminus\xi^-)G(\xi)$$

and

$$(L^\Delta_V k)(\eta) = -\sum_{x\in\eta^+}\int_{\Gamma_0^2} e_\lambda(\psi(x-\cdot);\xi^+)e_\lambda(\phi(x-\cdot);\xi^-)k(\eta\cup\xi)\mathrm{d}\lambda^2(\xi),$$

so $N^V_\alpha(\eta) = \exp(e^{\alpha^+}\langle\psi\rangle + e^{\alpha^-}\langle\phi\rangle)|\eta^+|$. Finally we see that the kinetic description is given by

$$\frac{\partial\rho_t^+}{\partial t}(x) = -\rho_t^+(x)e^{(\psi*\rho_t^+)(x)}e^{(\phi*\rho_t^-)(x)},\quad \frac{\partial\rho_t^-}{\partial t}(x) = 0.$$

Example 16. Let us look at the model with fecundity including interactions with both types of cells. The Markov generator is given by

$$(LF)(\gamma) = \sum_{x\in\gamma^+} e^{-E_\phi(x,\gamma^-)} e^{-E_\psi(x,\gamma^+\backslash x)} \int_{\mathbb{R}^d} a^+(x-y)(F(\gamma^+\cup y,\gamma^-) - F(\gamma))\mathrm{d}y,$$

where E_ϕ, E_ψ are given by the same expressions as in the previous example. In such case the operator on quasi-observables is given by

$$\begin{aligned}(\widehat{L}G)(\eta) = &\sum_{\xi\subset\eta}\sum_{x\in\xi^+} e^{-E_\phi(x,\xi^-)} e^{-E_\psi(x,\xi^+\backslash x)} \\ &\times \int_{\mathbb{R}^d} a(x-y) e_\lambda(e^{-\phi(x-\cdot)}-1;\eta^-\backslash\xi^-) e_\lambda(e^{-\psi(x-\cdot)}-1;\eta^+\backslash\xi^+) \\ &\times (G(\xi^+\backslash x\cup y,\xi^-) + G(\xi^+\cup y,\xi^-))\mathrm{d}y\end{aligned}$$

and on correlation functions by

$$\begin{aligned}(L^\Delta k)(\eta) = &\sum_{y\in\eta^+}\int_{\Gamma_0^2}\int_{\mathbb{R}^d} a(x-y)k(\eta\cup\xi\backslash y\cup x) e^{-E_\phi(x,\eta^-)} e^{-E_\psi(x,\eta^+\backslash y)} \\ &\times e_\lambda(e^{-\phi(x-\cdot)}-1;\xi^-) e_\lambda(e^{-\psi(x-\cdot)}-1;\xi^+)\mathrm{d}x\mathrm{d}\lambda^2(\xi) \\ &+ \sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a(x-y) e^{-E_\phi(y,\eta^-)} e^{-E_\psi(y,\eta^+\backslash y)} \\ &\times \int_{\Gamma_0^2} k(\eta\cup\xi\backslash y) e_\lambda(e^{-\phi(x-\cdot)}-1;\xi^-) e_\lambda(e^{-\psi(x-\cdot)}-1;\xi^+)\mathrm{d}\lambda^2(\xi).\end{aligned}$$

Hence M_α can be chosen as

$$\begin{aligned}M_\alpha(\eta) = &\beta_\psi(\alpha^+)\beta_\phi(\alpha^-)\sum_{y\in\eta^+}\int_{\mathbb{R}^d} a(x-y)e^{\psi(x-y)} e^{-E_\phi(x,\eta^-)} e^{-E_\psi(x,\eta^+)}\mathrm{d}x \\ &+ e^{-\alpha^+}\beta_\psi(\alpha^+)\beta_\phi(\alpha^-)\sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a(x-y) e^{-E_\phi(y,\eta^-)} e^{-E_\psi(y,\eta^+\backslash y)},\end{aligned}$$

where $\beta_\psi(\alpha^+) = \exp\Big(e^{\alpha^+}\int_{\mathbb{R}^d}(1-e^{-\psi(x)})\mathrm{d}x\Big)$ and $\beta_\phi(\alpha^-) = \exp\Big(e^{\alpha^-}\int_{\mathbb{R}^d}(1-e^{-\phi(x)})\mathrm{d}x\Big)$. Rescaling $a\to\varepsilon a$, $\phi\to\varepsilon\phi$, $\psi\to\varepsilon\psi$, putting $\frac{1}{\varepsilon}$ in front of the generator and renormalizing, we arrive at

$$\begin{aligned}&(\widehat{L}_{\varepsilon,ren}G)(\eta) \\ &= \sum_{\xi\subset\eta}\sum_{x\in\xi^+} e^{-\varepsilon E_\phi(x,\xi^-)} e^{-\varepsilon E_\psi(x,\xi^+\backslash x)} \int_{\mathbb{R}^d} a(x-y) e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\eta^-\backslash\xi^-\right) \\ &\quad\times e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\eta^+\backslash\xi^+\right)(G(\xi^+\backslash x\cup y,\xi^-) + \varepsilon G(\xi^+\cup y,\xi^-))\mathrm{d}y\end{aligned}$$

and on correlation functions at

$$\begin{aligned}
&(L^{\Delta}_{\varepsilon,ren}k)(\eta)\\
&= \sum_{y\in\eta^+}\int_{\Gamma_0^2}\int_{\mathbb{R}^d} a(x-y)k(\eta\cup\xi\backslash y\cup x)e^{-\varepsilon E_\phi(x,\eta^-)}e^{-\varepsilon E_\psi(x,\eta^+\backslash y)}\\
&\qquad \times e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\xi^-\right)e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\xi^+\right)\mathrm{d}x\mathrm{d}\lambda^2(\xi).\\
&+\varepsilon\sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a(x-y)e^{-\varepsilon E_\phi(y,\eta^-)}e^{-\varepsilon E_\psi(y,\eta^+\backslash y)}\\
&\qquad \times\int_{\Gamma_0^2} k(\eta\cup\xi\backslash y)e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\xi^-\right)e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\xi^+\right)\mathrm{d}\lambda^2(\xi).
\end{aligned}$$

This yields

$$N_\alpha(\eta) = \exp(e^{\alpha^+}\langle\psi\rangle + e^{\alpha^-}\langle\phi\rangle)\Big(\langle a e^{\psi}\rangle|\eta^+| + e^{-\alpha^+}\sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a(x-y)\Big).$$

Taking the limit $\varepsilon\to 0$, we arrive at

$$\begin{aligned}
(\widehat{L}_V G)(\eta) = \sum_{\xi\subset\eta}\sum_{x\in\xi^+}\int_{\mathbb{R}^d} & a(x-y)e_\lambda(-\phi(x-\cdot);\eta^-\backslash\xi^-)\\
&\times e_\lambda(-\psi(x-\cdot);\eta^+\backslash\xi^+)G(\xi^+\backslash x\cup y,\xi^-)\mathrm{d}y
\end{aligned}$$

and

$$\begin{aligned}
(L^{\Delta}_V k)(\eta) = \sum_{y\in\eta^+}\int_{\Gamma_0^2}\int_{\mathbb{R}^d} & a(x-y)k(\eta\cup\xi\backslash y\cup x)\\
&\times e_\lambda(-\phi(x-\cdot);\xi^-)e_\lambda(-\psi(x-\cdot);\xi^+)\mathrm{d}x\mathrm{d}\lambda^2(\xi)
\end{aligned}$$

and hence $N_\alpha^V(\eta) = \exp(e^{\alpha^+}\langle\psi\rangle + e^{\alpha^-}\langle\phi\rangle)\langle a\rangle|\eta^+|$. Thus the kinetic description is given by

$$\frac{\partial\rho_t^+}{\partial t}(x) = (a*\rho_t^+)(x)e^{-(\psi*\rho_t^+)(x)}e^{-(\phi*\rho_t^-)(x)},\qquad \frac{\partial\rho_t^-}{\partial t}(x) = 0.$$

Example 17. Another possibility is, where each $-$-cell creates a new $+$-cell independent of all other cells. Such free branching is described by the formal Markov generator

$$(LF)(\gamma) = \sum_{x\in\gamma^-}\int_{\mathbb{R}^d} a(x-y)(F(\gamma^+\cup y,\gamma^-) - F(\gamma))\mathrm{d}y.$$

On quasi-observables it is described via

$$\begin{aligned}
(\widehat{L}G)(\eta) &= \sum_{x\in\eta^-}\int_{\mathbb{R}^d} a(x-y)G(\eta^+\cup y,\eta^-\backslash x)\mathrm{d}y\\
&\quad + \sum_{x\in\eta^-}\int_{\mathbb{R}^d} a(x-y)G(\eta^+\cup y,\eta^-)\mathrm{d}y
\end{aligned}$$

and on correlation functions via

$$(L^{\Delta}k)(\eta) = \sum_{y\in\eta^+}\int_{\mathbb{R}^d} a(x-y)k(\eta^+\backslash y, \eta^- \cup x)\mathrm{d}x + \sum_{x\in\eta^-}\sum_{y\in\eta^+} a(x-y)k(\eta^+\backslash y, \eta^-).$$

Hence the functions $M_\alpha = N_\alpha$ can be chosen as $M_\alpha(\eta) = e^{-\alpha^+ + \alpha^-}\langle a\rangle|\eta^+| + e^{-\alpha^+}\sum\limits_{x\in\eta^-}\sum\limits_{y\in\eta^+} a(x-y)$. After scaling we arrive at

$$(\widehat{L}_V G)(\eta) = \sum_{x\in\eta^-}\int_{\mathbb{R}^d} a(x-y)G(\eta^+ \cup y, \eta^-\backslash x)\mathrm{d}y$$

and

$$(L_V^{\Delta}k)(\eta) = \sum_{y\in\eta^+}\int_{\mathbb{R}^d} a(x-y)k(\eta^+\backslash y, \eta^- \cup x)\mathrm{d}x$$

so that $M_\alpha^V(\eta) = e^{-\alpha^+ + \alpha^-}\langle a\rangle|\eta^+|$. Finally the kinetic description is given by

$$\frac{\partial\rho_t^+}{\partial t}(x) = (a * \rho_t^-)(x), \quad \frac{\partial\rho_t^-}{\partial t}(x) = 0.$$

Example 18. Let us investigate here the case of jumping particles. For simplicity let us only consider the case of additive intensities, i.e.

$$(LF)(\gamma) = \sum_{x\in\gamma^+} E(x,\gamma^-)\int_{\mathbb{R}^d} c(x-y)(F(\gamma^+\backslash x \cup y, \gamma^-) - F(\gamma))\mathrm{d}y,$$

where $0 \le a \in L^1(\mathbb{R}^d)$ is symmetric. In such case the operator on quasi-observables is given by

$$\begin{aligned}
&(\widehat{L}G)(\eta)\\
&= -\langle c\rangle \sum_{x\in\eta^+}\sum_{w\in\eta^-} a(x-w)G(\eta) - \langle c\rangle \sum_{x\in\eta^+}\sum_{w\in\eta^-} a(x-w)G(\eta^+, \eta^-\backslash w)\\
&\quad + \sum_{x\in\eta^+}\sum_{w\in\eta^-} a(x-w)\int_{\mathbb{R}^d} c(x-y)(G(\eta^+ \cup y\backslash x, \eta^-\backslash w) + G(\eta^+ \cup y\backslash x, \eta^-))\mathrm{d}y
\end{aligned}$$

and on correlation functions by

$$\begin{aligned}
(L^{\Delta}k)(\eta) &= -\langle c\rangle \sum_{x\in\eta^+}\sum_{w\in\eta^-} a(x-w)k(\eta) - \langle c\rangle \sum_{x\in\eta^+}\int_{\mathbb{R}^d} a(x-w)k(\eta^+, \eta^- \cup w)\mathrm{d}w\\
&\quad + \sum_{y\in\eta^+}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x-w)c(x-y)k(\eta^+\backslash y \cup x, \eta^- \cup w)\mathrm{d}x\mathrm{d}w\\
&\quad + \sum_{y\in\eta^+}\sum_{w\in\eta^-}\int_{\mathbb{R}^d} a(x-w)c(x-w)k(\eta^+\backslash y \cup x, \eta^-)\mathrm{d}x,
\end{aligned}$$

thus $M_\alpha = N_\alpha$ with

$$M_\alpha(\eta) = \langle c\rangle \sum_{x\in\eta^+}\sum_{w\in\eta^-} a(x-w) + 2\langle c\rangle\langle a\rangle e^{\alpha^-}|\eta^+| + \sum_{x\in\eta^+}\sum_{w\in\eta^-}(a*c)(x-w).$$

Scaling the potentials means $a \to \varepsilon a$ and after renormalization and limit transition $\varepsilon \to 0$ we arrive at

$$(\widehat{L}_V G)(\eta) = -\langle c\rangle \sum_{x\in\eta^+}\sum_{w\in\eta^-} a(x-w)G(\eta^+,\eta^-\backslash w)$$
$$+ \sum_{x\in\eta^+}\sum_{w\in\eta^-} a(x-w)\int_{\mathbb{R}^d} c(x-y)G(\eta^+\cup y\backslash x,\eta^-)\mathrm{d}y$$

and

$$(L_V^\Delta k)(\eta) = -\langle c\rangle \sum_{x\in\eta^+}\int_{\mathbb{R}^d} a(x-w)k(\eta^+,\eta^-\cup w)\mathrm{d}w$$
$$+ \sum_{y\in\eta^+}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x-w)c(x-y)k(\eta^+\backslash y\cup x,\eta^-\cup w)\mathrm{d}x\mathrm{d}w,$$

so $N_\alpha^V(\eta) = 2\langle c\rangle\langle a\rangle e^{\alpha^-}|\eta^+|$. Therefore the kinetic description is given by

$$\frac{\partial\rho_t^+}{\partial t}(x) = (c*((a*\rho_t^-)\rho_t^+))(x) - \langle c\rangle(a*\rho_t^-)(x)\rho_t^+(x), \quad \frac{\partial\rho_t^-}{\partial t}(x) = 0.$$

Let us now look at interactions, where it is allowed to change the type of cells. We will only investigate the change from + to − cells, whereas the other case can be obtained, by simply exchanging all + with − and vice versa.

Example 19. In the simplest case, the intensity to change from + to − is constant, here $q > 0$. In such case the Markov generator has the form

$$(LF)(\gamma) = q\sum_{x\in\gamma^+}(F(\gamma^+\backslash x,\gamma^-\cup x) - F(\gamma)).$$

It is not difficult to see, that in this case the operator on quasi-observables will have the form

$$(\widehat{L}G)(\eta) = -q|\eta^+|G(\eta) + q\sum_{x\in\eta^+} G(\eta^+\backslash x,\eta^-\cup x)$$

and on correlation functions it will be given by

$$(L^\Delta k)(\eta) = -q|\eta^+|k(\eta) + q\sum_{x\in\eta^-} k(\eta^+\cup x,\eta^-\backslash x),$$

so $M_\alpha(\eta) = N_\alpha(\eta) = N_\alpha^V(\eta) = q|\eta^+| + qe^{\alpha^+-\alpha^-}|\eta^-|$. Since on scaling is necessary here, we immediately obtain the kinetic description

$$\frac{\partial\rho_t^+}{\partial t}(x) = -q\rho_t^+(x), \quad \frac{\partial\rho_t^-}{\partial t}(x) = q\rho_t^+(x).$$

Example 20. Let us consider density dependent changes of types, where the intensity depends on the same type of particles, in such case the Markov generator is given by

$$(LF)(\gamma) = \sum_{x\in\gamma^+} E(x,\gamma^+)(F(\gamma^+\backslash x, \gamma^- \cup x) - F(\gamma)).$$

The generator on quasi-observables is given by

$$\begin{aligned}(\widehat{L}G)(\eta) = -E(\eta^+)G(\eta) &- \sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a(x-y)G(\eta^+\backslash y, \eta^-)\\ &+ \sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a(x-y)G(\eta^+\backslash x\backslash y, \eta^-\cup x)\\ &+ \sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a(x-y)G(\eta^+\backslash x, \eta^-\cup x)\end{aligned}$$

where $E(\eta^+) = \sum\limits_{x\in\eta^+}\sum\limits_{y\in\eta^+\backslash x} a(x-y)$. Similarly we can compute the operator for correlation functions and obtain

$$\begin{aligned}(L^\Delta k)(\eta) = -E(\eta^+)G(\eta) &- \sum_{x\in\eta^+}\int_{\mathbb{R}^d} a(x-y)k(\eta^+\cup y, \eta^-)\mathrm{d}y\\ &+ \sum_{x\in\eta^-}\int_{\mathbb{R}^d} a(x-y)k(\eta^+\cup x\cup y, \backslash x)\mathrm{d}y\\ &+ \sum_{x\in\eta^-}\sum_{y\in\eta^+} a(x-y)k(\eta^+\cup x, \eta^-\backslash x),\end{aligned}$$

which implies $M_\alpha = N_\alpha$ given by

$$M_\alpha(\eta) = E(\eta^+) + e^{\alpha^+}\langle a\rangle|\eta^+| + e^{2\alpha^+-\alpha^-}\langle a\rangle|\eta^-| + e^{\alpha^+-\alpha^-}\sum_{x\in\eta^-}\sum_{y\in\eta^+} a(x-y).$$

Scaling $a \to \varepsilon a$ and renormalizing we arrive at

$$\begin{aligned}(\widehat{L}_V G)(\eta) = &- \sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a(x-y)G(\eta^+\backslash y, \eta^-)\\ &+ \sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a(x-y)G(\eta^+\backslash x\backslash y, \eta^-\cup x)\end{aligned}$$

and

$$\begin{aligned}(L_V^\Delta k)(\eta) = &- \sum_{x\in\eta^+}\int_{\mathbb{R}^d} a(x-y)k(\eta^+\cup y, \eta^-)\mathrm{d}y\\ &+ \sum_{x\in\eta^-}\int_{\mathbb{R}^d} a(x-y)k(\eta^+\cup x\cup y, \eta^-\backslash x)\mathrm{d}y,\end{aligned}$$

so $N_\alpha^V(\eta) = e^{\alpha^+}\langle a\rangle|\eta^+| + e^{2\alpha^+-\alpha^-}\langle a\rangle|\eta^-|$. Therefore the kinetic description is given by

$$\frac{\partial\rho_t^+}{\partial t}(x) = -\rho_t^+(x)(a*\rho_t^+)(x), \quad \frac{\partial\rho_t^-}{\partial t}(x) = \rho_t^+(x)(a*\rho_t^+)(x).$$

Example 21. In this case the intensity to change the type dependent on the collection of cells of different type, here the Markov generator has the form

$$(LF)(\gamma) = \sum_{x\in\gamma^+} E(x,\gamma^-)(F(\gamma^+\backslash x, \gamma^- \cup x) - F(\gamma)).$$

Some computations yield

$$\begin{aligned}(\widehat{L}G)(\eta) = &- \sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)G(\eta) - \sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)G(\eta^+, \eta^-\backslash y)\\ &+ \sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)G(\eta^+\backslash x, \eta^- \cup x\backslash y)\\ &+ \sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)G(\eta^+\backslash x, \eta^- \cup x)\end{aligned}$$

and

$$\begin{aligned}(L^\Delta k)(\eta) = &- \sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)k(\eta) - \sum_{x\in\eta^+}\int_{\mathbb{R}^d} a(x-y)k(\eta^+, \eta^- \cup y)\mathrm{d}y\\ &+ \sum_{x\in\eta^-}\int_{\mathbb{R}^d} a(x-y)k(\eta^+ \cup x, \eta^- \cup y\backslash x)\mathrm{d}y\\ &+ \sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x} a(x-y)k(\eta^+ \cup y, \eta^-\backslash y).\end{aligned}$$

This yields $M_\alpha = N_\alpha$ with

$$M_\alpha(\eta) = \sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y) + e^{\alpha^-}\langle a\rangle|\eta^+| + e^{\alpha^+}\langle a\rangle|\eta^-| + e^{\alpha^+-\alpha^-}\sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x} a(x-y).$$

Scaling $a \to \varepsilon a$ and renormalizing we obtain

$$\begin{aligned}(\widehat{L}_V G)(\eta) = &- \sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)G(\eta^+, \eta^-\backslash y)\\ &+ \sum_{x\in\eta^+}\sum_{y\in\eta^-} a(x-y)G(\eta^+\backslash x, \eta^- \cup x\backslash y)\end{aligned}$$

and

$$\begin{aligned}(L^\Delta k)(\eta) = &- \sum_{x\in\eta^+}\int_{\mathbb{R}^d} a(x-y)k(\eta^+, \eta^- \cup y)\mathrm{d}y\\ &+ \sum_{x\in\eta^-}\int_{\mathbb{R}^d} a(x-y)k(\eta^+ \cup x, \eta^- \cup y\backslash x)\mathrm{d}y,\end{aligned}$$

so $N_\alpha^V(\eta) = e^{\alpha^-}\langle a\rangle|\eta^+| + e^{\alpha^+}\langle a\rangle|\eta^-|$. Finally the kinetic equation is given by

$$\frac{\partial\rho_t^+}{\partial t}(x) = -\rho_t^+(x)(a*\rho_t^-)(x), \quad \frac{\partial\rho_t^-}{\partial t}(x) = \rho_t^+(x)(a*\rho_t^-)(x).$$

Example 22. Let us take here exponential decaying intensities for changing the type. More precisely the Markov generator is given by

$$(LF)(\gamma) = \sum_{x\in\gamma^+} e^{-E_\phi(x,\gamma^-)} e^{-E_\psi(x,\gamma^+\backslash x)} (F(\gamma^+\backslash x, \gamma^-\cup x) - F(\gamma)).$$

The operator on quasi-observables is given by

$$\begin{aligned}(\widehat{L}G)(\eta) = &\sum_{\xi\subset\eta}\sum_{x\in\xi^+} e^{-E_\phi(x,\xi^-)} e^{-E_\psi(x,\xi^+\backslash x)} (G(\xi^+\backslash x, \xi^-\cup x) - G(\xi))\\ &\times e_\lambda(e^{-\phi(x-\cdot)}-1;\eta^-\backslash\xi^-) e_\lambda(e^{-\psi(x-\cdot)}-1;\eta^+\backslash\xi^+)\end{aligned}$$

and on correlation functions by

$$\begin{aligned}&(L^\Delta k)(\eta)\\ &= \sum_{x\in\eta^-} e^{-E_\phi(x,\eta^-\backslash x)} e^{-E_\psi(x,\eta^+)}\\ &\quad\times\int_{\Gamma_0^2} e_\lambda(e^{-\phi(x-\cdot)}-1;\xi^-) e_\lambda(e^{-\psi(x-\cdot)}-1;\xi^+) k(\eta^+\cup\xi^+\cup x, \eta^-\cup\xi^-\backslash x)\mathrm{d}\lambda^2(\xi)\\ &\quad-\sum_{x\in\eta^+} e^{-E_\phi(x,\eta^-)} e^{-E_\psi(x,\eta^+\backslash x)}\\ &\quad\times\int_{\Gamma_0^2} e_\lambda(e^{-\phi(x-\cdot)}-1;\xi^-) e_\lambda(e^{-\psi(x-\cdot)}-1;\xi^+) k(\eta\cup\xi)\mathrm{d}\lambda^2(\xi),\end{aligned}$$

which implies

$$\begin{aligned}M_\alpha(\eta) = &e^{\alpha^+-\alpha^-}\beta_\psi(\alpha^+)\beta_\phi(\alpha^-)\sum_{x\in\eta^-} e^{-E_\psi(x,\eta^+)} e^{-E_\phi(x,\eta^-\backslash x)}\\ &+\beta_\psi(\alpha^+)\beta_\phi(\alpha^-)\sum_{x\in\eta^+} e^{-E_\psi(x,\eta^+\backslash x)} e^{-E_\phi(x,\eta^-)}\end{aligned}$$

with

$$\beta_\psi(\alpha^+) = \exp\Bigg(e^{\alpha^+}\int_{\mathbb{R}^d}(1-e^{-\psi(x)})\mathrm{d}x\Bigg), \quad \beta_\phi(\alpha^-)\exp\Bigg(e^{\alpha^-}\int_{\mathbb{R}^d}(1-e^{-\phi(x)})\mathrm{d}x\Bigg).$$

Scaling $\phi, \psi \to \varepsilon\phi, \varepsilon\psi$ and renormalize we obtain

$$\begin{aligned}(\widehat{L}_{\varepsilon,ren}G)(\eta) = &\sum_{\xi\subset\eta}\sum_{x\in\xi^+} e^{-\varepsilon E_\phi(x,\xi^-)} e^{-\varepsilon E_\psi(x,\xi^+\backslash x)} (G(\xi^+\backslash x, \xi^-\cup x) - G(\xi))\\ &\times e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\eta^-\backslash\xi^-\right) e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\eta^+\backslash\xi^+\right)\end{aligned}$$

and

$$
\begin{aligned}
&(L^{\Delta}_{\varepsilon,ren}k)(\eta)\\
&= \sum_{x\in\eta^-} e^{-\varepsilon E_\phi(x,\eta^-\setminus x)} e^{-\varepsilon E_\psi(x,\eta^+)}\\
&\quad \times \int_{\Gamma_0^2} e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\xi^-\right) e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\xi^+\right)\\
&\quad \times k(\eta^+\cup\xi^+\cup x, \eta^-\cup\xi^-\setminus x)\mathrm{d}\lambda^2(\xi)\\
&\quad - \sum_{x\in\eta^+} e^{-\varepsilon E_\phi(x,\eta^-)} e^{-\varepsilon E_\psi(x,\eta^+\setminus x)}\\
&\quad \times \int_{\Gamma_0^2} e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon};\xi^-\right) e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\xi^+\right) k(\eta\cup\xi)\mathrm{d}\lambda^2(\xi)
\end{aligned}
$$

so that

$$
N_\alpha(\eta) = \exp(e^{\alpha^+}\langle\psi\rangle + e^{\alpha^-}\langle\phi\rangle)(e^{\alpha^+-\alpha^-}|\eta^-| + |\eta^+|) = N_\alpha^V(\eta).
$$

In the limit $\varepsilon \to 0$ we arrive at

$$
\begin{aligned}
&(\widehat{L}_V G)(\eta)\\
&= \sum_{\xi\subset\eta}\sum_{x\in\xi^+} (G(\xi^+\setminus x, \xi^-\cup x) - G(\xi)) e_\lambda(-\phi(x-\cdot);\eta^-\setminus\xi^-) e_\lambda(-\psi(x-\cdot);\eta^+\setminus\xi^+)
\end{aligned}
$$

and

$$
\begin{aligned}
&(L^{\Delta}_V k)(\eta)\\
&= \sum_{x\in\eta^-}\int_{\Gamma_0^2} e_\lambda(-\phi(x-\cdot);\xi^-) e_\lambda(-\psi(x-\cdot);\xi^+) k(\eta^+\cup\xi^+\cup x, \eta^-\cup\xi^-\setminus x)\mathrm{d}\lambda^2(\xi)\\
&\quad - \sum_{x\in\eta^+}\int_{\Gamma_0^2} e_\lambda(-\phi(x-\cdot);\xi^-) e_\lambda(-\psi(x-\cdot);\xi^+) k(\eta\cup\xi)\mathrm{d}\lambda^2(\xi)
\end{aligned}
$$

and hence the kinetic description is given by

$$
\frac{\partial\rho_t^+}{\partial t}(x) = -\rho_t^+(x) e^{-(\phi*\rho_t^-)(x)} e^{-(\psi*\rho_t^+)(x)}, \quad \frac{\partial\rho_t^-}{\partial t}(x) = \rho_t^+(x) e^{-(\phi*\rho_t^-)(x)} e^{-(\psi*\rho_t^+)(x)}.
$$

5.1 Cell-death model

Let us start with the analysis of the first model stated in the context of two-component systems, the heuristic Markov generator is given by, c.f. (23),

$$
(LF)(\gamma^+,\gamma^-) = (AF)(\gamma^+,\gamma^-) + (BF)(\gamma^+,\gamma^-) + (VF)(\gamma^+,\gamma^-).
$$

The first operator A is the Contact Model for usual cells and has the form

$$(L_{CM}F)(\gamma^+,\gamma^-) = m_0 \sum_{x\in\gamma^+} (F(\gamma^+\backslash x, \gamma^- \cup x) - F(\gamma^+,\gamma^-)) \\ + \lambda \sum_{x\in\gamma^+} \int_{\mathbb{R}^d} a(x-y)(F(\gamma^+\cup y, \gamma^-) - F(\gamma^+,\gamma^-))\mathrm{d}y.$$

The operator B describes the evolution of $-$ cells, which can only disappear from the system, so it has the simple form

$$(BF)(\gamma^+,\gamma^-) = m_1 \sum_{x\in\gamma^-} (F(\gamma^+,\gamma^-\backslash x) - F(\gamma^+,\gamma^-)).$$

The last part describes the interaction of both types and is assumed to be of the form

$$(VF)(\gamma^+,\gamma^-) = \lambda^- \sum_{x\in\gamma^+}\sum_{y\in\gamma^-} \varphi(x-y)(F(\gamma^+\backslash x, \gamma^-\cup x) - F(\gamma^+,\gamma^-)).$$

The intensities $m_0, m_1, \lambda, \lambda^-$ are strictly positive and the potentials $0 \le a, \varphi \in L^1(\mathbb{R}^d)$ are symmetric and normalized to 1. In [9] the general form of $\widehat{L} = \widehat{A} + \widehat{B} + \widehat{V}$ was computed for $G \in B_{bs}(\Gamma_0^2)$. In this special case we get

$$(\widehat{A}G)(\eta^+,\eta^-) = -m_0|\eta^+|G(\eta^+,\eta^-) + m_0 \sum_{x\in\eta^+} G(\eta^+\backslash x, \eta^-\cup x) \\ + \lambda \sum_{x\in\eta^+} \int_{\mathbb{R}^d} a(x-y)G(\eta^+\backslash x \cup y, \eta^-)\mathrm{d}y \\ + \lambda \sum_{x\in\eta^+} \int_{\mathbb{R}^d} a(x-y)G(\eta^+ \cup y, \eta^-)\mathrm{d}y$$

for the first part

$$(BG)(\eta^+,\eta^-) = -m_1|\eta^-|G(\eta^+,\eta^-)$$

for the second part, and finally

$$(\widehat{V}G)(\eta^+,\eta^-) = \lambda^- \sum_{x\in\eta^+}\sum_{y\in\eta^-} \varphi(x-y)(G(\eta^+\backslash x, \eta^-\cup x\backslash y) - G(\eta^+,\eta^-\backslash y)) \\ + \lambda^- \sum_{x\in\eta^+}\sum_{y\in\eta^-} \varphi(x-y)(G(\eta^+\backslash x, \eta^-\cup x) - G(\eta^+,\eta^-)).$$

Let us first realize this operator on the Banach space $\mathbb{B}_\alpha$.

Lemma 5.1. *The corresponding function $M_\alpha = M_\alpha^A + M^B + M_\alpha^V$ is given by*

$$M_\alpha^A(\eta^+,\eta^-) = (m_0 + e^{\alpha^+-\alpha^-}m_0 + \lambda)|\eta^+| + \lambda e^{-\alpha^+} \sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a(x-y)$$

$$M^B(\eta^+,\eta^-) = m_1|\eta^-|$$

$$M_\alpha^V(\eta^+,\eta^-) = \lambda^-(e^{\alpha^-}|\eta^+| + e^{\alpha^+}|\eta^-|) + \lambda^- e^{\alpha^+-\alpha^-} \sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x} \varphi(x-y) \\ + \lambda^- \sum_{x\in\eta^+}\sum_{y\in\eta^-} \varphi(x-y).$$

If $a, \varphi \in L^\infty(\mathbb{R}^d)$, then $\widehat{L} \in L(\mathbb{B}_\alpha, \mathbb{B}_{\alpha'})$ for any $\alpha' < \alpha$.

Proof. Let $G \in D(M_\alpha)$, then clearly $\widehat{A}G, \widehat{B}G \in \mathbb{B}_\alpha$, so we will only check $\widehat{V}G \in \mathbb{B}_\alpha$, which follows from

$$\begin{aligned}
&\int_{\Gamma_0^2} \sum_{x\in\eta^+} \sum_{y\in\eta^-} \varphi(x-y)|G(\eta^+\backslash x, \eta^- \cup x\backslash y)| e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}\lambda(\eta^+,\eta^-) \\
&= e^{\alpha^+ + \alpha^-} \int_{\Gamma_0^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x-y)|G(\eta^+, \eta^- \cup x)| e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}x\mathrm{d}y\mathrm{d}\lambda(\eta^+,\eta^-) \\
&= e^{\alpha^+} \int_{\Gamma_0^2} \sum_{x\in\eta^-} \int_{\mathbb{R}^d} \varphi(x-y)|G(\eta^+, \eta^-)| e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}y\mathrm{d}\lambda(\eta^+,\eta^-) \\
&= e^{\alpha^+} \int_{\Gamma_0^2} |\eta^-||G(\eta^+,\eta^-)| \mathrm{d}\lambda^2(\eta^+,\eta^-)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Gamma_0^2} \sum_{x\in\eta^+} \sum_{y\in\eta^-} \varphi(x-y)|G(\eta^+\backslash x, \eta^- \cup x)| e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}\lambda(\eta^+,\eta^-) \\
&= e^{\alpha} \int_{\Gamma_0^2} \sum_{y\in\eta^-} \int_{\mathbb{R}^d} \varphi(x-y)|G(\eta^+, \eta^- \cup x) e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}x\mathrm{d}\lambda(\eta^+,\eta^-) \\
&= \int_{\Gamma_0^2} \sum_{x\in\eta^-} \sum_{y\in\eta^-\backslash x} \varphi(x-y)|G(\eta^+, \eta^-)| e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}\lambda(\eta^+,\eta^-).
\end{aligned}$$

The contributions from the negative parts can be dealt in the same way and the estimate for $\|\widehat{L}\|_{\alpha\alpha'}$ can be shown like in the one-component case. □

Again the computation of the operator L^Δ was done for a more general case in [9] which shows that for $|k(\eta)| \leq |\eta|!C^{|\eta|}$ for some $C > 0$ the operator $L^\Delta = A^\Delta + B^\Delta + V^\Delta$ is given by

$$\begin{aligned}
(A^\Delta k)(\eta^+,\eta^-) = -m_0|\eta^+|k(\eta^+,\eta^-) &+ \sum_{x\in\eta^-} k(\eta^+ \cup x, \eta^-\backslash x) \\
&+ \lambda \sum_{x\in\eta^+} \int_{\mathbb{R}^d} a(x-y)k(\eta^+\backslash x \cup y, \eta^-)\mathrm{d}y \\
&+ \lambda \sum_{x\in\eta^+} \sum_{y\in\eta^+\backslash x} a(x-y)k(\eta^+\backslash x, \eta^-)
\end{aligned}$$

and

$$(Bk)(\eta^+,\eta^-) = -m_1|\eta^-|k(\eta^+,\eta^-)|$$

and

$$\begin{aligned}(V^{\Delta}k)(\eta^+,\eta^-) = \lambda^- \sum_{x\in\eta^-} \int_{\mathbb{R}^d} \varphi(x-y)k(\eta^+\cup x, \eta^-\cup y\backslash x)\mathrm{d}y \\ - \lambda^- \sum_{x\in\eta^+} \int_{\mathbb{R}^d} \varphi(x-y)k(\eta^+, \eta^-\cup y)\mathrm{d}y \\ + \lambda^- \sum_{x\in\eta^-} \sum_{y\in\eta^-\backslash x} \varphi(x-y)k(\eta^+\cup x, \eta^-\backslash x) \\ - \lambda^- \sum_{x\in\eta^+} \sum_{y\in\eta^-} \varphi(x-y)k(\eta^+,\eta^-).\end{aligned}$$

As before (43) can be used to realize L^{Δ} on $\mathbb{B}_{\alpha}^*$.

Scaling

For scaling let us scale the potentials a, φ to εa and $\varepsilon\varphi$, then the renormalized operator will have the form $\widehat{L}_{\varepsilon,ren} = \widehat{L}_V + \varepsilon C$ given by

$$\begin{aligned}(\widehat{L}_V G)(\eta^+,\eta^-) = -m_0|\eta^+|G(\eta^+,\eta^-) - m_1|\eta^-|G(\eta^+,\eta^-) \\ + \sum_{x\in\eta^+} G(\eta^+\backslash x, \eta^-\cup x) + \lambda \sum_{x\in\eta^+} \int_{\mathbb{R}^d} a(x-y)G(\eta^+\backslash x\cup y)\mathrm{d}y \\ + \sum_{x\in\eta^+}\sum_{y\in\eta^-} \varphi(x-y)(G(\eta^+\backslash x, \eta^-\cup x\backslash y) - G(\eta^+,\eta^-\backslash y))\end{aligned}$$

and

$$\begin{aligned}(CG)(\eta) = \lambda \sum_{x\in\eta^+} \int_{\mathbb{R}^d} a(x-y)G(\eta^+\cup y, \eta^-)\mathrm{d}y \\ + \lambda^- \sum_{x\in\eta^+}\sum_{y\in\eta^-} \varphi(x-y)(G(\eta^+\backslash x, \eta^-\cup x) - G(\eta^+,\eta^-))\end{aligned}$$

Therefore the function the function N_α is given by M_α. Concerning convergence of the generators we obtain the following.

Theorem 5.2. *For each $G \in D(N_\alpha)$*

$$\widehat{L}_{\varepsilon,ren}G \to \widehat{L}_V G, \quad \varepsilon \to 0$$

in $\mathbb{B}_\alpha$. If $a, \varphi \in L^\infty(\mathbb{R}^d)$, then for all $\alpha' < \alpha$

$$\|\widehat{L}_{\varepsilon,ren} - \widehat{L}_V\|_{\alpha\alpha'} \to 0, \quad \varepsilon \to 0$$

holds.

The dual operators can be simply computed and are given by:

$$\begin{aligned}(L_V^{\Delta}k)(\eta^+,\eta^-) &= -m_0|\eta^+|k(\eta^+,\eta^-) - m_1|\eta^-|k(\eta^+,\eta^-) + \sum_{x\in\eta^-} k(\eta^+\cup x,\eta^-\backslash x)\\ &+ \lambda \sum_{x\in\eta^+}\int_{\mathbb{R}^d} a(x-y)k(\eta^+\backslash x\cup y,\eta^-)\mathrm{d}y\\ &+ \lambda^- \sum_{x\in\eta^-}\int_{\mathbb{R}^d} \varphi(x-y)k(\eta^+\cup x,\eta^-\cup y\backslash x)\mathrm{d}y\\ &- \lambda^- \sum_{x\in\eta^+}\int_{\mathbb{R}^d} \varphi(x-y)k(\eta^+,\eta^-\cup y)\mathrm{d}y\end{aligned}$$

and

$$\begin{aligned}(C^{\Delta}k)(\eta^+,\eta^-) &= -\lambda^- \sum_{x\in\eta^+}\sum_{y\in\eta^-} \varphi(x-y)k(\eta^+,\eta^-)\\ &+ \lambda^- \sum_{x\in\eta^-}\sum_{y\in\eta^-} \varphi(x-y)k(\eta^+\cup x,\eta^-\backslash x)\\ &+ \lambda \sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a(x-y)k(\eta^+\backslash x,\eta^-).\end{aligned}$$

If $a,\varphi \in L^\infty(\mathbb{R}^d)$, then

$$\|L^{\Delta}_{\varepsilon,ren} - L^{\Delta}_V\|_{\alpha'\alpha} \to 0, \quad \varepsilon \to 0.$$

Let us finally compute $L_V^{\Delta}e_\lambda(\rho^+)e_\lambda(\rho^-)$ and derive from this the kinetic description.

$$\begin{aligned}&(L_V^{\Delta}e_\lambda(\rho^+)e_\lambda(\rho^-))(\eta^+,\eta^-)\\ &= \sum_{x\in\eta^-}\int_{\mathbb{R}^d} \varphi(x-y)\rho^+(x)\rho^-(y)e_\lambda(\rho^+;\eta^+)e_\lambda(\rho^-;\eta^-\backslash x)\\ &- \sum_{x\in\eta^+}\int_{\mathbb{R}^d} \varphi(x-y)\rho^+(x)e_\lambda(\rho^+;\eta^+\backslash x)\rho^-(y)e_\lambda(\rho^-;\eta^-)\\ &- \sum_{x\in\eta^+} m_0\rho^+(x)e_\lambda(\rho^+;\eta^+\backslash x)e_\lambda(\rho^-;\eta^-)\\ &- \sum_{x\in\eta^-} m_1\rho^-(x)e_\lambda(\rho^-;\eta^-\backslash y)e_\lambda(\rho^+;\eta^+)\\ &+ \sum_{x\in\eta^+} \lambda\int_{\mathbb{R}^d} a(x-y)\rho^+(y)\mathrm{d}y e_\lambda(\rho^+;\eta^+\backslash x)e_\lambda(\rho^-;\eta^-)\\ &+ \sum_{x\in\eta^-} m_0\rho^+(x)e_\lambda(\rho^+;\eta^+)e_\lambda(\rho^-;\eta^-\backslash x)\end{aligned}$$

and thus the system of equations for ρ_t^+ and ρ_t^- is given by, c.f. (24)

$$\begin{cases} \dfrac{\partial \rho_t^-}{\partial t}(x) = -m_1\rho_t^-(x) + \rho_t^+(x)(\varphi * \rho_t^-)(x) + m_0\rho_t^+(x) \\ \dfrac{\partial \rho_t^+}{\partial t}(x) = -(m_0 + (\varphi * \rho_t^-)(x))\rho_t^+(x) + (a * \rho_t^+)(x) \end{cases}.$$

5.2 Go-or-grow models

First model

Here the first model is given by $L = L_{CM} + L_{hop} + V$, where L_{CM} is given by

$$\begin{aligned}(L_{CM}F)(\gamma^+,\gamma^-) &= m \sum_{x\in\gamma^-} (F(\gamma^+,\gamma^-\backslash x) - F(\gamma^+,\gamma^-)) \\ &\quad + \lambda \sum_{x\in\gamma^-} \int_{\mathbb{R}^d} a(x-y)(F(\gamma^+,\gamma^-\cup x) - F(\gamma^+,\gamma^-))\mathrm{d}y\end{aligned}$$

and is describing the proliferation of the −-cells. The density independent intensity of death is given by $m > 0$ and the proliferation intensity by $\lambda > 0$. The kernel $0 \le a \in L^1(\mathbb{R}^d)$ is again symmetric and normalized to 1. The motion of the moving +-cells is described by

$$\begin{aligned}(L_{hop}F)(\gamma^+,\gamma^-) &= d \sum_{x\in\gamma^+} (F(\gamma^+\backslash x,\gamma^-) - F(\gamma^+,\gamma^-)) \\ &\quad + \sum_{x\in\gamma^+} \int_{\mathbb{R}^d} c(x-y)(F(\gamma^+\backslash x\cup y,\gamma^-) - F(\gamma^+,\gamma^-))\mathrm{d}y.\end{aligned}$$

Here we included also density independent mortality of the moving cells with intensity $d > 0$. The microscopic behaviour to change from one type (state) to another is given by

$$\begin{aligned}(VF)(\gamma^+,\gamma^-) &= q \sum_{x\in\gamma^+} (F(\gamma^+\backslash x,\gamma^-\cup y) - F(\gamma^+,\gamma^-)) \\ &\quad + \sum_{x\in\gamma^-} \left(p + \sum_{y\in\gamma^-\backslash x} \varphi(x-y)\right)(F(\gamma^+\cup x,\gamma^-\backslash x) - F(\gamma^+,\gamma^-)).\end{aligned}$$

The operator for quasi-observables $\widehat{L} = \widehat{L}_{CM} + \widehat{L}_{hop} + \widehat{V}$ is given by, c.f. [9]

$$\begin{aligned}(\widehat{V}G)(\eta^+,\eta^-) &= q \sum_{x\in\eta^+} (G(\eta^+\backslash x,\eta^-\cup x) - G(\eta^+,\eta^-)) \\ &\quad + p \sum_{x\in\eta^-} (G(\eta^+\cup x,\eta^-\backslash x) - G(\eta^+,\eta^-)) \\ &\quad + \sum_{x\in\eta^-} \sum_{y\in\eta^-\backslash x} \varphi(x-y)(G(\eta^+\cup x,\eta^-\backslash x\backslash y) - G(\eta^+,\eta^-\backslash y)) \\ &\quad + \sum_{x\in\eta^-} \sum_{y\in\eta^-\backslash x} \varphi(x-y)(G(\eta^+\cup x,\eta^-\backslash x) - G(\eta^+,\eta^-))\end{aligned}$$

and

$$(\widehat{L}_{hop}G)(\eta^+,\eta^-) = -d|\eta^+|G(\eta^+,\eta^-) + \sum_{x\in\eta^+}\int_{\mathbb{R}^d} c(x-y)(G(\eta^+\backslash x\cup y,\eta^-) - G(\eta^+,\eta^-))\mathrm{d}y.$$

The expression for $\widehat{L}_{CM}$ is similar to those before and is given by

$$(\widehat{L}_{CM}G)(\eta^+,\eta^-) = -m|\eta^-|G(\eta^+,\eta^-) + \lambda\sum_{x\in\eta^-}\int_{\mathbb{R}^d} a(x-y)G(\eta^+,\eta^-\backslash x\cup y)\mathrm{d}y + \lambda\sum_{x\in\eta^-}\int_{\mathbb{R}^d} a(x-y)G(\eta^+,\eta^-\cup y)\mathrm{d}y.$$

Lemma 5.3. *The function* $M_\alpha = M_\alpha^{CM} + M^{hop} + M_\alpha^V$ *is given by*

$$\begin{aligned} M_\alpha^{CM}(\eta^+,\eta^-) &= (m+\lambda)|\eta^-| + \lambda e^{-\alpha^-}\sum_{x\in\eta^-}\sum_{x\in\eta^-\backslash y} a(x-y)\\ M^{hop}(\eta^+,\eta^-) &= (d+2\langle c\rangle)|\eta^+|\\ M_\alpha^V(\eta^+,\eta^-) &= |\eta^+|(q+pe^{\alpha^--\alpha^+} + e^{2\alpha^--\alpha^+}\langle\varphi\rangle)\\ &\quad + |\eta^-|(p+qe^{\alpha^+-\alpha^-} + e^{2\alpha^--\alpha^+}\langle\varphi\rangle)\\ &\quad + \sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x}\varphi(x-y) + e^{\alpha^--\alpha^+}\sum_{x\in\eta^+}\sum_{y\in\eta^-}\varphi(x-y). \end{aligned}$$

If $a,\varphi\in L^\infty(\mathbb{R}^d)$ *then* $\widehat{L}\in L(\mathbb{B}_\alpha,\mathbb{B}_{\alpha'})$ *for any* $\alpha'<\alpha$.

Proof. We will only compute the function M_α^V for three terms, the rest can be done in the same way.

$$\begin{aligned} &\int_{\Gamma_0^2}\sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x}\varphi(x-y)|G(\eta^+\cup x,\eta^-\backslash x\backslash y)|e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}\mathrm{d}\lambda(\eta^+,\eta^-)\\ &= e^{2\alpha^-}\int_{\Gamma_0^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(x-y)|G(\eta^+\cup x,\eta^-)|e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}\mathrm{d}x\mathrm{d}y\mathrm{d}\lambda(\eta^+,\eta^-)\\ &= e^{2\alpha^--\alpha^+}\int_{\Gamma_0^2}\int_{\mathbb{R}^d}\sum_{x\in\eta^+}\varphi(x-y)|G(\eta^+,\eta^-)|e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}\mathrm{d}y\mathrm{d}\lambda(\eta^+,\eta^-)\\ &= e^{2\alpha^--\alpha^+}\langle\varphi\rangle\int_{\Gamma_0^2}|\eta^+||G(\eta^+,\eta^-)|e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}\mathrm{d}\lambda(\eta^+,\eta^-) \end{aligned}$$

and

$$\int_{\Gamma_0^2}\sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x}\varphi(x-y)|G(\eta^+,\eta^-\backslash y)|e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}\mathrm{d}\lambda(\eta^+,\eta^-)$$

$$= e^{2\alpha^-} \int_{\Gamma_0^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x-y)|G(\eta^+, \eta^- \cup x)| e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}x \mathrm{d}y \mathrm{d}\lambda^2(\eta^+, \eta^-)$$

$$= e^{\alpha^-} \int_{\Gamma_0^2} \sum_{x \in \eta^-} \int_{\mathbb{R}^d} \varphi(x-y)|G(\eta^+, \eta^-)| e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}y \mathrm{d}\lambda^2(\eta^+, \eta^-)$$

$$\leq e^{\alpha^-} \langle \varphi \rangle \int_{\Gamma_0^2} |\eta^-||G(\eta^+, \eta^-)| e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}\lambda(\eta^+, \eta^-)$$

and, finally,

$$\int_{\Gamma_0^2} \sum_{x \in \eta^-} \sum_{y \in \eta^- \setminus x} \varphi(x-y)|G(\eta^+ \cup x, \eta^- \setminus x)| e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}\lambda(\eta^+, \eta^-)$$

$$= e^{\alpha^-} \int_{\Gamma_0^2} \sum_{y \in \eta^-} \int_{\mathbb{R}^d} \varphi(x-y)|G(\eta^+ \cup x, \eta^-)| e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}x \mathrm{d}\lambda^2(\eta^+, \eta^-)$$

$$= e^{\alpha^- - \alpha^+} \int_{\Gamma_0^2} \sum_{x \in \eta^+} \sum_{y \in \eta^-} \varphi(x-y)|G(\eta^+, \eta^-)| e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} \mathrm{d}\lambda(\eta^+, \eta^-). \quad \square$$

Next we easily see that

$$(L_{hop}^{\Delta} k)(\eta^+, \eta^-) = -d|\eta^+| k(\eta^+, \eta^-) + \sum_{x \in \eta^+} \int_{\mathbb{R}^d} c(x-y)(k(\eta^+ \setminus x \cup y, \eta^-) - k(\eta^+, \eta^-)) \mathrm{d}y$$

and

$$\begin{aligned}(V^{\Delta} k)(\eta^+, \eta^-) = {} & q \sum_{x \in \eta^-} k(\eta^+ \cup x, \eta^- \setminus x) - q|\eta^+| k(\eta^+, \eta^-) \\ & + p \sum_{x \in \eta^+} k(\eta^+ \setminus x, \eta^- \cup x) - p|\eta^-| k(\eta^+, \eta^-) \\ & + \sum_{x \in \eta^+} \int_{\mathbb{R}^d} \varphi(x-y) k(\eta^+ \setminus x, \eta^- \cup x \cup y) \mathrm{d}y \\ & + \sum_{x \in \eta^+} \sum_{y \in \eta^-} \varphi(x-y) k(\eta^+ \setminus x, \eta^- \cup x) \\ & - \sum_{x \in \eta^-} \int_{\mathbb{R}^d} \varphi(x-y) k(\eta^+, \eta^- \cup y) \mathrm{d}y \\ & - \sum_{x \in \eta^-} \sum_{y \in \eta^- \setminus x} \varphi(x-y) k(\eta^+, \eta^-).\end{aligned}$$

Again under the conditions $a, \varphi \in L^\infty(\mathbb{R}^d)$ this expression can be well-defined as an element of $L(\mathbb{B}_{\alpha'}^*, \mathbb{B}_\alpha^*)$ with the same norm estimate as $\|\widehat{L}\|_{\alpha\alpha'}$.

In previous section the kinetic description for each term contained in $\widehat{L}$ was derived, so let us give only a short outline how it works in this particular case.

Since the jumping part is free, c.f. $\phi = 0 = \psi$ from previous section, the operator L_{hop} will not change after renormalization. So let us scale the potential φ by $\varepsilon\varphi$. This will lead to the renormalized operator

$$\begin{aligned}(\widehat{V}_{\varepsilon,ren}G)(\eta^+,\eta^-) &= q\sum_{x\in\eta^+}(G(\eta^+\backslash x,\eta^-\cup x)-G(\eta^+,\eta^-))\\ &+p\sum_{x\in\eta^-}(G(\eta^+\cup x,\eta^-\backslash x)-G(\eta^+,\eta^-))\\ &+\sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x}\varphi(x-y)(G(\eta^+\cup x,\eta^-\backslash x\backslash y)-G(\eta^+,\eta^-\backslash y))\\ &+\varepsilon\sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x}\varphi(x-y)(G(\eta^+\cup x,\eta^-\backslash x)-G(\eta^+,\eta^-))\end{aligned}$$

and thus we get.

Theorem 5.4. *For each $G\in D(M_\alpha)$ we have $\widehat{L}_{\varepsilon,ren}G\in\mathbb{B}_\alpha$ and*

$$\widehat{L}_{\varepsilon,ren}G\to\widehat{L}_VG,\quad \varepsilon\to 0$$

in $\mathbb{B}_\alpha$, where $\widehat{L}_V=\widehat{A}+\widehat{L}_{hop}+\widehat{V}_V$ is a superposition of the limiting part for the contact model, the operator $\widehat{L}_{hop}$ and

$$\begin{aligned}(\widehat{V}_VG)(\eta^+,\eta^-) &= q\sum_{x\in\eta^+}(G(\eta^+\backslash x,\eta^-\cup x)-G(\eta^+,\eta^-))\\ &+p\sum_{x\in\eta^-}(G(\eta^+\cup x,\eta^-\backslash x)-G(\eta^+,\eta^-))\\ &+\sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x}\varphi(x-y)(G(\eta^+\cup x,\eta^-\backslash x\backslash y)-G(\eta^+,\eta^-\backslash y)).\end{aligned}$$

Assume $a,\varphi\in L^\infty$, then for all $\alpha'<\alpha$

$$\|\widehat{L}_{\varepsilon,ren}-\widehat{L}_V\|_{\alpha\alpha'}\to 0,\quad \varepsilon\to 0.$$

and $\widehat{A}$ was given above.

The same result holds for correlation function operators with

$$\begin{aligned}(V_V^{\triangle}k)(\eta^+,\eta^-) &= q\sum_{x\in\eta^-}k(\eta^+\cup x,\eta^-\backslash x)-q|\eta^+|k(\eta^+,\eta^-)\\ &= p\sum_{x\in\eta^+}k(\eta^+\backslash x,\eta^-\cup x)-p|\eta^-|k(\eta^+,\eta^-)\\ &\quad+\sum_{x\in\eta^+}\int_{\mathbb{R}^d}\varphi(x-y)k(\eta^+\backslash x,\eta^-\cup x\cup y)\mathrm{d}y\\ &\quad-\sum_{x\in\eta^-}\int_{\mathbb{R}^d}\varphi(x-y)k(\eta^+,\eta^-\cup y)\mathrm{d}y.\end{aligned}$$

Let us now compute $(\widehat{V}_V e_\lambda(\rho^+)e_\lambda(\rho^-))(\eta^+,\eta^-)$. This is given by

$$\begin{aligned}
&(\widehat{V}_V e_\lambda(\rho^+)e_\lambda(\rho^-))(\eta^+,\eta^-)\\
&= q\sum_{x\in\eta^-}\rho^+(x)e_\lambda(\rho^+;\eta^+)e_\lambda(\rho^-;\eta^-\backslash x)\\
&\quad - q\sum_{x\in\eta^+}\rho^+(x)e_\lambda(\rho^+;\eta^+\backslash x)e_\lambda(\rho^-;\eta^-)\\
&\quad + p\sum_{x\in\eta^+}\rho^-(x)e_\lambda(\rho^-;\eta^-)e_\lambda(\rho^+;\eta^+\backslash x)\\
&\quad - p\sum_{x\in\eta^-}\rho^-(x)e_\lambda(\rho^-;\eta^-\backslash x)e_\lambda(\rho^+;\eta^+)\\
&\quad + \sum_{x\in\eta^+}e_\lambda(\rho^+;\eta^+\backslash x)e_\lambda(\rho^-;\eta^-)\rho^-(x)\int_{\mathbb{R}^d}\varphi(x-y)\rho^-(y)\mathrm{d}y\\
&\quad - \sum_{x\in\eta^-}e_\lambda(\rho^+;\eta^+)e_\lambda(\rho^-;\eta^-)\rho^-(x)\int_{\mathbb{R}^d}\varphi(x-y)\rho^-(y)\mathrm{d}y
\end{aligned}$$

and hence the kinetic description is given by

$$\begin{aligned}
\frac{\partial\rho^+}{\partial t}(x) &= -(\langle c\rangle + q + d)\rho^+(x) + (c*\rho^+)(x) + p\rho^-(x) + \rho^-(x)(\varphi*\rho^-)(x)\\
\frac{\partial\rho^-}{\partial t}(x) &= -(m+p)\rho^-(x) + \lambda(a*\rho^-)(x) - \rho^-(x)(\varphi*\rho^-)(x) + q\rho^+(x).
\end{aligned}$$

Second model

Now let us investigate the second model. Here $L = L_{CM} + L_{hop} + V$ with the operator $V = V_1 + V_2$ slightly changed to

$$\begin{aligned}
(VF)(\gamma^+,\gamma^-) &= q\sum_{x\in\gamma^+}\exp\Big(-\sum_{y\in\gamma^-}\psi(x-y)\Big)(F(\gamma^+\backslash x,\gamma^-\cup y) - F(\gamma^+,\gamma^-))\\
&\quad + \sum_{x\in\gamma^-}\Big(p + \sum_{y\in\gamma^-\backslash x}\varphi(x-y)\Big)(F(\gamma^+\cup x,\gamma^-\backslash x) - F(\gamma^+,\gamma^-)).
\end{aligned}$$

and therefore the rate of changing from + to − cells is also density dependent. Clearly all results except these concerning V_1 still hold true, so let us only investigate this part. The expression for quasi-observables is given by

$$\begin{aligned}
&(\widehat{V}_1 G)(\eta^+,\eta^-)\\
&= \sum_{x\in\eta^+}\sum_{\xi^-\subset\eta^-} e^{-E_\psi(x,\xi^-)}e_\lambda(e^{-\psi(x-\cdot)}-1;\eta^-\backslash\xi^-)(G(\eta^+\backslash x,\xi^-\cup x) - G(\eta^+,\xi^-)).
\end{aligned}$$

Lemma 5.5. *The function M_α is given by $M_\alpha = M_\alpha^{CM} + M^{hop} + M_\alpha^V$, where M_α^{CM} and M^{hop} are given as in the Cell-death model and*

$$\begin{aligned} M_\alpha^V(\eta^+,\eta^-) &= |\eta^+|(pe^{\alpha^- - \alpha^+} + e^{2\alpha^- - \alpha^+}\langle\varphi\rangle) + |\eta^-|(p + e^{2\alpha^- - \alpha^+}\langle\varphi\rangle) \\ &\quad + \sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x}\varphi(x-y) + e^{\alpha^- - \alpha^+}\sum_{x\in\eta^+}\sum_{y\in\eta^-}\varphi(x-y). \\ &\quad + q\beta_\psi(\alpha^-)e^{\alpha^+ - \alpha^-}\sum_{x\in\eta^-} e^{-E_\psi(x,\eta^-\backslash x)} + q\beta_\psi(\alpha^-)\sum_{x\in\eta^+} e^{-E_\psi(x,\eta^-)}, \end{aligned}$$

where

$$\beta_\psi(\alpha^-) = \exp\Big(e^{\alpha^-}\int_{\mathbb{R}^d}(1-e^{-\psi(x)})\mathrm{d}x\Big).$$

If $\varphi, a \in L^\infty(\mathbb{R}^d)$, then $\widehat{L} \in L(\mathbb{B}_\alpha, \mathbb{B}_{\alpha'})$ for all $\alpha' < \alpha$.

Proof. This follows from

$$\begin{aligned} &\int_{\Gamma_0^2}\sum_{x\in\eta^+}\sum_{\xi^-\subset\eta^-} e^{-E_\psi(x,\xi^-)}e_\lambda(|e^{-\psi(x-\cdot)}-1|;\eta^-\backslash\xi^-) \\ &\qquad \times|G(\eta^+\backslash x,\xi^-\cup x)|e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}\mathrm{d}\lambda^2(\eta^+,\eta^-) \\ &= e^{\alpha^+}\int_{\Gamma_0^2}\int_{\mathbb{R}^d}\int_{\Gamma_0} e^{-E_\psi(x,\xi^-)}e_\lambda(1-e^{-\psi(x-\cdot)};\eta^-) \\ &\qquad \times|G(\eta^+,\xi^-\cup x)|e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}e^{\alpha^-|\xi^-|}\mathrm{d}\lambda(\xi^-)\mathrm{d}x\mathrm{d}\lambda^2(\eta^+,\eta^-) \\ &= \beta_\psi(\alpha^-)e^{\alpha^+-\alpha^-}\int_{\Gamma_0^2}\Big(\sum_{x\in\xi^-} e^{-E_\psi(x,\xi^-\backslash x)}\Big)|G(\eta^+,\xi^-)|e^{\alpha|\eta^+|}e^{\alpha|\xi^-|}\mathrm{d}\lambda^2(\eta^+,\xi^-) \end{aligned}$$

and

$$\begin{aligned} &\int_{\Gamma_0^2}\sum_{x\in\eta^+}\sum_{\xi^-\subset\eta^-} e^{-E_\psi(x,\xi^-)}e_\lambda(|e^{-\psi(x-\cdot)}-1|;\eta^-\backslash\xi^-) \\ &\qquad \times|G(\eta^+,\xi^-)|e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}\mathrm{d}\lambda^2(\eta^+,\eta^-) \\ &= \int_{\Gamma_0^2}\sum_{x\in\eta^+}\int_{\Gamma_0} e^{-E_\psi(x,\xi^-)}e_\lambda(1-e^{-\psi(x-\cdot)};\eta^-) \\ &\qquad \times|G(\eta^+,\xi^-)|e^{\alpha^+|\eta^+|}e^{\alpha^-|\eta^-|}e^{\alpha^-|\xi^-|}\mathrm{d}\lambda(\xi^-)\mathrm{d}\lambda^2(\eta^+,\eta^-) \\ &= \beta_\psi(\alpha^-)\int_{\Gamma_0^2}\Big(\sum_{x\in\eta^+} e^{-E_\psi(x,\xi^-)}\Big)|G(\eta^+,\xi^-)|e^{\alpha^+|\eta^+|}e^{\alpha^-|\xi^-|}\mathrm{d}\lambda^2(\eta^+,\xi^-). \quad \square \end{aligned}$$

Since ψ is non-negative we can skip the terms containing q in the definition of the domain, i.e. if $M_\alpha^V = M_\alpha^{V_1} + qM_\alpha^{V_2}$, then

$$D(M_\alpha) = \left\{G \in \mathbb{B}_\alpha \;:\; M^{hop}G, M_\alpha^{CM}G, M_\alpha^{V_1}G \in \mathbb{B}_\alpha\right\},$$

where $M_{alpha}^{V_1}$ contains the terms for switching $-$ to $+$ cells and V_2 corresponds to the switching of $+$ to $-$ cells. The operator for correlation functions is

$$\begin{aligned}
&(V_2^{\Delta}k)(\eta^+,\eta^-)\\
&= \sum_{x\in\eta^-} e^{-E_\psi(x,\eta^-\backslash x)} \int_{\Gamma_0} e_\lambda(e^{-\psi(x-\cdot)}-1;\xi^-)k(\eta^+\cup x,\eta^-\cup\xi^-\backslash x)\mathrm{d}\lambda(\xi^-)\\
&\quad - \sum_{x\in\eta^+} e^{-E_\psi(x,\eta^-)} \int_{\Gamma_0} e_\lambda(e^{-\psi(x-\cdot)}-1;\xi^-)k(\eta^+,\eta^-\cup\xi^-)\mathrm{d}\lambda(\xi^-).
\end{aligned}$$

The scaling $a,\varphi,\psi\to\varepsilon a,\varepsilon\varphi,\varepsilon\psi$ leads to the new renormalized expression for $\widehat{V}_{2,\varepsilon,ren}$

$$\begin{aligned}
(\widehat{V}_{2,\varepsilon,ren}G)(\eta^+,\eta^-) = \sum_{x\in\eta^+}\sum_{\xi^-\subset\eta^-} e^{-\varepsilon E_\psi(x,\xi^-)} e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\eta^-\backslash\xi^-\right)\\
\times\left(G(\eta^+\backslash x,\xi^-\cup x)-G(\eta^+,\xi^-)\right)
\end{aligned}$$

and thus to the limiting hierarchical operator

$$(\widehat{V}_{1,V}G)(\eta^+,\eta^-) = \sum_{x\in\eta^+}\sum_{\xi^-\subset\eta^-} e_\lambda(-\psi(x-\cdot);\eta^-\backslash\xi^-)(G(\eta^+\backslash x,\xi^-\cup x)-G(\eta^+,\xi^-)).$$

Theorem 5.6. *Assume $\psi\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, then for all $G\in D(M_\alpha)$ such that $\sum\limits_{x\in\xi^-}\sum\limits_{y\in\xi^-\backslash x}\psi(x-y)G\in\mathbb{B}_\alpha$ and $\sum\limits_{x\in\eta^+}\sum\limits_{y\in\xi^-}\psi(x-y)G\in\mathbb{B}_\alpha$ the convergence $\widehat{L}_{\varepsilon,ren}G\to\widehat{L}_VG$ for $\varepsilon\to 0$ holds in $\mathbb{B}_\alpha$. If in addition $a,\varphi,\psi\in L^\infty(\mathbb{R}^d)$ then for all $\alpha'<\alpha$*

$$\|\widehat{L}_{\varepsilon,ren}-\widehat{L}_V\|_{\alpha\alpha'}\to 0,\quad \varepsilon\to 0.$$

Proof. Let us first estimate

$$\begin{aligned}
&|(\widehat{V}_{1,\varepsilon,ren}G)(\eta)-(\widehat{V}_{1,V}G)(\eta)|\\
&\le \sum_{x\in\eta^+}\sum_{\xi^-\subset\eta^-} |G(\eta^+\backslash x,\xi^-\cup x)-G(\eta^+,\xi^-)|\\
&\qquad\times |e^{-\varepsilon E_\psi(x,\xi^-)}e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\eta^-\backslash\xi^-\right)-e_\lambda(-\psi(x-\cdot);\eta^-\backslash\xi^-)|
\end{aligned}$$

and then the modulus in the sum by

$$\begin{aligned}
&|e^{-\varepsilon E_\psi(x,\xi^-)}e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\eta^-\backslash\xi^-\right)-e_\lambda(-\psi(x-\cdot);\eta^-\backslash\xi^-)|\\
&\le |1-e^{-\varepsilon E_\psi(x,\xi^-)}|e_\lambda\left(\frac{|e^{-\varepsilon\psi(x-\cdot)}-1|}{\varepsilon};\eta^-\backslash\xi^-\right)\\
&\quad+\left|e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\eta^-\backslash\xi^-\right)-e_\lambda(-\psi(x-\cdot);\eta^-\backslash\xi^-)\right|\\
&\le \varepsilon E_\psi(x,\xi^-)e_\lambda(\psi(x-\cdot);\eta^-\backslash\xi^-)
\end{aligned}$$

$$+ \sum_{w\in\eta^-\backslash\xi^-} \left| \frac{e^{-\varepsilon\psi(x-w)}-1}{\varepsilon} + \psi(x-w) \right| e_\lambda(\psi(x-\cdot);\eta^-\backslash\xi^-\backslash w)$$
$$\le \varepsilon E_\psi(x,\xi^-)e_\lambda(\psi(x-\cdot);\eta^-\backslash\xi^-) + \varepsilon \sum_{w\in\eta^-\backslash\xi^-} \psi(x-w)^2 e_\lambda(\psi(x-\cdot);\eta^-\backslash\xi^-\backslash w).$$

Integrating over Γ_0^2 with respect to $e^{\alpha'^+|\eta^+|}e^{\alpha'^-|\eta^-|}\mathrm{d}\lambda^2(\eta^+,\eta^-)$ we obtain for the part containing $G(\eta^+\backslash x,\xi^-\cup x)$

$$\varepsilon e^{\alpha'^+} \int_{\Gamma_0^2}\int_{\mathbb{R}^d}\int_{\Gamma_0} |G(\eta^+,\xi^-\cup x)|E_\psi(x,\xi^-)e_\lambda(\psi(x-\cdot);\eta^-)$$
$$\times e^{\alpha'^-|\xi^-|}e^{\alpha'^-|\eta^-|}e^{\alpha'^+|\eta^+|}\mathrm{d}x\mathrm{d}\lambda^3(\eta\text{'}\xi^-)$$
$$\le \varepsilon e^{e^{\alpha'^-}\langle\psi\rangle} \int_{\Gamma_0^2} \Big(\sum_{x\in\eta^-} E_\psi(x,\xi^-\backslash x) \Big) |G(\eta^+,\xi^-)|e^{\alpha'^+|\eta^+|}e^{\alpha'^-|\xi^-|}\mathrm{d}\lambda^2(\eta^+,\xi^-)$$

and for the second term

$$\varepsilon e^{\alpha'} \int_{\Gamma_0^2}\int_{\Gamma_0} |G(\eta^+,\xi^-)|e^{\alpha'|\eta^+|}e^{\alpha'|\xi^-|}$$
$$\times \sum_{x\in\eta^+}\int_{\mathbb{R}^d} \psi(x-w)^2 e_\lambda(e^{\alpha'}\psi(x-\cdot);\eta^-)\mathrm{d}w\mathrm{d}\lambda(\eta^+,\eta^-,\xi^-)$$
$$\le \varepsilon e^{\alpha'\langle\psi\rangle}\langle\psi^2\rangle e^{\alpha'} \int_{\Gamma_0^2} |\eta^+||G(\eta^+,\xi^-)|e^{\alpha'|\eta^+|}e^{\alpha'|\xi^-|}\mathrm{d}\lambda(\eta^+,\xi^-).$$

Similar estimations for the parts containing $G(\eta^+,\xi^-)$ show together with above computations the first part of the assertion. The second part follows from $E_\psi(x,\xi) \le \|\psi\|_{L^\infty}|\xi|$. □

The operator for correlation function is changed only at the new operator V^Δ and the rescaled version has the form

$$(V^\Delta_{1,\varepsilon,ren}k)(\eta^+,\eta^-)$$
$$= \sum_{x\in\eta^-} e^{-\varepsilon E_\psi(x,\eta^-\backslash x)} \int_{\Gamma_0} e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\xi^-\right) k(\eta^+\cup x,\eta^-\cup\xi^-\backslash x)\mathrm{d}\lambda(\xi^-)$$
$$- \sum_{x\in\eta^+} e^{-\varepsilon E_\psi(x,\eta^-)} \int_{\Gamma_0} e_\lambda\left(\frac{e^{-\varepsilon\psi(x-\cdot)}-1}{\varepsilon};\xi^-\right) k(\eta^+,\eta^-\cup\xi^-)\mathrm{d}\lambda(\xi^-)$$

and the limiting operator

$$(V^\Delta_{1,V}k)(\eta^+,\eta^-) = \sum_{x\in\eta^-}\int_{\Gamma_0} e_\lambda(-\psi(x-\cdot);\xi^-)k(\eta^+\cup x,\eta^-\cup\xi^-\backslash x)\mathrm{d}\lambda(\xi^-)$$
$$- \sum_{x\in\eta^+}\int_{\Gamma_0} e_\lambda(-\psi(x-\cdot);\xi^-)k(\eta^+,\eta^-\cup\xi^-)\mathrm{d}\lambda(\xi^-).$$

Again if $a, \varphi, \psi \in L^\infty(\mathbb{R}^d)$, then the convergence

$$\|L^\Delta_{\varepsilon,ren} - L^\Delta_V\|_{\alpha'\alpha} \to 0, \quad \varepsilon \to 0$$

holds. Computing $V_{1,V} e_\lambda(\rho^+) e_\lambda(\rho^-)$ one sees that the equations for the local densities will have the prescribed form (18),(19).

Last two models

Here the changes of types are density independent, i.e. $\varphi = \psi = 0$, but the proliferation is changed either to density dependent mortality or to density dependent birth. Both models were analysed in the one-component case. Since the changes of types are prescribed by constant intensities they do not influence the construction of an evolution and only contribute by additional terms in the kinetic description. It is not difficult to combine all results and derive from them the corresponding kinetic description stated before.

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