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Backward Stochastic Differential Equations driven by Lévy noise with applications in Finance

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Abstract. The main goal of this paper is to provide a coincise and self-contained introduction to treat financial mathematical models driven by noise of Lévy type in the framework of the backward stochastic differential equations (BSDEs) theory. We shall present techniques and results which are relevant from a mathematical point of views as well in concrete market applications, since they allow to overcome the discrepancies between *real world* financial data and classical models which are based on Brownian diffusions.

BSEDs' techniques in presence of Lévy perturbations actually play a major role in the solution of hedging and pricing problems especially with respect to non-linear scenarios and for incomplete markets.

In particular, we provide an analogue of the celebrated Black–Scholes formula, but the *Lévy market* case, with a clear economical interpretation for all the involved financial parameters, as well as an introduction to the emerging field of dynamic risk measures, for Lévy driven BSDEs, making use of the concept of g-expectation in presence of a Lipschitz driver.

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1 Introduction

Background and motivation

In this paper we give a review of some recent applications in finance based on techniques coming from the theory of stochastic differential equations in backward form, see, e.g., [15, 20]. The main goal of the work consists in the analysis of problems characterizing modern financial markets. In particular we aim at dealing with concrete models arising in contexts which are not covered by the standard approach originated by Black and Scholes, see [12], and then widely used in a pletora of financial contexts, see, e.g., [58] and references therein for a comprehensive introduction to main results in this framework.

A key ingredient of our analysis will be an extensive use of the theory of backward stochastic differential equations (BSDEs) introduced by Bismut

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(1973), for the linear case, and generalized by Pardoux and Peng (1990) in the general non-linear case in the Brownian framework, see, e.g. [44]. In [45] Pardoux and Peng provided also a Feynman–Kac type theorem for solution of non-linear parabolic partial differential equation (PDE). We would like to underline that BSDEs techinques provide a powerful instruments to analyze an heterogeneous class of real problems, namely they are not a mere toy for mathematicians. In particular BSDEs approach are succesfully used, e.g., in finance, physics, biology, etc.; we refer to [33], [29], respectively.

In the mathematical finance framework BSDEs techiques gained a great attention by both practitioners and academics in particular with respect to the option pricing and utilities problem, see, e.g., [33] for one of the first review on the subject, [15, 20] for a more extensive introduction to recent developments.

Nevertheless, empirical evidence has pointed out that the traditional setup where the geometric Brownian motion is assumed as the model for the stock prices' behaviours, it is not fully satisfactory since it lacks of an accurate description of financial data, see, e.g., [16] and [57].

Discrepancies between the Black–Scholes forecasted prices and real data, for instance, arise in the study of implied volatility surface in option market, kurtosis and skewness of asset returns, see, e.g. [16] and [61].
Latter issues has promoted the development of more flexible models and led to an explosion of the literature on this subject starting from the late 90s, see e.g. [15, 19, 20, 46] and references therein and also [11] where more emphasis is put on interest rate models. A great improvement towards more realistic models able to describe and, possibly, forecast movements of relevant financial quantities, has been achieved taking into account Lévy processes which allow for random jump perturbations in the asset price motion.

Such stochastic processes are characterized by random jumps, hence allowing to capture sudden variations of prices happen, for example, in presence of turbulent economics dynamics originated by unexpected political events, natural disasters, abrupt variations of commodities' prices, etc.

We shall consider the wealth processes dynamic of a portfolio, composed by a riskless asset and a risky security, modeled by a BSDE driven by a Lévy process, hence generalizing the classic approach based on Brownian stochastic driver.

We list some references which include great improvements in this direction. We recall that a Lévy process consists of three stochastically independent parts: a purely deterministic linear part, a Brownian motion and a pure jump process. In [60], Situ studies BSDEs driven by a Brownian motion plus a Poisson point process, and Ouknine, in [43], consider the case of a BSDEs driven by a Poisson random measure. Latter works are all based on the integral representation of a square-integrable random variables in terms of a Poisson random measure.

Nualart and Schoutens in [40] proved a martingale representation theorem for Lévy processes satisfying some exponential moment condition and Feynman–Kac formula by Teugels orthonormalization procedure.

The Feynman–Kac formula and the related partial differential integral equation (PDIE) also play an important role in finance applications: they provide an analogue of the famous Black–Scholes partial differential equation and thus can be used for the purpose of option pricing in a Lévy market.

Structure

The paper is organized as follows: in Sect.2 basic definitions for establishing the appropriate frameworks where to develop BSDEs' theory, are stated. Sect.3 deals with BSDEs driven by pure jump noise, while Sect.4 treats the case of market models driven by Lévy noise of general type, moreover it contains a subsection about pricing/hedging problems. Finally, in Sect.5, an overview about dynamic risk measures is presented.

2 Mathematical framework

In this section we shall give basic definitions and results in order to define a suitable framework for both the BSDEs' techinques developed in Sect.3 and related applications to hedging/pricing problems analyzed in Sect.4. Let us start defining a setup where the preference filtration is generated by two mutually independent processes, a Brownian Motion and a Poisson random measure, following the approach developed in [48].

For any $T > 0$, $t \in [0, T]$, and $p \in \mathbb{N}$ with $p > 1$, let

- W_t be a one dimensional Brownian motion on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$;
- $N(dt, du)$ be a Poisson random measure on $\mathbb{R}^+ \times \mathbb{R}^*$ with compensator $\nu(du)dt$ such that it is a σ-finite Lévy measure on $(\mathbb{R}^*, \mathcal{B}(\mathbb{R}^*))$, and let $\widetilde{N}(dt, du)$ be its compensated process;
- $(\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated (jointly) by W_t and N_t.
- $\mathcal{P}$ is the predictable σ-algebra on $[0, T] \times \Omega$.
- $L^p(\mathcal{F}_T)$ is the set of random variables $\xi : \Omega \longrightarrow \mathbb{R}$, which are $\mathcal{F}_T$ measurable and p-integrable, namely

$$\|\xi\|_p := \left(\int_\Omega |\xi|^p \, d\mathbb{P} \right)^{1/p} < +\infty \, .$$

- $\mathscr{H}^{p,T}$ is the set of real-valued predictable processes ϕ such that

$$\|\phi\|^p_{\mathscr{H}^{p,T}} := \mathbb{E} \left[\left(\int_0^T \phi_t^2 dt \right)^{p/2} \right] < +\infty.$$

- L^p_ν is the set of Borelian functions $l : \mathbb{R}^* \longrightarrow \mathbb{R}$ such that

$$\int_{\mathbb{R}^*} |l(u)|^p \nu(du) < +\infty,$$

  in particular the set L^2_ν is a Hilbert space equipped with the scalar product

$$\langle \delta, \, l \rangle_\nu := \int_{\mathbb{R}^*} \delta(u) l(u) \nu(du), \qquad \forall \, \delta, l \in L^2_\nu \times L^2_\nu,$$

  and the norm

$$\|l\|^2_\nu := \int_{\mathbb{R}^*} |l(u)|^2 \nu(du) < +\infty.$$

- $\mathscr{H}_\nu^{p,T}$ is the set of *predictable* stochastic processes, namely $l \in \mathscr{H}_\nu^{p,T}$ iff
$$l : ([0,T] \times \Omega \times \mathbb{R}^*, \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^*)) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$
$$(t, w, u) \longrightarrow l_t(w, u)$$
and
$$\|l\|^p_{\mathscr{H}_\nu^{p,T}} := \mathbb{E}\left[\left(\int_0^T \|l_t\|^2_\nu \, dt\right)^{p/2}\right] < +\infty \, .$$
- $S^{p,T}$ is the set of real-valued right continuous, left limited (RCLL) adapted processes ϕ such that
$$\|\phi\|^p_{S^p} := \mathbb{E}\left[\sup_{0 \le t \le T} |\phi_t|^p\right] < +\infty.$$
- τ_0 denotes the set of stopping times τ, such that $\tau \in [0,T]$, a.s.

If T is fixed and there is no ambiguity, we shall adopt the notation $\mathscr{H}^p$ for $\mathscr{H}^{p,T}$, respectively $\mathscr{H}^p_\nu$ for $\mathscr{H}_\nu^{p,T}$ and S^p for $S^{p,T}$.

We introduce the notion of driver prior of dealing with definitions, properties and results for BSDEs and related solutions.

Definition 2.1. A function f
$$f : \; [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times L^2_\nu \longrightarrow \mathbb{R}$$
$$(t, w, x, \pi, l(\cdot)) \longrightarrow f(t, w, x, \pi, l(\cdot))$$
is said to be a driver if
- is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}(L^2_\nu)$-measurable,
- $f(\cdot, 0, 0, 0) \in \mathscr{H}^2$.

Moreover, f is said to be a Lipschitz driver if it is Lipschitz with respect to x, π and l, namely if there exists a constant $C \ge 0$ such that $d\mathbb{P} \otimes dt$ a.s., for each $(x_1, \pi_1, l_1(\cdot))$, $(x_2, \pi_2, l_2(\cdot)) \in \mathbb{R} \times \mathbb{R} \times L^2_\nu$, the following holds
$$|f(t, w, x_1, \pi_1, l_1) - f(t, w, x_2, \pi_2, l_2)| \le C \left(|x_1 - x_2| + |\pi_1 - \pi_2| + ||l_1 - l_2||_\nu\right) .$$

Definition 2.2. A BSDE with jumps with driver f, in the unknowns X_t, π_t, l_t, reads as follows
$$-dX_t = f(t, X_{t^-}, \pi_t, l_t(\cdot)) \, dt - \pi_t \, dW_t - \int_{\mathbb{R}^*} l_t(u) \, \widetilde{N}(dt, du), \tag{1}$$
with terminal time T and terminal condition ξ
$$X_T = \xi,$$
where
- W_t is a one dimensional Brownian Motion,
- $N(dt, du)$ is a Poisson random measure on $\mathbb{R}^+ \times \mathbb{R}^*$,
- $\xi \in L^2(\mathcal{F}_T)$,

Remark 2.1. We would like to underline that the terminal condition $X_T = \xi$ is the responsible for the appellation *backward* given to equation (1).

Definition 2.3. A solution of (1) is a triple of processes (X_t, π_t, l_t) satisfying (1) and such that

- X_t is a RCLL optional process,
- π_t is an $\mathbb{R}$-valued predictable process defined on $\Omega \times [0,T]$ such that the stochastic integral with respect to W_t is well defined,
- l_t is an $\mathbb{R}$-valued predictable process defined on $\Omega \times [0,T] \times \mathbb{R}^*$ such that the stochastic integral with respect to $\widetilde{N}_t$

$$\int_{\mathbb{R}^*} l_t(u)\, \widetilde{N}(dt, du),$$

  is well defined.

Under suitable conditions on the final point ξ and the driver f, the following existence and uniqueness result, due to Tang e Li, holds, see [62] for details.

Theorem 2.1 (Existence and uniqueness)**.** *Let $T > 0$. For each Lipschitz driver f, in the sense of Def.(2.1) and for every terminal condition $\xi \in L^2(\mathcal{F}_T)$, there exists a unique solution (X_t, π_t, l_t) of (1) and $(X_t, \pi_t, l_t) \in S^{2,T} \times \mathscr{H}^{2,T} \times \mathscr{H}_\nu^{2,T}$.*

3 Linear BSDEs with jumps

A linear driver f for (1), is of the form

$$f(t, X_t, \pi_t, l_t(\cdot)) = \varphi_t + \delta_t X_t + \beta_t \pi_t + \langle \gamma_t, l_t \rangle_\nu , \tag{2}$$

where

- δ_t and β_t are a real valued predictable processes, supposed to be a.s. integrable with respect to dt and dW_t,
- $(\gamma_t(\cdot))_{t\in[0,T]}$ a real-valued predictable process defined on $[0,T] \times \Omega \times \mathbb{R}^*$, that is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^*)$-measurable, and integrable with respect to $\widetilde{N}(dt, du)$,
- $\varphi_t \in \mathscr{H}^{2,T}$.

The linear BSDE driven by f as in (2), reads as follows

$$\begin{cases} -dX_t = (\varphi_t + \delta_t\, X_t + \beta_t\, \pi_t + \langle \gamma_t, \, l_t \rangle_\nu)\; dt \\ \qquad\qquad\qquad\qquad - \pi_t\, dW_t - \displaystyle\int_{\mathbb{R}^*} l_t(u) \widetilde{N}(dt, du), \\ X_T = \xi. \end{cases} \tag{3}$$

In what follows, we will study particular cases of linear BSDEs for which, see Sect3.1, explicit formulae for the solution will be provided. In particular, we will prove that the solution of a linear BSDE can be expressed as a conditional expectation of some specified known processes.

For each $t \in [0,T]$, $T > 0$, let us introduce the process $(\Gamma_s^t)_{s\in[t,T]}$ defined by

$$\begin{cases} d\Gamma_s^t = \Gamma_{s^-}^t \left[\delta_s\, ds + \beta_s\, dW_s + \int_{\mathbb{R}^*} \gamma_s(u)\, \widetilde{N}(ds,du)\right], \\ \Gamma_t^t = 1. \end{cases} \tag{4}$$

For the SDE (4) there exists a unique solution, see, e.g., Chapter 5 in [3], given by

$$\Gamma_s^t = e^{\int_s^t \delta_u du}\, Z_s^t, \tag{5}$$

where the process $(Z_s^t)_{s\in[t,T]}$ solves the following SDE

$$\begin{cases} dZ_s^t = Z_{s^-}^t \left[\beta_s dW_s + \int_{\mathbb{R}^*} \gamma_s(u) \widetilde{N}(ds,du)\right], \\ Z_t^t = 1. \end{cases} \tag{6}$$

Remark 3.1. It can be useful to see at the process Γ_s^t as it plays the role of "δ-discounted" process of Z_s^t.

For future use, it is useful to define the stochastic process, which appear in (6), as

$$M_t := \int_0^t \beta_s dW_s + \int_0^t \int_{\mathbb{R}^*} \gamma_s(u) \widetilde{N}(ds,du), \tag{7}$$

and provide some useful results for exponential local martingales of M_t driven by a Brownian Motion and a Poisson random measure.

In particular, it is useful for application to show that M_t is a *local martingale*, indeed M_t in eq.(7) can be rewritten as a Lévy-type stochastic integral then proving that it is a local martingale. In fact considering the differential of M_t

$$\begin{aligned} dM_t = {} & \beta_t\, dW_t + \int_{\mathbb{R}^*} \gamma_t(u)\, \widetilde{N}(dt,du) = \\ = {} & \beta_t\, dW_t + \int_{\mathbb{R}^*} \gamma_t(u)\, N(ds,du) - \int_{\mathbb{R}^*} \gamma_t(u)\, \nu(du) dt = \\ = {} & \beta_t\, dW_t + \int_{|u|\geq 1} \gamma_t(u)\, N(dt,du) + \int_{|u|<1} \gamma_t(u)\, N(dt,du) \\ & - \int_{|u|\geq 1} \gamma_t(u) \nu(du) dt \; - \int_{|u|<1} \gamma_t(u)\, \nu(du)\, dt = \\ = {} & \beta_t\, dW_t + \int_{|u|<1} \gamma_t(u)\, \widetilde{N}(dt,du) + \int_{|u|\geq 1} \gamma_t(u)\, N(dt,du) \\ & - \int_{|u|\geq 1} \gamma_t(u)\, \nu(du) dt, \end{aligned}$$

and we can apply the following Theorem, see Chapter 5 in [3] for details, to obtain the desired result:

Theorem 3.1. *If Y_t is a Lévy-type stochastic integral of this form*

$$dY_t = G_t\,dt + F_t\,dW_t + \int_{|x|<1} H(t,x)\,\widetilde{N}(dt,dx) + \int_{|x|\geq 1} K(t,x)\,N(dt,dx), \quad (8)$$

then Y_t is a local martingale if and only if

$$G_t \,+\, \int_{|x|\geq 1} K(t,x)\,\nu(dx) \,=0 \quad a.s.$$

for (Lebesgue) almost all $t \geq 0$.

For a complete review on properties and representation of Lévy-type stochastic integral we refer to [3] and [47].

We would like to underline that we have defined the process Z_s^t in eq.(6) as

$$\begin{cases} dZ_t = Z_{t^-}\left[\beta_t dW_t + \int_{\mathbb{R}^*} \gamma_s(u)\widetilde{N}(ds,du)\right] = Z_{t^-}\,dM_t, \\ Z_0 = 1, \end{cases} \quad (9)$$

hence Z_t, satisfying eq.(9), is given by the Doléans–Dade formula, see [21]. The solution of (9) exists and is unique, see, e.g., [3] for details, being the so-called exponential local martingale associated with the local martingale M_t and it is denoted by $\mathcal{E}(M)_t := Z_t$. In particular $\mathcal{E}(M)_t$ has the following expression

$$\mathcal{E}(M)_t = \exp\left\{M_t - \frac{1}{2}\,[M_t^c, M_t^c]\right\}\prod_{0\leq s\leq t}[1+\Delta M_s]\,e^{-\Delta M_s}\,.$$

It is interesting to note that the differential of the process M_t can be divided in its continuous part, resp. discontinuous part, denoted by dM_t^c, resp. by dM_t^d, and such that the following expressions hold

$$dM_t^c := \beta_t\,dW_t - \int_{|u|\geq 1}\gamma_t(u)\,\nu(du)\,dt,$$

respectively

$$dM_t^d := \int_{|u|<1}\gamma_t(u)\,\widetilde{N}(dt,du) \,+\, \int_{|u|\geq 1}\gamma_t(u)\,N(dt,du)\,.$$

The quadratic variation of the continuous part of M_t is given by

$$[M_t^c, M_t^c] = \int_0^t \beta_s^2\,ds\,.$$

Finally, in order to show an explicit representation of Doléans–Dade formula for Z_t, we also define

$$Y_t := \int_{\mathbb{R}^*} uN([0,t],du), \quad (10)$$

which is a compound Poisson process, hence the following relation, between ΔM_t and ΔY_t, holds

$$\Delta M_t = \gamma_t(\Delta Y_t) = \int_0^t\int_{\mathbb{R}^*}\gamma_s(u)\,N(ds,du),$$

and we also have

$$\begin{aligned}\mathcal{E}(M)_t = \exp\Bigg\{\int_0^t \beta_s dW_s + \int_0^t\int_{\mathbb{R}^*} \gamma_s(u)\,\widetilde{N}(ds,du) - \frac{1}{2}\int_0^t \beta_s^2 ds + \\ - \int_0^t\int_{\mathbb{R}^*} \gamma_s(u)\,N(ds,du)\Bigg\} \prod_{0\le s\le t} (1+\gamma_s(\Delta Y_s)) = \\ = \exp\Bigg\{\int_0^t \beta_s dW_s - \frac{1}{2}\int_0^t \beta_s^2\,ds - \int_0^t\int_{\mathbb{R}^*} \gamma_t(u)\nu(du)ds\Bigg\} \\ \times \prod_{0\le s\le t} (1+\gamma_s(\Delta Y_s)) .\end{aligned} \tag{11}$$

Remark 3.2. Note that, if the following condition holds

$$\gamma_t(\Delta Y_t) \ge -1, \qquad \forall t \in [0,T] \; a.s.,$$

then $\mathcal{E}(M)_t \ge 0, \;\; \forall t \in [0,T]$ a.s.

It is interesting to note that, if M_t is driven by a Brownian Motion and a Poisson random measure, as in our case, the following stronger property holds.

Proposition 3.2. *Let β_t, $\gamma_t(\cdot)$ be a real valued predictable process, and M_t be the local martingale defined by (7).*

Then, the following assertions are equivalent:

(i) if $(T_n)_n$ is a sequence of stopping times corresponding to the jumps times of Y_t, then for every $n \in \mathbb{N}$

$$\gamma_{T_n}(\Delta Y_{T_n}) \ge -1, \qquad \mathbb{P} \; a.s.,$$

(ii) $\gamma_t(u) \ge -1, \qquad d\mathbb{P} \otimes dt \otimes d\nu(u) \;\; a.s.$

Moreover, if (i) and/or (ii) is satisfied, then we have

$$\mathcal{E}(M)_t \ge 0, \qquad 0 \le t \le T, \;\; a.s.$$

In the light of financial applications, we now provide a sufficient condition for the square integrability property of $\mathcal{E}(M)_t$, also giving a detailed proof for such a condition following [48].

Proposition 3.3. *Let β_t and $\gamma_t(\cdot)$ be real valued predictable processes and M_t be the local martingale defined by (7). Suppose that*

$$\int_0^T \beta_s^2\,ds + \int_0^T ||\gamma_s||_\nu^2\,ds < +\infty, \tag{12}$$

then

$$\mathbb{E}\left[\mathcal{E}(M)_T^2\right] < +\infty,$$

namely, for any $T > 0$, the random variable $\mathcal{E}(M)_T^2 \in L^2(\mathcal{F}_T)$.

Proof. By the Itô product formula we get

$$\mathcal{E}(M)_t^2 = \mathcal{E}(2M_t + [M,M]_t),$$

where

$$[M,M]_t = [M^c,M^c]_t + \sum_{0<s\le t} \Delta M_s^2 = \int_0^t \beta_s^2\, ds + \sum_{0<s\le t} l_s^2(\Delta Y_s).$$

Using the formula (10) we also have

$$\begin{aligned}[M,M]_t &= \int_0^t \beta_s^2\, ds + \int_0^t \int_{|x|<1} l_s^2(u)\, N(ds,du) + \int_0^t \int_{|x|\ge 1} l_s^2(u)\, N(ds,du) \\ &= \int_0^t \beta_s^2\, ds + \int_0^t \int_{\mathbb{R}^*} l_s^2(u)\, N(ds,du).\end{aligned}$$

Moreover, since

$$\begin{aligned}\sum_{s\le t} l_s^2(\Delta Y_s) = \int_0^t \int_{\mathbb{R}^*} l_s^2(u)\, N(ds,du) = \\ &= \int_0^t \int_{\mathbb{R}^*} l_s^2(u)\, \widetilde{N}(ds,du) + \int_0^t \int_{\mathbb{R}^*} l_s^2(u)\, \nu(du)ds \\ &= \int_0^t \int_{\mathbb{R}^*} l_s^2(u)\, \widetilde{N}(ds,du) + \int_0^t ||\gamma_s||_\nu^2\, ds,\end{aligned}$$

it follows that

$$\begin{aligned}&\mathcal{E}(M)_t^2 = \mathcal{E}(2M_t + [M,M]_t) = \\ &= \mathcal{E}\left[2M_t + \int_0^t \beta_s\, ds + \int_0^t ||\gamma_s||_\nu^2\, ds + \int_0^t \int_{\mathbb{R}^*} \gamma_s^2(u)\widetilde{N}(ds,du)\right] = \\ &= \mathcal{E}\left[N_t + \int_0^t \beta_s^2\, ds + \int_0^t ||\gamma_s||_\nu^2\, ds\right],\end{aligned} \tag{13}$$

where $N_t := 2M_t + \int_0^t \int_{\mathbb{R}^*} \gamma_s^2(u)\widetilde{N}(ds,du)$.

Note that $\int_0^t \beta_s^2\, ds + \int_0^t ||\gamma_s||_\nu^2\, ds$ does not give a contribute in the computation of $[N^c,N^c]_t$ and ΔN_t, therefore

$$\mathcal{E}(M)_t^2 = \mathcal{E}(N)_t \exp\left\{\int_0^t \beta_s^2\, ds + \int_0^t ||\gamma_s||_\nu^2\, ds\right\}. \tag{14}$$

By Th.(3.1) we can rewrite N_t as a Lévy-type stochastic integral then N_t is a local martingale and, exploiting a classical result (see, e.g., Chapter 7 in [35]), the latter implies that $\mathcal{E}(N)_t$ is a local martingale, too.

By assumption, there exists $K>0$ such that

$$\exp\left\{\int_0^T \beta_s^2\, ds + \int_0^T ||\gamma_s||_\nu^2\, ds\right\} \le K \quad a.s.,$$

moreover, by (12), $\mathcal{E}(M)_t^2 \ge 0$, and $\mathcal{E}(N)_t \ge 0$. Since any nonnegative local martingale is a supermartingale, we have that $\mathcal{E}(N)_t$ is a supermartingale hence it has a non-increasing expectation, namely

$$\mathbb{E}\left[\mathcal{E}(M)_T^2\right] \le K\,\mathbb{E}\left[\mathcal{E}(N)_T\right] \le K,$$

which ends the proof. $\square$

Remark 3.3.

1. If the assumptions of Proposition (3.3) are satisfied, we have

$$\mathcal{E}(M)_t \in \; S^{2,T} \; .$$

2. Taking β_t bounded, and $\psi_t \in L^2_\nu$ such that

$$|\gamma_t(u)| \leq \psi(u), \qquad dt \otimes dP \otimes d\nu(u) \;\; \text{a.s.},$$

it follows that the process $||\gamma_t||^2_\nu$ is bounded, indeed

$$||\gamma_t||^2_\nu = \int_{\mathbb{R}^*} |\gamma_t(u)|^2 \, \nu(du) \leq \int_{\mathbb{R}^*} \psi^2(u) \, \nu(du) < +\infty,$$

hence

$$\int_0^T \beta_s^2 \, ds + \int_0^T ||\gamma_s||^2_\nu \, ds \; < +\infty,$$

and, by Prop.(3.3) we have $\mathcal{E}(M)_T \in L^2(\mathcal{F}_T)$. Latter result will be exploited to study linear BSDE and to state a *comparison theorem* which turn out to be useful in order to obtain results in the framework of *dynamic risk measures*, see Sect.5, Th.5.1.

3.1 Properties of linear BSDEs with jumps

The main aim of this section is to show that the solution of a linear BSDE with jumps can be written as a conditional expectation via an exponential semimartingale.

Theorem 3.4. *Let γ_t, β_t, $\gamma_t(\cdot)$ be real valued, bounded, predictable processes and $\varphi_t \in \mathscr{H}^{2,T}$.*

Suppose that $\Gamma_t \in S^2$, where Γ_t is the solution of the SDE (4).

Let (X_t, β_t, l_t) be the solution in $S^{2,T} \times \mathscr{H}^{2,T} \times \mathscr{H}^{2,T}_\nu$ of the following linear BSDE

$$\begin{cases} -dX_t = (\varphi_t + \delta_t \, X_t + \beta_t \, \pi_t + \langle \gamma_t, \, l_t \rangle_\nu) \; dt \\ \qquad\qquad\qquad - \pi_t \, dW_t - \displaystyle\int_{\mathbb{R}^*} l_t(u) \widetilde{N}(dt, du), \\ X_T = \xi, \end{cases} \tag{15}$$

then, the process X_t satisfies

$$X_t = \mathbb{E}\left[\Gamma^t_T \, \xi + \int_t^T \Gamma^t_s \, \varphi_s \, ds \, | \, \mathcal{F}_t \right], \qquad 0 \leq t \leq T \quad a.s. \tag{16}$$

Proof. We follow the approach in [48]. Fix $t \in [0, T]$ and, in order to simplify the notation used, let us denote Γ_s instead of Γ^t_s for $s \in [t, T]$. By the Itô-product formula we have

$$
\begin{aligned}
-\,d(X_s\,\Gamma_s) =& -X_{s^-}\,d\Gamma_s + \Gamma_{s^-}\,dX_s - d[X,\Gamma]_s = \\
= & -X_{s^-}\left[\Gamma_{s^-}\,\delta_s ds + \Gamma_{s^-}\,\beta_s dW_s + \Gamma_{s^-}\int_{\mathbb{R}^*}\gamma_s(u)\widetilde{N}(ds,du)\right] \\
& + \Gamma_{s^-}\,\left[\varphi_s\,ds \,+\, \delta_s\,X_s\,ds \,+\, \beta_s\,\pi_s\,ds \,+\, \langle\gamma_s,\,l_s\rangle_\nu\,ds \,-\, \pi_s\,dW_s\right] \\
& - \Gamma_{s^-}\int_{\mathbb{R}^*} l_s(u)\widetilde{N}(ds,du) \,-d[X,\Gamma]_s.
\end{aligned}
\tag{17}
$$

We postpone the estimate of the contribute given by the quadratic variation $d[X,\Gamma]$ starting, instead, to rewrite the processes X_t and Γ_t as general Lévy-type stochastic integral. In particular, the differential of X_s is given by

$$
\begin{aligned}
dX_s =& -\underbrace{(\varphi_s + \delta_s\,X_s + \beta_s\,\pi_s + \langle\gamma_s,\,l_s\rangle_\nu)}_{Y_s}\,ds + \pi_s\,dW_s \,+ \int_{\mathbb{R}^*} l_s(u)\,\widetilde{N}(ds,du) = \\
=& -Y_s\,ds + \pi_s\,dW_s + \int_{|u|<1} l_s(u)\,\widetilde{N}(ds,du) + \int_{|u|\geq 1} l_s(u)\,N(ds,du) + \\
& - \int_{|u|\geq 1} l_s(u)\nu(du)\,ds = \\
=& -\underbrace{\left(Y_s + \int_{|u|\geq 1} l_s(u)\nu(du)\right)}_{\widetilde{Y}_s} ds + \pi_s\,dW_s + \int_{|u|<1} l_s(u)\,\widetilde{N}(ds,du) + \\
& + \int_{|u|\geq 1} l_s(u)N(ds,du) = \\
=& -\widetilde{Y}_s\,ds + \pi_s\,dW_s + \int_{|u|<1} l_s(u)\,\widetilde{N}(ds,du) + \int_{|u|\geq 1} l_s(u)\,N(ds,du),
\end{aligned}
\tag{18}
$$

while for the differential of Γ_s we have

$$
\begin{aligned}
d\Gamma_s =& \left(\Gamma_{s^-}\,\delta_s \,-\, \Gamma_{s^-}\int_{|u|\geq 1}\Gamma_s(u)\nu(du)\right) ds \,+\, \Gamma_{s^-}\,\beta_s\,dW_s + \\
& + \Gamma_{s^-}\int_{|u|<1}\Gamma_s(u)\,\widetilde{N}(ds,du) \,+\, \Gamma_{s^-}\int_{|u|\geq 1}\Gamma_s(u)\,N(ds,du)\,.
\end{aligned}
\tag{19}
$$

For the quadratic variation term, namely $[X,\Gamma]_s$, in (17) we have

$$
[X,\Gamma]_s = \int_0^s \Gamma_{s^-}\,\pi_s\,\beta_s\,ds + \int_0^s\int_{\mathbb{R}^*} l_s(u)\,\gamma_s(u)\,\Gamma_{s^-}\,N(ds,du), \tag{20}
$$

then, combining (18), (19), (20), we obtain

$$\begin{aligned} -d(X_s\,\Gamma_s) =\; & -X_s\Gamma_{s^-}\,\delta_s\,ds + \Gamma_{s^-}\,\left[\varphi_s + \delta_s\,X_s + \beta_s\,\pi_s + \langle\gamma_s,\,l_s\rangle_\nu\right]\,ds \\ & - \Gamma_{s^-}\,(X_{s^-}\,\beta_s + \pi_s)\,dW_s \\ & - \Gamma_{s^-}\,\pi_s\,\beta_s\,ds \;-\; \Gamma_{s^-}\int_{\mathbb{R}^*} l_s(u)\gamma_s(u)\,\widetilde{N}(ds,du) + \\ & - \Gamma_{s^-}\int_{\mathbb{R}^*} l_s(u)\gamma_s(u)\nu(du)ds - \Gamma_{s^-}\int_{\mathbb{R}^*} l_s(u)\widetilde{N}(ds,du) \\ =\; & \Gamma_s\,\varphi_s\,ds - dM_s, \end{aligned}$$

where

$$dM_s = -\Gamma_{s^-}\;\left(X_{s^-}\beta_S + \pi_s\right)\,dW_s - \Gamma_{s^-}\int_{\mathbb{R}^*} l_s(u)(1+\gamma_s(u))\,\widetilde{N}(ds,du),$$

and, integrating between t and T, we get

$$-X_T\,\Gamma_T^t \;+\; X_t\,\Gamma_t^t = \int_t^T \Gamma_s^t\,\varphi_s\,ds - M_T + M_t\,.$$

Therefore, exploiting the terminal conditions, it follows that

$$X_t - \xi\,\Gamma_T^t = \int_t^T \Gamma_s^t\,\varphi_s ds - M_T + M_t\,. \tag{21}$$

Since $\Gamma_t \in S^2$, $X_t \in S^{2,T}$, $\pi_t \in \mathscr{H}^{2,T}$, $l_t \in \mathscr{H}_\nu^{2,T}$ and the processes δ_t, β_t and γ_t are bounded, it follows that the local martingale M_t is in fact a martingale, hence, taking the conditional expectation in (21), we have

$$\mathbb{E}\left[X_t\,|\,\mathcal{F}_t\right] = \mathbb{E}\left[\Gamma_T^t\,\xi + \int_t^T \Gamma_s^t\,\varphi_s\,ds - M_T + M_t\,|\,\mathcal{F}_t\right], \quad a.s.,$$

and since X_t is an $\mathcal{F}_t$-adapted process, by assumption, we conclude the proof, namely the following holds

$$X_t = \mathbb{E}\left[\Gamma_T^t\,\xi + \int_t^T \Gamma_s^t\,\varphi_s\,ds\,|\,\mathcal{F}_t\right], \quad a.s. \qquad \square$$

Two non trivial corollaries of Th.3.4, which will be used in Sect.4 to guarantee a non-arbitrage condition in a Lévy market model, are the following, see [48] for details.

Corollary 3.5. *Suppose that the assumptions of Th.(3.4) are satisfied, then if*

(i) $\quad\gamma_t(u) \geq -1,\ d\mathbb{P}\otimes dt\otimes d\nu(u)\quad a.s.,$

$\varphi_t \geq 0 \qquad t\in[0,T],\ d\mathbb{P}\otimes dt,\quad a.s.,$

$X_T = \xi \geq 0\quad a.s.$

then $X_t \geq 0,\quad 0\leq t\leq T\;\; a.s.$

(ii) $\gamma_t(u) > -1 \qquad d\mathbb{P} \otimes dt \otimes d\nu(u) \ \ a.s.,$

$\varphi_t \geq 0 \qquad t \in [0, T],\ d\mathbb{P} \otimes dt \ \ a.s.,$

$X_{t_0} = 0 \ \ a.s.\ for\ some\ t_0 \in [0, T]$

then $\varphi_t = 0 \quad d\mathbb{P} \otimes dt \ \ a.s.\ on\ [t_0, T]$, *and* $\xi = 0 \ \ a.s.\ on\ A \in \mathcal{F}_{t_0}$.

Corollary 3.6. *Suppose that*

- *the assumptions of Theorem (3.4) are satisfied,*
- C *is the bound of the process* δ_t,
- $\gamma_t(u) \geq -1, d\mathbb{P} \otimes dt \otimes d\nu(u) \ \ a.s.$

If, for some $\varepsilon \geq 0$,

$$\xi \geq 0 \ \ a.s. \quad and \quad \varphi_t \geq -\varepsilon,\ 0 \leq t \leq T \quad d\mathbb{P} \otimes dt \ \ a.s.$$

then

$$X_t \geq -\varepsilon T e^{CT} \quad a.s. \quad \forall\, t \in [0, T].$$

In this section we have provided general results coming from the theory of BSDEs with jump component. We decided to choose, between all possible, those techniques which are widely exploited in financial applied literature to efficiently deal with financial market perturbed by noise of Lévy type. In particular a complete treatment of the backward stochastic differential equations theory is far from the aim of this work and we refer the interested reader to, e.g., [20, 32, 36, 63].

4 Market model driven by Lévy processes

It has been shown, see, e.g., [16] and [10], that Lévy processes are relevant in mathematical finance, in particular in modelling of stock prices.

In this section we give a brief introduction to financial markets where the asset prices behaviour are represented by Itô–Lévy processes. We will discuss, using the results obtained in the previous sections, the problem of pricing and hedging contingent claim which is written on an underlying that is subjected to a risk of both diffusive and *jump* type, hence allowing for underlyings whose behaviors can be characterized by random discontinuities.

The latter allows to model the financial evolution (in time) of quantities of interest, taking into account the possibility for abrupt change of their value. In this case a perfect hedge does not exist, namely it is not always possible to replicate the derivative payoff by a controlled portfolio of the basic securities, see, e.g. [33].

We will derive the pricing relation by the risk-neutral valuation and by change of measure approach, which may be considered a generalization of the approach made in the Black–Scholes framework. For further details we refer to [8, 10, 59].

4.1 The model

We begin introducing the typical setup for continuous-time asset pricing.

Let us fix $T > 0$ and define a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let

- W_t be a one dimensional Brownian Motion,
- $N(dt, du)$ be a Poisson Random measure, indipendent from W_t, with compensator $\nu(du)dt$,
- $\mathcal{F}_t$ be the filtration generated by both W_t and N.

In order to not overload notation we will assume that the financial market consists of two kinds of securities: a locally riskless asset (or bond) and a risky security (or stock). Such an approach is not restrictive for our purposes, since it contains all the necessary and sufficient features which differentiate the Lévy case from the classical Black–Schoeles framework. In particular we have the following setup

1. the bond price is indicated by S_t^0 and its behaviour is governed by the equation
$$\begin{cases} dS_t^0 = S_t^0\, r_t\, dt, \qquad 0 \leq t \leq T, \\ S_0^0 = 1, \end{cases} \tag{22}$$
where r_t is the risk-free interest rate.

2. the dynamic of risky asset, S_t^1, at time t is given by
$$\begin{cases} dS_t^1 = S_{t^-}^1 \left[\mu_t\, dt + \sigma_t\, dW_t + \displaystyle\int_{\mathbb{R}^*} l_t(u)\, \widetilde{N}(dt, du) \right], \\ \qquad\qquad 0 \leq t \leq T, \\ S_0^1 \in \mathbb{R}, \end{cases} \tag{23}$$

where μ_t, σ_t are predictable processes and $l_t(u) \in \mathscr{H}_\nu^{2,T}$ which represent the drift, resp. the diffusion, of the Brownian component of the noise.

Analogously to the case where the asset price is modeled by a geometric Brownian motion, we still have that the stock price S_t^1 is a càdlàg process, which is now described by a geometric Lévy–Itô process.

The latter implies that the investor in the risky asset is exposed to two different type of risks, namely a *diffusion risk*, caused by the Brownian randomness, and the risk due to the pure jump component $\int_{\mathbb{R}^*} l_t(u)\, \widetilde{N}(dt, du)$, with random jump amplitudes $l_t(u)$, characterizing (23). In particular l_t is a source of randomness associated to the volatility coefficient of a Poisson process and it is responsible for the jumps of S_t^1, therefore its jump times are those of N.

It is natural to suppose that an investor entering in a market which behaves according to equations (22) and (23), would like to make a profitable investment. Obvious related question shall sounds like: *How much money should I invest?* Or: *How do I have to choose the right investment portfolio at time a certain given time?*. In order to give a useful answer to latter type of questions, let us introduce the following standard quantities and processes:

- we call V_t the wealth (stochastic) process obtained summing the amounts of our investment (at time t) which we hold in every single asset composing the market. We assume to start with an initial wealth amount $V_0 > 0$, at time $t = 0$.

- at any time $t \in [0, T]$, being $T > 0$ the time-horizon of our investment, the investor use the whole amount V_t, dividing it into two parts:
  - π_t is the amount of wealth V_t invested in the risky stock at time $t \in [0, T]$,
  - while the remaining part of the money is invested in the riskless asset and its value is given by $\pi^0_t = V_t - \pi_t$.

A reasonable requirement that we have to fulfill is the so called *self-financing* property for our investment strategy. Namely a portfolio is self-financing if there is no exogenous infusion or withdrawal of money. In particular the latter implies that we can augment the investment in the risky asset only subtracting money from the quantity financing the bond.

Otherwise we can easily obtain an arbitrage opportunity, see, e.g., [9, page 87]. Mathematically speaking a *self-financed* portfolio for our model is defined as follows

Definition 4.1. A self-financing strategy is a pair (V_t, π_t) where π_t is a predictable process such that

$$V_t = V_0 + \int_0^t \left(\pi^0_t \, \frac{dS^0_t}{S^0_t} + \pi_t \, \frac{dS^1_t}{S^1_t} \right), \tag{24}$$

with

$$\mathbb{E}\left[\int_0^T |\, \sigma_t \, \pi_t \,|^2 dt \right] < +\infty,$$

and

$$\mathbb{E}\left[\int_0^T \pi_t^2 \; l_t^2(u) \, \nu(du) \, dt \right] < +\infty \, .$$

From a heuristic point of view, def.(4.1) implies that the instantaneous variation of the wealth value is caused uniquely by assets' prices variations, and not by injecting or withdrawing funds from outside.

In particular an investor cannot use funds other that the initial wealth to finance his position in the market, moreover he is not allowed to spend money outside of the market or the consumption is only financed with the profits realized by the portfolio and not by outside benefits.

A (self-financing) strategy which satisfies eq.(24), equivalently has to satisfies (in our particular market model) the following linear SDE

$$dV_t = r_t \, V_t \, dt \; + \; \pi_t \, (\mu_t - r_t) \, dt \; + \; \pi_t \, \sigma_t \, dW_t \; + \; \pi_t \int_{\mathbb{R}^*} l_t(u) \, \widetilde{N}(dt, du) \, .$$

As in the Black–Scholes scenario it is of great relevance, for princing and hedging given derivatives, see, e.g. [42], to determine if at least one equivalent martingale measure exists, with respect to *real world probability measure*, namely the measure derived from time series for the financial quantities composing our model. A key result by which addressing such a quest is the Girsanov Theorem, see [30], for jump diffusion process, see, e.g., [39, 42]. In particular for a Geometric Itô–Lévy process, we recall that it has the following form:

Theorem 4.1. *(Girsanov Theorem for geometric Itô–Lévy processes)*
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space. Let X_t a Geometric Itô–Lévy process of the form

$$dX_t = X_{t^-}\left[a_t\, dt + b_t\, dW_t + \int_{\mathbb{R}^*} c_t(u)\, \widetilde{N}(dt, du)\right].$$

Let u_t, $v_t(u)$ be two predictable processes with respect to $\mathcal{F}_t$ such that

(i) $a_t - r_t + u_t\,\beta_t + \int_{\mathbb{R}^*} c_t(u)\, v_t(u)\, \nu(du) = 0,$

(ii) the process $v_t(u)$ satisfies $v_t(u) > -1$,

(iii) the process Z_t defined by the solution of the following SDE

$$\begin{cases} dZ_t = Z_{t^-}\left[u_t\, dW_t + \int_{\mathbb{R}^*} v_t(u)\, \widetilde{N}(dt, du)\right], \\ Z_0 = 1, \end{cases} \tag{25}$$

is well-defined and satisfies

$$\mathbb{E}\left[Z_T\right] = 1.$$

Then there exists a probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ which is equivalent to $\mathbb{P}$ and such that

$$d\mathbb{Q} = Z_T\, d\mathbb{P}\,.$$

Moreover the discounted process of X_t is a local martingale with respect to $\mathbb{Q}$.

Th.(4.1) straightforward implies the following result.

Corollary 4.2. *Let u_t and $v_t(u) > -1$ be a predictable processes such that the process Z_t satisfying*

$$dZ_t = Z_{t^-}\left[u_t\, dW_t + \int_{\mathbb{R}^*} v_t(u)\, \widetilde{N}(dt, du)\right],$$

is well-defined for $0 \le t \le T$. Suppose that

$$\mathbb{E}[Z_T] = 1,$$

and define the probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ by

$$d\mathbb{Q} = Z_T\, d\mathbb{P},$$

then

(i) the process $W_t^{\mathbb{Q}}$ defined by

$$W_t^{\mathbb{Q}} = W_t - \int_0^t u_s\, ds, \tag{26}$$

is a $\mathbb{Q}$-Brownian Motion and

(ii) the random measure $\widetilde{N}^{\mathbb{Q}}(dt, du)$ defined by

$$\widetilde{N}^{\mathbb{Q}}(dt, du) = \widetilde{N}(dt, du) - v_t(u)\, \nu(du)dt, \tag{27}$$

is such that

$$\int_0^t \int_{\mathbb{R}^*} \widetilde{N}^{\mathbb{Q}}(ds, du) = \int_0^t \int_{\mathbb{R}^*} \widetilde{N}(ds, du) - \int_0^t \int_{\mathbb{R}^*} v_s(u)\, \nu(du)ds\,.$$

is a $\mathbb{Q}$-local martingale.

In the following we exploit results given in Sec. 3.1, together with Th.(4.1), to characterize a financial market whose dynamic is described by a BSDE with jumps, see, e.g. [8]. In particular we will interpret eq.(15) as the equation of the price of a contingent claim on the underlying asset S^1_t.

Let us consider the Geometric Itô–Lévy process (23) modelling the behaviour of the risk asset S^1_t and the process Z_t describing by (9).

If β_t and γ_t are predictable processes such that

$$- \quad \gamma_t(u) > -1, \quad d\mathbb{P} \otimes dt \otimes \nu(du) \text{ a.s.}, \tag{28}$$

$$- \quad \mathbb{E}[Z_T] = 1 \qquad \text{on } \mathcal{F}_T\,, \tag{29}$$

$$- \quad \mu_t - r_t + \sigma_t\,\beta_t + \int_{\mathbb{R}^*} l_t(u)\,\gamma_t(u)\,\nu(du) = 0. \tag{30}$$

it is possible to define an equivalent martingale measure $\mathbb{Q}$ such that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T.$$

By Doléans–Dade formula the Radon–Nikodym density Z_T is given by

$$Z_T = \mathcal{E}(M)_T = \exp\left\{\int_0^T \beta_s dW_s - \frac{1}{2}\int_0^T \beta_s^2\,ds - \int_0^T \int_{\mathbb{R}^*} \gamma_t(u)\nu(du)ds\right\}$$
$$\times \prod_{0\le s\le T} (1 + \gamma_s(\Delta Y_s))\,. \tag{31}$$

In eq.(31) the first term is the usual Radon–Nikodym derivative for the Brownian Motion. The change in the distributions of the Brownian motion components follows from standard Girsanov Theorem. The remaining terms, again independent of each other, provide the Radon–Nikodym derivatives of the Poisson random measure. Details on Girsanov's measure transformation for Poisson random measures can be found, e.g., in [42].

We would like to underline that, with respect to necessary conditions to define the equivalent martingale measure, we have that

- since the expectation of the process Z_t, at time T, satisfies

  $$\mathbb{E}[Z_T] = 1, \tag{32}$$

  and M_T is a local martingale, then we have that $\mathcal{E}(M)_T = Z_T$ has to be a martingale.

  Note that, if the random variable β_t, γ_t satisfies

  $$\int_0^T \beta_s^2\,ds + \int_0^T ||\gamma_s||^2_\nu\,ds \quad \text{is bounded,}$$

  condition (32) is assured by Proposition (3.3).
- by the condition

  $$\mu_t - r_t + \sigma_t\,\beta_t + \int_{\mathbb{R}^*} l_t(u)\,\gamma_t(u)\,\nu(du) = 0, \tag{33}$$

we have that the parameters in the Radon–Nikodym derivative determine a family of equivalent martingale measures for the model, from which a suitable choice is made, moreover the terms β_t and γ_t represent the market price of risk associated with the Brownian risk, respectively the *new* component associated to the jump risk.

Let us now consider the problem of pricing and hedging a contingent claim whose payoff at maturity time T is given by ξ as in the framework described in Sect.4.1.

A contingent claim is said to be hedgeable if there exists a self-financing strategy (V_t, π_t) such that

$$\begin{cases} dV_t = r_t V_t dt + \pi_t\,(\mu_t - r_t)\,dt + \pi_t\,\sigma_t\,dW_t + \int_{\mathbb{R}^*} \pi_t\,l_t(u)\,\widetilde{N}(dt, du), \\ V_T = \xi. \end{cases} \tag{34}$$

Exploiting condition (33), we can rewrite eq.(34) as

$$\begin{aligned} dV_t &= r_t\,V_t dt + \pi_t\,(\mu_t - r_t)\,dt + \pi_t\,\sigma_t\,dW_t + \int_{\mathbb{R}^*} \pi_t\,l_t(u)\,\widetilde{N}(dt, du) = \\ &= r_t\,V_t dt \;-\; \pi_t\,\sigma_t\,\beta_t\,dt + \pi_t\,\sigma_t\,dW_t - \int_{\mathbb{R}^*} \pi_t\,\gamma_t(u)\,l_t(u)\,\nu(du)\,dt \;+ \\ &\quad + \int_{\mathbb{R}^*} \pi_t\,l_t(u)\,\widetilde{N}(dt, du) = \\ &= r_t\,V_t dt \;-\; \pi_t\,\sigma_t\,\beta_t\,dt - \langle \gamma_t,\, \pi_t\,l_t \rangle_\nu\,dt + \pi_t\,\sigma_t\,dW_t + \int_{\mathbb{R}^*} \pi_t\,l_t(u)\,\widetilde{N}(dt, du)\,. \end{aligned}$$

Hence, eq.(15) can be interpreted as a model for the value of an hedging strategy against a contingent claim ξ, choosing

- $\delta_t = -r_t$ the risk free rate,
- $\pi_t = \pi_t\,\sigma_t$, which includes the volatility for Brownian component,
- $l_t = \pi_t\,l_t$, which includes the volatility for jump component.

We can also consider that the total wealth process is also dependent on a certain consumption function c_t which expresses the consumer spending. The resulting dynamic satisfies

$$\begin{aligned} dV_t = &-c_t\,dt + r_t\,V_t dt \;-\; \pi_t\,\sigma_t\,\beta_t\,dt - \langle \gamma_t,\, \pi_t\,l_t \rangle_\nu\,dt \\ &+ \pi_t\,\sigma_t\,dW_t + \int_{\mathbb{R}^*} \pi_t\,l_t(u)\,\widetilde{N}(dt, du), \end{aligned}$$

and we can apply Th.(3.4) to obtain a formula for pricing a contingent claim in a market consisting of a risky asset driven by a jump-diffusion dynamic, namely we have

$$X_t = \mathbb{E}\left[\Gamma^t_T\,\xi + \int_t^T \Gamma^t_s\,\varphi_s\,ds \,|\, \mathcal{F}_t\right], \quad 0 \le t \le T \text{ a.s.},$$

where the process Γ_t is defined by

$$\Gamma_t = e^{-\int_0^t r_s\,ds}\,Z_t,$$

and it represents the discounted value of Z_t.

If $c_t = \varphi_t = 0$, then eq.(3) for a jump-diffusion market, becomes

$$X_t = \mathbb{E}\left[e^{-\int_t^T r_s\,ds}\;\; Z_T^t\,\xi\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s\,ds}\,\xi\right], \quad 0 \le t \le T \text{ a.s.}, \tag{35}$$

therefore the market price of a contingent claim in a jump-diffusion market can be computed as in the Black–Scholes context, see, e.g. [10].

In particular such a price results in a discounted expectation of the terminal payoff with respect to a martingale measure $\mathbb{Q}$.

Remark 4.1. Let us emphasize some relevant aspects.

- A fundamental difference between Black–Scholes and jump-diffusion setting concerns the equivalent martingale measure $\mathbb{Q}$. In the first case it is uniquely defined instead in the second there are many equivalent martingale measures to choose from, each yielding different distributions for the jump components. In fact, if we consider eq.(33) for market prices of risk, an equivalent martingale measure can be obtained by specifying choices of β_t and γ_t.
  Hence, when the underlying stock price dynamics are modeled by a jump-diffusion model, the market is incomplete. The latter can be interpreted in the light of Arrow's work, see [4], as a result of lack of Arrow–Debreu securities compared with the number of (possibly stochastic) states of nature. One can then apply the risk-neutral pricing formula to price derivative securities, but this formula can no longer be justified by a hedging argument. More precisely, pricing a contingent claim ξ using an equivalent martingale measure does not longer correspond to the initial price of a hedging strategy.
  Methods of evaluating contingent claims in an incomplete market are, e.g., the local risk-minimizing trading strategy approach, see Schweizer [56], or the minimum entropy martingale measure analysis, see [57], or the risk indifference approach, see, e.g., [42]. See also [37, 38] for a comprehensive treatment of the incomplete markets' theory.
- In finance, the existence of an equivalent martingale measure is linked with the absence of arbitrage property. If the process β_t and γ_t satisfies condition (28) and (29), the set of equivalent martingale measures is not empty but not reduced to a singleton. Indeed, the second fundamental Theorem of asset pricing, see, e.g. [58], guarantees that the market is free of arbitrage. However the lack of uniqueness for the martingale measure implies that the market is incomplete, so in this case perfect hedging is not possible.

We would also like to underline that some financial considerations can be made with respect to Corollary (3.5), in particular

- the choice of parameter γ_t, which represents the market price of risk for jump process, such that

  $$\gamma_t(u) \ge -1, \quad d\mathbb{P} \otimes dt \otimes d\nu(u) \text{ a.s.},$$

  guarantees, by eq.(11), the non negativity of Z_t.

Moreover, by Cor.(3.5), if $c_t \geq 0$ and $\xi \geq 0$, then the price of a contingent claim X_t satisfies

$$X_t \geq 0, \quad 0 \leq t \leq T \text{ a.s.}$$

- the choise of parameter γ_t such that

$$\gamma_t(u) \, > \, -1, \quad d\mathbb{P} \otimes dt \otimes d\nu(u) \text{ a.s.},$$

assures, by Th.(4.1), the existence of, at least, one a martingale measure. This fact implies, by the first Theorem of asset pricing, the non existence of arbitrage condition, indeed the second statement in Cor.(3.5) is clearly a non-arbitrage condition.

5 Dynamic risk measures

In this section we shall establish links between continuous-time dynamic risk measures and BSDEs with jumps. We would like to recall Sec.13 of [20] and references therein, for a detailed treatment of the subject with related actuarial applications. In finance, we are often exposed to risk in capital, whether as investors, traders or corporations. It seems therefore useful to quantify the riskiness of our financial position and hence to decide if such a risk is acceptable or not. The latter need naturally leads to the definition of *measures of risk*. During years, several classes of risk measures were proposed in literature. Let us give a brief résumé of their importance, see, e.g., [18, 22, 27] for further details.

Measuring and managing risks is one of the key activities in financial frameworks. Risk management provides methods to determine how to best handle different risk exposures, identify acceptable positions and determine minimum capital requirement that are required by financial institutions in order to ensure their stability.

One possible tool to measure risk is the Value at Risk (VaR), the most diffuse risk measure accepted in the financial industry. Nevertheless such a tool has been controversial at least from the end of last century, both from a *qualitative* and *quantitative* point of view. As examples it useful to recover the so called *Jorion–Taleb debate* (1997), where major criticisms addressed the VaR claimed ability to give *accurate* estimates for the rare events risks which is impossible by nature, the discussion within the Global Association of Risk Professionals Review, see, e.g., [13] and the conclusions obtained in [5] with respect to the VaR underestimation of extreme events probabilities, see also [41].

Moreover VaR is not a *coherent risk meassure* since it is not subadditive, see, e.g., [22]. In particular the VaR of a combined portfolio can be larger than the sum of its components' VaRs. Possible generalizations which allow to answer such criticisms are provided by the *conditional VaR* (CVaR), see [51] and references therein, and the *Entropic VaR* (EVaR), see, e.g., [2].

To overcome previously recalled shortcomings, an axiomatic approach to risk measure has been shown to be a key point for further developments in risk management and mathematical finance.

In particular, the concept of coherent and convex measures can provide an axiomatic approach resulting in a robust theory. First steps towards this direction are in the papers by Artzner, Delbaen, Eber and Heath, [5], while Föllmer and Schied [24, 25], and Fritelli and Rosazza Gianin [27, 28], have developed the theory of convex risk measures. Aximonatization of dynamic risk measures has been given by Riedle [50], see also [6, 7] and [53].

Links between dynamic risk measures and BSDEs are given in [54], see also [23, 26] and the *Dynamic Risk Measures* chapter, by Acciaio e Penner in [1]. Other authors have exploited dynamic risk measures induced by a BSDE in the Brownian case, see, e.g., [49] and [55], while in what follows, we give an overview about connections between dynamic risk measures and BSDEs with Lévy perturbations, see, e.g. [52].

5.1 Static risk measure

Risk measures were introduced in the literature to "evaluate future losses" providing criteria on the acceptability of risk exposures, and also for pricing purposes, see, e.g., [55].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $T > 0$ be a fixed future date and $\mathcal{X}$ be the space of all financial positions in which we are interested; for simplicity assume that $\mathcal{X} = L^p(\Omega, \mathcal{F}, \mathbb{P})$, with $1 \le p \le +\infty$.

For instance, an element of $\mathcal{X}$ may be the net worth at the maturity time T of a financial contract.

Definition 5.1. A static risk measure is a functional $\rho : \mathcal{X} \longrightarrow \mathbb{R}$, satisfying some properties induced by financial considerations. We present a list of axioms for ρ:

1. convexity
$$\rho(\alpha X + (1-\alpha) Y) \;\le\; \alpha \rho(X) + (1-\alpha) \rho(Y), \qquad \forall \alpha \in (0,1),\ \forall X, Y \in \mathcal{X};$$
2. positivity
$$\text{if} \quad X \ge 0 \quad \text{then} \quad \rho(X) \le \rho(0);$$
3. constancy
$$\rho(\alpha) = -\alpha \qquad \forall \alpha \in \mathbb{R};$$
4. translability
$$\rho(X + \beta) = \rho(X) - \beta \qquad \forall \beta \in \mathbb{R},\ \forall X \in \mathcal{X};$$
5. sublinearity
$$\rho(\alpha X) = \alpha \rho(X), \qquad \forall X \in \mathcal{X}, \quad \forall \alpha \ge 0;$$
$$\rho(X + Y) \le \rho(X) + \rho(Y), \qquad \forall X, Y \in \mathcal{X};$$
6. lower semi-continuity
$$\{X \in \mathcal{X} \,:\, \rho(X) \le \gamma\} \quad \text{is close in } \mathcal{X} \text{ for any } \gamma \in \mathbb{R}.$$

In the following, we will interpret ρ as follows: given a financial position X, the quantity $\rho(X)$ represents the riskiness of X and, by convention, X is acceptable when $\rho(X) < 0$, unacceptable otherwise.

Under translability, property (4) above, ρ has an extra interpretation. In particular translability implies

$$\rho(X + \rho(X)) = 0,$$

hence $\rho(X)$ is the amount of money that makes the position X *neutral-acceptable*. In other words, for an unacceptable position X, the quantity $\rho(X) \geq 0$, represents the minimum capital that we have to add to the initial position X in order to get an acceptable new position.

Between the set of all static risk measure, standard literature gives a special attention to coherent and convex measures.

Definition 5.2. A functional $\rho : \mathcal{X} \longrightarrow \mathbb{R}$, is a coherent risk measure if it satisfies properties (2), (4), and (5).

Definition 5.3. A functional $\rho : \mathcal{X} \longrightarrow \mathbb{R}$, is a convex risk measure if it satisfies properties (1), (6), and $\rho(0) = 0$.

Risk measures, at this moment, are defined starting from a set of financial positions, however they can be also generated alternatively. In particular, following [49] it is also possible to establish a link between "g-expectations", a particular expectation depending on a functional g, and risk measures for BSDEs. A financial interpretation of g will be given and motivated exploiting particular BSDEs which provides well established financial model. In particular g could depend on the preferences of the investor and on some parameters to which our financial position is characterized.

This new way of defining risk measures has been widely studied in the classical Black–Scholes framework, see, e.g. [49] and [54], in what follows we shall cover the same subject, but in a jump-diffusion framework following [48], see also Sec.13 of [20]. Let us consider the non linear BSDE given in eq.(3) with driver g,

$$\begin{cases} -dX_t = g(t, X_{t^-}, \pi_t, l_t(\cdot))\, dt - \pi_t\, dW_t - \displaystyle\int_{\mathbb{R}^*} l_t(u)\, \widetilde{N}(dt, du), \\ X_T = \xi, \end{cases} \tag{36}$$

In this equation, see also Sect. (4.1), X_T is the payoff of the contingent claim at maturity T, while X_t represents the value of the replicanting strategy at time $t \in [0, T]$. We focus our attention on the first component X_t of the solution (36), i.e., the so-called "g-expectation".

Definition 5.4. For every $\xi \in L^2(\mathcal{F}_T)$, the g-expectation of ξ is defined as

$$\varepsilon_g[\xi] := X_0\,.$$

The conditional g-expectation of ξ under $\mathcal{F}_t$ for every $\xi \in L^2(\mathcal{F}_T)$ is defined as

$$\varepsilon_g\left[\xi \,|\, \mathcal{F}_t\right] := X_t\,.$$

In order to define a risk measure via g-expectation let us consider $\mathcal{X} = L^2(\mathcal{F}_T)$, g a Lipschitz driver and set

$$\rho_g : L^2(\mathcal{F}_T) \longrightarrow \mathbb{R}$$

as follows:

$$\rho_g(\xi) := \varepsilon_g\left[-\xi\right], \qquad \xi \in L^2(\mathcal{F}_T).$$

ρ is a static risk measure satisfying axioms 2, 3 of Def (5.1).

Remark 5.1. (Interpretation of g)
As already pointed out, the functional g plays an important role in the "construction" of risk measures, so its choice is crucial. Its financial interpretation will be given and motivated through some BSDEs well known in finance, as the dynamic of a replicating strategy.

We emphasize that any $X \in \mathcal{X}$ denotes a general random variable representing the net worth of a financial position with maturity T .

In particular, it may happen that X "comes from" a stochastic process $(X_t)_{t\in[0,T]}$, the dynamic of which is known. For instance, X_t may represent the net worth of a portfolio of shares at time t and X_T its net worth at (horizon) time T, namely when the portfolio is sold out. Furthermore, g could depend on the preferences of the investor.

Let us consider a typical hedging problem that we have already study in Sect.4.1.1: we are interested in pricing a contingent claimX_t whose payoff at final time T is specified by the condition $X_T = \xi$.

$$\begin{cases} dV_t = r_t V_t dt + \pi_t\left(\mu_t - r_t\right)dt + \pi_t\,\sigma_t\,dW_t + \displaystyle\int_{\mathbb{R}^*} \pi_t\,l_t(u)\,\widetilde{N}(dt,du), \\ V_T = \xi. \end{cases} \tag{37}$$

It is well-known that the price at time 0 of a claim is simply the value V_0 of the self-financing portfolio that replicates X at the maturity T.
Then the corresponding static risk measure

$$\rho_g(\xi) := \varepsilon_g\left[-\xi\right] = -V_0,$$

represents the initial value of the replicating strategy of $(-\xi)$.

Nevertheless it is clear that the case just presented results in a very particular scenario. In general situations, namely when markets are not complete, the case of risk measures coming from acceptable sets formed with super-replicable positions was analysed by Föllmer and Schied, see, e.g., [25].

5.2 Dynamic risk measure and BSDEs with jumps

Risk measures discussed so far deal with the problem of quantify today the riskiness of financial position with maturity at future date T. In this sense, such risk measure can be considered as static.
However, most investors are making portfolio decisions dynamically and a further problem consists in monitoring the riskiness of our financial position at different times between today and the final date T. In order to deal with this problem we need to treat risk measures in a dynamic setting.

A dynamic risk measure $(\rho_t)_{t\in[0,T]}$, is defined, see, e.g. [55], as a map such that at any instant t, ρ_t is a random variable wich represent the riskiness of our financial position at time t, conditionally to the information avaible up to that time t.

Moreover, dynamic risk measures should fulfill some boundary conditions at time 0 and T.

Definition 5.5. We call a dynamic risk measure any map such that
- $\rho_t : \mathcal{X} \longrightarrow L^0(\mathcal{F}_t), \quad \forall\, t \in [0,T]$,
- ρ_0 is a static risk measure,
- $\rho_T(\xi) = -\xi, \qquad \forall\, \xi \in \mathcal{X}$.

In particular $(\rho_t)_t$ satisfies a set of desirable properties which are the same of the static case, see, e.g., [28]. Similarly to the static case, we consider strong connection between dynamic risk measure and conditional g-expectation.

Definition 5.6. We define the dynamic risk measure $(\rho_t)_{t\in[0,T]}$, induced by BSDE (36) with driver g, by the g-conditional expectation of ξ, that is actually the first component of the solution, namely

$$\rho_t^g(\xi,T) := \varepsilon_g\left[-\xi \,|\, \mathcal{F}_t\right] := -X_t(\xi,T), \qquad \xi \in L^2(\mathcal{F}_T),\, t \in [0,T]\,. \tag{38}$$

We follow a typical axiomatic approach, see, e.g., [56]. In particular, a dynamic risk measures, generated via conditional expectation, has to satisfies a given list of axioms. Assumptions on the driver of BSDE and properties of the related solution both induce such a list whose components are relevant from a financial point of view (see, e.g., [31] for a detailed financial treatment of the following axioms and [14] for other useful properties):

- **Consistency**
  Let $S \in \tau_{0,T}$ be a stopping time, for all $t \le S$,
  $$\rho_t(\xi,T) = \rho_t(-\rho_S(\xi,T),S) \quad \text{a.s.},$$
  or, equivalently
  $$X_t(\xi,T) = X_t(X_S(\xi,T),S) \quad \text{a.s.}\,.$$
  For each t smaller that S, the risk measure associated with position ξ and maturity T coincides with the risk measure associated with maturity S and position $-\rho_S(\xi,T) = X_S(\xi,T)$.
- **Continuity**
  Let $(\theta^\alpha)_{\alpha\in\mathbb{R}}$ be a family of stopping times in $\tau_{0,T}$, converging a.s. to a stopping time $\theta \in \tau_{0,T}$, namely
  $$\theta^\alpha \underset{\alpha\to\,\alpha_0}{\longrightarrow} \theta\,.$$
  Moreover $(\xi^\alpha)_{\alpha\in\mathbb{R}}$ be a family of random variables such that
  $$\mathbb{E}\left[\operatorname*{ess\,sup}_{\alpha}(\xi^\alpha)^2\right] < +\infty,$$
  and for each α, ξ^α is $\mathcal{F}_{\theta^\alpha}$-measurable. Suppose also that ξ^α converges a.s. to an $\mathcal{F}_\theta$-measurable random variable, i.e.
  $$\xi^\alpha \underset{\alpha\to\alpha_0}{\longrightarrow} \xi,$$

then, for each $S \in \tau_{0,T}$, the random random variable

$$\rho_S(\xi^\alpha, \theta^\alpha) \underset{\alpha \to \alpha_0}{\longrightarrow} \rho_S(\xi, \theta) \quad \text{a.s.}$$

- **Zero-one law**
  If $g(t, 0, 0) = 0$, then the risk-measure associated to the null position is equal to 0. More precisely, the risk-measure satisfies the Zero-one law property:

$$\rho_t(\mathbb{1}_A \, \xi, T) = \mathbb{1}_A \, \rho_t(\xi, T), \quad \text{a.s. } t \leq T, \ A \in \mathcal{F}_t, \ \xi \in L^2(\mathcal{F}_T)\,.$$

- **Translation invariance**
  If g does not depend on x, then the associated risk-measure satisfies the translation invariance property:

$$\rho_t(\xi + \xi', T) = \rho_t(\xi, T) - \xi', \qquad \xi \in L^2(\mathcal{F}_T), \, \xi' \in L^2(\mathcal{F}_t)\,.$$

  Latter property requires, indeed, translation invariance not only with respect to constants, but also with respect to any $\mathcal{F}_t$-measurable random variable or, roughly speaking, to any risky position that is completely determined by the information available at time t.
- **Homogeneous property**
  If g is positively homogenous with respect to (x, π, l), then the risk measure ρ is positively homogenous with respect to ξ, that is, for all $\lambda \geq 0$, the following holds

$$\rho_t(\lambda \, \xi, T) = \lambda \, \rho(\xi, T), \qquad \xi \in L^2(\mathcal{F}_T)\,.$$

  Special attention should be paid to the financial interpretation of sublinearity. This axiom, which was originally motivated by liquidity reasons, since it assures that the riskiness of a number λ of identical positions ξ is λ times the riskiness of ξ.
  For concrete applications in finance a further assumption on the driver g of eq. (36), is required in order to enrich the set of properties possessed by the risk measure ρ, see, e.g., [31]. In particular we require the following

Definition 5.7. (***Assumption A***) Let $T > 0$. We assume that for each $(x, \pi, l_1, l_2) \in [0, T] \times \Omega \times \mathbb{R}^2 \times (L^2_\nu)^2$

$$f(t, x, \pi, l^1) - f(t, x, \pi, l^2) \geq \left\langle \theta_t^{x,\pi,l^1,l^2}, \, l^1 - l^2 \right\rangle_\nu, \quad d\mathbb{P} \otimes dt \ \text{a.s.}$$

where

$$\theta : \, [0, T] \times \Omega \times \mathbb{R}^2 \times (L^2_\nu)^2 \longrightarrow L^2_\nu$$

$$(t, w, x, \pi, l^1, l^2) \longrightarrow \theta_t^{x,\pi,l^1,l^2}(w, \cdot),$$

has to satisfies the following conditions:
  - $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}((L^2_\nu)^2)$-measurable,
  - bounded,

– for each $(x, \pi, l^1, l^2) \in \mathbb{R}^2 \times (L^2_\nu)^2$

$$\theta_t^{x,\pi,l^1,l^2}(u) \geq -1 \quad \textit{and} \quad |\theta_t^{x,\pi,l^1,l^2}(u)| \leq \psi(u), \quad d\mathbb{P}\otimes dt \otimes d\nu(u) \text{ a.s.} \tag{39}$$

where $\psi \in L^2_\nu$.

- **Monotonicity**
ρ is non increasing with respect to ξ, that is, for each ξ^1, $\xi^2 \in \mathcal{F}_T$,

$$\text{if} \quad \xi^1 \geq \xi^2 \text{ a.s., then } \rho_t(\xi^1, T) \leq \rho_t(\xi^2, T), \quad 0 \leq t \leq T \text{ a.s.}$$

Note that the monotonicity of ρ implies that if two financial position are such that $\xi^1 \geq \xi^2$, then their risk measures have to satisfy $\rho(\xi^1) \leq \rho(\xi^2)$. Note that the opposite inequality holds because of financial interpretation of the risk measure.
The proof of this result follows from a comparison theorem between the solutions of two non linear BSDEs imposing different hypotesis on terminal conditions, see [48] and [52] for further details. Before show how this result can be proven, let us underline that,contrary to the Brownian case, the monotonicity property does not generally holds, and requires an additional assumption.

Theorem 5.1. *(Comparison theorem for BSDEs with jumps)*
Let ξ_1 and $\xi_2 \in L^2(\mathcal{F}_T)$. Let f_1 be a Lipschitz driver and f_2 be a driver. For $i = 1, 2$, let (X^i_t, π^i_t, l^i_t) be a solution in $S^{2,T} \times \mathscr{H}^{2,T} \times \mathscr{H}^{2,T}_\nu$ of BSDE

$$\begin{cases} -dX^i_t = f_i(t, X^i_t, \pi^i_t, l^i_t)dt \,-\, \pi^i_t\, dW_t \,-\, \displaystyle\int_{\mathbb{R}^*} l^i_t(u)\, \widetilde{N}(dt, du), \\ X^i_T = \xi_i. \end{cases} \tag{40}$$

Assume that there exists a bounded predictable process γ_t such that

$$- \gamma_t(u) \geq -1, \quad d\mathbb{P} \otimes dt \otimes \nu(du) \quad a.s., \tag{41}$$

$$- |\gamma_t(u)| \leq \psi(u), \quad d\mathbb{P} \otimes dt \otimes \nu(du) \quad a.s. \textit{ with } \psi \in L^2_\nu, \tag{42}$$

$$- f_1(t, X^2_t, \pi^2_t, l^1_t) - f_1(t, X^2_t, \pi^2_t, l^2_t) \geq < \gamma_t, l^1_t - l^2_t >_\nu, \quad d\mathbb{P} \otimes dt \quad a.s. \tag{43}$$

Assume that

$$- \xi_1 \geq \xi_2 \quad a.s., \tag{44}$$

$$- f_1(t, X^2_t, \pi^2_t, l^2_t) \geq f_2(t, X^2_t, \pi^2_t, l^2_t), \quad 0 \leq t \leq T \quad d\mathbb{P} \otimes dt \quad a.s., \tag{45}$$

then
(i) $X^1_t \geq X^2_t, \quad 0 \leq t \leq T, \quad d\mathbb{P} \otimes dt \quad a.s.$
(ii) Moreover, if inequality (43) is satisfied for (X^1_t, π^1_t, l^1_t) instead of (X^2_t, π^2_t, l^2_t) and if f_2, instead of f_1, is Lipschitz and satisfies (45), then (i) still holds.

- **Convexity**
  If g is concave with respect to (x, π, l), then the dynamic risk measure ρ is also convex, that is for any $\lambda \in [0,1]$, ξ^1, $\xi^2 \in L^2(\mathcal{F}_T)$

  $$\rho_t(\lambda\,\xi^1 + (1-\lambda)\,\xi^2, T) \;\leq\; \lambda\,\rho_t(\xi^1, T) + (1-\lambda)\,\rho_t(\xi^2, T)\,.$$

  Subadditivity property encourages the diversification through portfolios of risks since, the riskiness of a portfolio $(\xi^1 + \xi^2)$ is smaller than the sum of the riskiness of the single positions ξ^1 and ξ^2 .
  On the contrary, convexity assures only diversification through portfolios originated by "ad hoc weighted" sums of single positions.
- **No arbitrage**
  Now we assume that in Assumption A the following additional requirement holds
  $$\theta_t^{x,\pi,l^1,l^2} \;>\; -1 \qquad d\mathbb{P} \otimes dt \otimes d\nu(u) \;\text{ a.s.}$$
  For each ξ^1, $\xi^2 \in L^2(\mathcal{F}_T)$, if
  - $\xi^1 \geq \xi^2$,
  - $\rho_t(\xi^1, T) = \rho_t(\xi^2, T)$ a.s. on an event $A \in \mathcal{F}_t$,

  then
  $$\xi^1 = \xi^2 \qquad \text{a.s. on } A\,.$$

  Note that generally, contrary to the monotonicity property, the no arbitrage property is not required for risk-measures.

We would like to underline that while static risk measures provide useful details for risk management purposes over a fixed period of time with fixed *boundary conditions* in that period, dynamic risk measures allow us to better follow how the riskiness of a financial position behaves continuously in time whit respect to continuous time-variations of data upon which it depends. The latter implies that we can take decisions, e.g. changing portfolio composition and/or modifying capital requirement for the liability, at time $t \in [0,T]$, if T is the expiration date of our investment, according to what is happening, hence resulting in a more efficient global financial strategy.

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