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On nonautonomous Markov evolutions in continuum

M. Friesen,[1] *O. Kutoviy*[2]

Abstract. The nonautonomous Cauchy problem in a scale of Banach spaces is investigated. The existence and uniqueness of solutions to this problem is proven. The obtained results are applied to several dynamics of Markov evolutions in continuum (e.g. spatial logistic model, Glauber dynamics, etc.).

Contents

1 Introduction

A possible way of describing dynamics of complex systems of interacting particle is to assume that the elementary acts of the evolution occur at random and the evolution itself is Markovian. Among the mentioned elementary acts one can distinguish birth, death and motion. The rates at which they occur may

[1] Fakultät für Mathematik, Universität Bielefeld, 33615 Bielefeld, Germany. martin.friesen@gmx.de
[2] Department of Mathematics, MIT, Cambridge, MA, USA; Fakultät für Mathematik, Universität Bielefeld, 33615 Bielefeld, Germany. kutovyi@mit.edu, kutoviy@math.uni-bielefeld.de

depend on the actual state of the system and on the environment. Among various problems coming from the natural and life sciences the existence of state evolutions for wide classes of intensities (e.g. time dependent intensities) seems to be one of the fundamental problem. The evolution of states is informally given as a solution to the initial value problem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle F, \mu_t\rangle = \langle LF, \mu_t\rangle, \quad \mu_t|_{t=0} = \mu_0,$$

provided, of course, that a solution exists. Here L is an informal generator which describes the functional evolution of the system

$$\frac{\partial}{\partial t}F_t = LF_t, \quad F_t|_{t=0} = F_0$$

and

$$\langle F, \mu\rangle := \int_\Gamma F(\gamma)\mathrm{d}\mu(\gamma).$$

One of the aims of the present paper is to develop methods to solve nonautonomous Cauchy problems in a scale of Banach spaces $\mathbb{B}_\alpha$, which will be used to treat systems with time or enviroment dependent intensities. Our main technical tool is a general theorem by M. Safonov from [26] and several conclusions, obtained in the present paper. Using this theorem we will prove the existence of solutions on a bounded time interval for several models and in some cases we will give conditions for the existence of solutions on unbounded time intervals. The first part will be devoted to the general theory of nonautonomous Cauchy problems on scales of Banach spaces. A version of the general theorem by Safonov for linear operators will be proven. Afterwards we will extend this theorem for weaker assumptions, where the generator consists of two parts $L = A + B$ and only the second part satisfies the assumptions of the general theorem of Safonov. This technique will be used to prove a continuous dependence of the solutions on parameters. Markov evolutions of continuous interacting particle systems were studied by many authors for time independent coefficients. In the present paper we are going to be focused on nonautonomous models of birth and death type. However, the abstract results obtained in this paper may be applied also to other classes of Markov evolution. In our approach, populations will appear as particle configurations forming the following phase spaces

$$\Gamma = \Gamma(\mathbb{R}^d) = \{\gamma \subset \mathbb{R}^d : \ |\gamma \cap K| < \infty, \ \ \forall K \subset \mathbb{R}^d \text{ compact}\}.$$

One of the most simplest models of birth and death type is the so-called Sourgailis model. The mechanism of its evolution is given by the following heuristic generator

$$(LF)(\gamma) = m\sum_{x\in\gamma}(F(\gamma\backslash x) - F(\gamma)) + \kappa\int_{\mathbb{R}^d}(F(\gamma\cup x) - F(\gamma))\,\mathrm{d}x \qquad (1)$$

with $m, \kappa > 0$, cf. [27, 28]. In (1), the first term describes the death of the particle located at $x \in \gamma$ occurring independently with the rate $m > 0$. The second term in (1) describes the birth of a particle at $x \in \mathbb{R}^d$ with the constant rate $\kappa > 0$, which is independent of $\gamma \in \Gamma$. The corresponding state evolution as well as ergodic properties of the process were recently studied in [3].

Another model for Markov evolution which includes interaction between particles in the birth mechanism is, for example, the continuous contact model. It can be described by the formal Markov generator

$$(LF)(\gamma) = m\sum_{x\in\gamma}(F(\gamma\backslash x) - F(\gamma)) + \int_{\mathbb{R}^d}\sum_{x\in\gamma} a(x,y)\,(F(\gamma\cup y) - F(\gamma))\,\mathrm{d}y,$$

where $m > 0$ and $a(x,y) > 0$. The first term (death) is the same as for Sourgailis model and the second term describes the birth of a new particle at $y \in \mathbb{R}^d$ given by the whole configuration γ with the rate $\sum_{x\in\gamma} a(x,y) > 0$. This model was studied in the translation invariant case, i.e. $a(x,y) \equiv a(x-y) = a(y-x)$, in [18] and [20]. In [20] the authors proved the existence of the corresponding process for a dispersion kernel $a \in L^p(\mathbb{R}^d)$, $p > 1$ with compact support. The evolution of correlation functions and invariant states for the contact model were studied in [18].

A generalization of the previous model which includes local regulation in death is described by

$$\begin{aligned}(LF)(\gamma) \quad &= \quad \sum_{x\in\gamma}\Big(m + \sum_{y\in\gamma\backslash x} a^-(x,y)\Big)\,(F(\gamma\backslash y) - F(\gamma)) \\ &\quad + \sum_{x\in\gamma}\int_{\mathbb{R}^d} a^+(x,y)(F(\gamma\cup y) - F(\gamma))\mathrm{d}y\end{aligned}$$

with a competition kernel $a^- : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}_+$, a dispersion kernel $a^+ : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}_+$ and a mortality rate $m > 0$. Such model is called spatial logistic model or Bolker-Dieckman-Law-Pacala (short BDLP) model. A detailed analysis of this model in the case of translation invariant kernels may be found in [5, 7].

Another example of birth and death type dynamics is a non-equilibrium Glauber-type dynamics, described by

$$(LF)(\gamma) = m\sum_{x\in\gamma}(F(\gamma\backslash x) - F(\gamma)) + z\int_{\mathbb{R}^d} e^{-E(x,\gamma)}(F(\gamma\cup x) - F(\gamma))\mathrm{d}x$$

with $E(x,\gamma) = \sum_{y\in\gamma}\phi(x,y)$, where $\phi : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$ is a pair potential. For non-negative translation invariant potentials this model was discussed in [4, 6, 11, 12, 17, 19]. The reversible states for these dynamics are grand canonical Gibbs measures. This fact gives a standard way to construct properly associated stationary Markov processes using the corresponding (non-local) Dirichlet forms related to the considered Markov generators and Gibbs measures. These processes describe the equilibrium Glauber dynamics which preserve the initial Gibbs state in the time evolution, see e.g. [19]. The construction of a non-equilibrium Glauber-type dynamics was done in [17]. It was based on a general approach for the construction of non-equilibrium evolutions developed in [16]. In [6] the authors have shown that the correlation functions corresponding to Glauber dynamics converge to the correlation functions of the equilibrium state. Using Ovsjannikov-type technique in [4] an evolution in a scale of Banach spaces for quasi-observables and correlation functions was proved. In contrast to [6] in the present paper no conditions on z and $\beta = \int\limits_{\mathbb{R}^d} 1 - e^{-\phi(x)}\mathrm{d}x$ are imposed. The same technique was used in [11] to analyze the evolution of Bogoliubov generating functionals. In the present paper the similar arguments will be used to generalize the existence results, although only existence and no further properties will be studied.

Chapter 3.5 of this paper is devoted to the general birth and death Markov dynamics, given by

$$(LF)(\gamma) = \sum_{x\in\gamma} d(x,\gamma\backslash x)(F(\gamma\backslash x) - F(\gamma)) + \int\limits_{\mathbb{R}^d} b(x,\gamma)(F(\gamma\cup x) - F(\gamma))\mathrm{d}x. \quad (2)$$

Using a semigroup approach the existence of a solution to the corresponding Cauchy problem for quasi-observables and correlation functions were proven, cf. [8]. The authors further have shown that under several conditions there exists a unique solution to the stationary equation $L^{\Delta}k = 0$, which can be constructed by the "generalized Kirkwood-Salzburg" operator. Here L^{Δ} denotes the generator for the evolution of correlation functions. In this paper we will also study these equations in the class of sub-Poissonian correlation functions.

The structure of the paper can be described as follows. At the beginning we give a brief outline on the continuous Sourgailis model. An explicit solution for correlation functions k_t will be given and differentiability on some Banach spaces will be proven, assuming the initial data are regular enough. The possibility to solve all equations explicitly suggests this model as a play model. Further questions concerning this model deal with random time dependent coefficients.

In sections 3 and 4 the existence of solutions for quasi-observables in the case of BDLP and Glauber dynamics will be proven and, further, the evolution of correlation functions and Bogoliubov generating functionals be considered for Glauber dynamics. The assumptions are likely the same as for the time independent results, despite all inequalities should hold uniformly in time.

In the last section we will prove existence of solutions for infinite time intervals for general birth and death dynamics with the time dependent coefficients. Here the time dependence will enter only multiplicatively, i.e. $d_t = m(t)d$ and $b_t = \kappa(t)b$ (cf. (2)), since we need precise information about the domains of the corresponding generators.

2 Evolution Equations in Scales of Banach Spaces

2.1 General setting

Let X be a Banach space, let $[0,T] = I \subset \mathbb{R}$ be a compact interval, and let $(L(t), D(L(t)))_{t\in[0,T]}$ be a family of operators on X. Our main object of investigation is the following nonautonomous Cauchy problem

$$\frac{\partial u}{\partial t}(t) = L(t)u(t), \quad t \geq s,\ t \in I, \quad u(s) = u_0 \tag{3}$$

on X for $0 \leq s < T$. Such equations were analyzed in, e.g. [13, 22, 23]. The aim is to construct an evolution family

$$\Delta \ni (t,s) \longmapsto U(t,s) \in L(X),$$

where $\Delta = \{(t,s) \in I \times I : \quad s \leq t\ \}$. This map should be strongly continuous and should have, instead of the usual semigroup property, the evolution family property

$$U(s,s) = id_X, \quad U(t,q)U(q,s) = U(t,s), \quad 0 \leq s \leq q \leq t \leq T.$$

In order to give sense to the right hand side of (3), i.e. $L(t)u(t)$, we should assume $u(t) \in D(L(t))$ or more generally

$$u(t) \in \bigcap_{s\in[0,T]} D(L(s)) \subset X.$$

In general it is difficult to characterize the explicit structure of $D(L(t))$, which is one of the major difficulties in this approach. Therefore one restricts to some smaller subspace. Assume there exists a Banach space $Y \subset \bigcap_{t\in I} \mathrm{Dom}(L(t)) \subset X$ such that for each $u \in Y$ the mapping

$$\Delta \ni (t,s) \longmapsto U(t,s)u \in X$$

is differentiable with derivatives

$$\frac{\partial U}{\partial t}(t,s)u = L(t)U(t,s)u, \quad \frac{\partial U}{\partial s}(t,s)u = -U(t,s)L(s)u.$$

Then we can formally write the solution to (3) as

$$u(t;s,u_0) = U(t,s)u_0.$$

Similarly, the expression $L(t)U(t,s)u_0$ would be well-defined if we assume $U(t,s)u_0 \in Y$, so $U(t,s)Y \subset Y$, which will be assumption in Theorem 2.3. This considerations motivate the following definition of a solution to the above nonautonomous Cauchy problem (3), which can be found, e.g., in [24].

Definition 2.1. Let X, Y be Banach spaces such that $Y \subset X$ is continuously and densely embedded. For a family of operators $(L(t), D(L(t)))_{t\in[0,T]}$ assume

$$Y \subset \bigcap_{t\in[0,T]} D(L(t)) \subset X.$$

A function $u = u(t)$ is called Y-valued solution of the nonautonomous Cauchy problem (3) with initial condition $u_0 \in Y$, if it has the following properties

1. $u \in C([0,T]; Y) \cap C^1([0,T]; X)$
2. u solves (3).

The derivatives at $t = 0$ and $t = T$ will be always defined by

$$\frac{\partial u}{\partial t}(0) = \lim_{h\to 0, h>0} \frac{u(h) - u(0)}{h}, \quad \frac{\partial u}{\partial t}(T) = \lim_{h\to 0,\ \ h>0} \frac{u(T) - u(T-h)}{h}.$$

Note that in contrast to a classical solution we impose continuity in the Y-norm, which is a stronger condition than just $u(t) \in Y \subset D(L(t))$. Contrary to the general semigroup theory, where the semigroup is always differentiable on the domain of its generator, it is possible that an evolution family is nowhere differentiable. Now we will state two results due to [24] for existence of evolution families under conditions know as "the hyperbolic case". For this let us recall the definition of admissibility.

Definition 2.2. Let $(L, D(L))$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X and $Y \subset X$ a subspace. Y is said to be L-admissible if $T(t)Y \subset Y$ holds and the restriction $T(t)|_Y$ is a C_0-semigroup on Y.

In [24] it was shown that this is equivalent to the condition that the part $\widetilde{L}$ of L on Y is again a generator of a C_0-semigroup. This semigroup is then given by restricting $T(t)$ to Y. The part $\widetilde{L}$ of L on Y is defined as

$$D(\widetilde{L}) = \{u \in Y \cap D(L) : \ \ Lu \in Y\}, \quad \widetilde{L}u = Lu, \quad \text{for } u \in D(\widetilde{L}).$$

Theorem 2.1 ([24])**.** *Let X, Y be Banach spaces such that Y can be densely embedded in X and let $(L(t), D(L(t)))_{t\in[0,T]}$ be generators of C_0-semigroups $((e^{\tau L(t)})_{\tau\geq 0})_{t\in[0,T]}$ on X. Assume that the following conditions are satisfied:*

1. *$L(t)$ is Kato-stable, i.e. $\exists M \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(L(t))$ for all $t \in [0,T]$ and*

$$\| e^{\tau_k L(t_k)} \cdots e^{\tau_1 L(t_1)} \|_X \leq M e^{\omega \sum\limits_{j=1}^{k} \tau_j}$$

   for all $0 \leq t_1 \leq \cdots \leq t_k \leq T$, $k \in \mathbb{N}$ and $\tau_1, \ldots, \tau_k \geq 0$, where $\rho(L(t))$ denotes the resolvent set of $L(t)$.

2. *$Y \subset \bigcap_{t\in[0,T]} D(L(t))$ and*

$$I \ni t \longmapsto L(t) \in L(Y, X)$$

   is continuous in the uniform operator topology.

3. Y is $L(t)$-admissible for all $t \in [0,T]$ and $\widetilde{L}(t)$ as the part of $L(t)$ in Y is Kato-stable.

Then there exists a unique evolution family $(U(t,s))_{(t,s)\in\Delta}$ with the properties:

1. $\|U(t,s)\|_{L(X)} \le Me^{\omega(t-s)}, \quad (t,s) \in \Delta$
2. $\left(\frac{\partial U}{\partial t}\right)^{+} (t,s)|_{t=s} u = L(s)u$
3. $\frac{\partial U}{\partial s}(t,s)u = -U(t,s)L(s)u$

for $u \in Y$. Here the derivatives are considered in the sense of the norm in X and $\left(\frac{\partial U}{\partial t}\right)^{+} (t,s)|_{t=s}u$ is the right-sided derivative evaluated at (s,s).

Remark 2.1.

1. Kato-stability is neither necessary nor a sufficient condition for the existence of an evolution family. In [23] the authors gave a counterexample, where an evolution family exists, so the Cauchy problem (3) is well-posed, but the stability condition is not satisfied.

2. The main idea of the proof is to consider a sequence of with respect to t piecewise constant operators $A_n(t)$ and define appropriate evolution families $U_n(t,s)$, which are piecewise continuously differentiable on X for $u \in Y$. After showing the existence of a limit $U(t,s)$ in the strong sense on $L(X)$ it remains to show that the differentiability property still holds.

3. It is possible to replace continuity of $t \longmapsto L(t) \in L(Y,X)$ by the weaker assumption
$$L(\cdot) \in L^1([0,T], L(Y,X)).$$
In this case the strong differentiability for $(t,s) \in \Delta$ holds only almost everywhere.

To obtain stronger differentiability properties for $U(t,s)$ we should know further properties of the evolution family. In a scale of Banach spaces these properties can be easily checked. As already mentioned we should assume $U(t,s)u \in Y$ for $u \in Y$ to give meaning to the expression $\frac{\partial U(t,s)}{\partial t}u = L(t)U(t,s)u$. This will be the content of the next theorem, cf. [24].

Theorem 2.2. *Let $X, Y, L(t), U(t,s)$ be as in Theorem 2.1. If $U(t,s)Y \subset Y$ holds and the mapping*
$$\Delta \ni (t,s) \longmapsto U(t,s)u$$
is continuous in Y for $u \in Y$, then $U(t,s)$ satisfies the stronger differentiability property
$$\frac{\partial U}{\partial t}(t,s)u = L(t)U(t,s)u, \quad 0 \le s < t \le T.$$
Consequently equation (3) *has a unique Y-valued solution given by $U(t,s)u_0 = u(t)$.*

2.2 Scales of Banach Spaces

In this section we will introduce the notion of a one-parameter family of Banach spaces and state some consequences for the corresponding nonautonomous Cauchy problems, which will be useful later.

Definition 2.3. A scale of Banach spaces of type 1 is a one-parameter family $(\mathbb{B}_\alpha, \|\cdot\|_\alpha)_{\alpha_* \leq \alpha \leq \alpha^*}$ with $\alpha_* < \alpha^*$ satisfying

$$\alpha' < \alpha \Longrightarrow \|\cdot\|_\alpha \leq \|\cdot\|_{\alpha'}, \quad \mathbb{B}_{\alpha'} \subset \mathbb{B}_\alpha.$$

Analogous, a scale of Banach spaces of type 2 is a one-parameter family $(\mathbb{B}'_\alpha, \|\cdot\|_\alpha)_{\alpha_* \leq \alpha \leq \alpha^*}$ with

$$\alpha' < \alpha \Longrightarrow \|\cdot\|_{\alpha'} \leq \|\cdot\|_\alpha, \quad \mathbb{B}'_\alpha \subset \mathbb{B}'_{\alpha'}.$$

$\mathbb{B}_\alpha$ will always denote a scale of Banach spaces of type 1 and $\mathbb{B}'_\alpha$ a scale of type 2.

The family of weighted L^p spaces is a natural example for scales of Banach spaces. Let (Ω, μ) be a measurable space and $\omega : \Omega \longrightarrow \mathbb{R}_+$ be a measurable function. Define the weighted L^p spaces by

$$\mathbb{B}_\alpha = \left\{ f : \Omega \longrightarrow \mathbb{K} : \quad \|f\|_\alpha^p = \int\limits_\Omega |f(x)|^p e^{-\alpha\omega(x)} \mathrm{d}\mu(x) < \infty \right\}$$

for $1 \leq p < \infty$ and for $p = \infty$ as the weighted Banach space with the norm

$$\|f\|_\alpha = \operatorname*{ess\,sup}_{x \in \Omega} |f(x)| e^{-\alpha\omega(x)}.$$

Clearly $(\mathbb{B}_\alpha, \|\cdot\|_\alpha)$ is a scale of Banach spaces of type 1 and $\mathbb{B}'_\alpha = \mathbb{B}_{-\alpha}$ a scale of Banach spaces of type 2.

Remark 2.2.

1. We do not impose conditions whether the embeddings from the smaller into the bigger Banach spaces are dense. In applications we will consider the scale of L^1- respectively L^∞-type spaces, so this condition would not hold for $\mathbb{B}_\alpha$. In [2] the author uses the density of embeddings to prove some sufficient conditions for the well-posedness of equation (3).

2. In general, the spaces $\mathbb{B}_\alpha$, $\bigcup\limits_{\alpha' < \alpha} \mathbb{B}_{\alpha'}$ and $\bigcap\limits_{\alpha'' > \alpha} \mathbb{B}_{\alpha''}$ are different for a scale of type 1. The same is valid for a scale of type 2.

Using this approach, one has the possibility to overcome the difficulty of the time dependence of the domain $D(L(t))$. More precisely one would like to consider the operators $L(t)$ as bounded operators acting from smaller into a bigger Banach space, cf. [2, 5, 11]. Using this, one could consider the operators $(L(t), \mathrm{Dom})_{t \in [0,T]}$ on $\mathbb{B}_\alpha$ with the domain

$$\mathrm{Dom} = \bigcup_{\alpha' < \alpha} \mathbb{B}_{\alpha'}$$

for a scale of type 1. In the case of a scale of type 2 one has $\mathrm{Dom} = \bigcup_{\alpha<\alpha'} \mathbb{B}'_{\alpha'}$. Except for Theorem 2.1 and 2.3 we do not need any conditions of closedness of the operators for this approach. Unfortunately, the solutions will only exist on a bounded time interval $[0, T_*)$. As a consequence of the proof we will see that the solutions evolve in this scale of Banach spaces.

Now assume $Y \hookrightarrow X$ are Banach spaces and $I \ni t \longrightarrow L(t) \in L(Y, X)$ is strongly continuous. Then $\sup_{t\in I} \|L(t)u\|_X < \infty$ holds for all $u \in Y$ and by Banach-Steinhaus theorem $L(t)$ is uniformly bounded in $t \in I$, i.e. $M := \sup_{t\in I} \|L(t)\|_{L(Y,X)} < \infty$. Moreover, for each function $u \in C([0, T]; Y)$ the mapping $I \ni t \longmapsto L(t)u(t) \in X$ is continuous, which follows for $t_0, t \in I$ from

$$\begin{aligned} &\|L(t)u(t) - L(t_0)u(t_0)\|_X \\ \leq\ &\|L(t)u(t) - L(t)u(t_0)\|_X + \|L(t)u(t_0) - L(t_0)u(t_0)\|_X \\ \leq\ &M\|u(t) - u(t_0)\|_Y + \|L(t)u(t_0) - L(t_0)u(t_0)\|_X. \end{aligned}$$

For our calculations, we will need the following product formula for evolution families, which proof shall be omitted.

Lemma 2.3. *Let $Y \hookrightarrow X$ be Banach spaces, $U : \Delta \longrightarrow L(X)$ strongly continuous in the second variable for fixed $t \in I$ and let $s \longmapsto U(t, s)u \in X$ be continuously differentiable for fixed $t \in I$ and $u \in Y$. Then for each $u \in C^1(I, X)$ such that $u(t) \in Y$ with $t \in I$ the equation*

$$\frac{\partial}{\partial s}\left(U(t, s)u(s)\right) = \frac{\partial U}{\partial s}(t, s)u(s) + U(t, s)\frac{\partial u}{\partial s}(s), \quad (t, s) \in \Delta \tag{4}$$

holds on X.

Remark 2.3.

1. Of course, we can apply this lemma for strongly continuously differentiable evolution families as in Theorem 2.1 and 2.3.

2. In many applications the so-called exponential growth condition

   $$\|U(t, s)\|_{L(X)} \leq Ce^{\omega(t-s)}$$

   is satisfied. Nevertheless there are evolution families that do not have exponential growth. For example denote by X the space of all continuous bounded functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ and let $0 < p : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be bounded. The expression

   $$U(t, s)f(x) = \frac{p(t)}{p(s)}f(x), \quad x \in \mathbb{R}$$

   defines an operator $U(t, s) \in L(X)$ with $\|U(t, s)\|_{L(X)} = \frac{p(t)}{p(s)}$. If p is not bounded away from 0, then clearly $U(t, s)$ cannot be exponentially bounded. Note that a strongly continuous semigroup $(T(t))_{t\geq 0}$ always obeys a bound $\|T(t)\|_{L(X)} \leq Ce^{\omega t}$.

2.3 The Space of Solutions

At first, we will give a formal definition of a solution to equation (3) in a scale of Banach spaces. The idea is to consider solutions in some Banach space $\mathbb{B}_{\alpha^*}$ with the property that for each t there exist α_t such that $u(t) \in \mathbb{B}_{\alpha_t}$ holds. Additionally we would like to have the differentiability property for each α in the space $\mathbb{B}_\alpha$. In other words a solution is a consistent family of solutions in the spaces $\mathbb{B}_\alpha$.

Definition 2.4. Given a scale of Banach spaces of type 1 and $L(t) \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$ for $\alpha' < \alpha$ and $t \in [0,T]$. A solution in the scale $\mathbb{B}_\alpha$ to the Cauchy problem

$$\frac{\partial u}{\partial t}(t) = L(t)u(t), \quad u(0) = u_0 \in \mathbb{B}_{\alpha_*}, \quad t \in [0,T] \tag{5}$$

is given by a continuous, monotonically increasing function $(\alpha_*, \alpha^*] \ni \alpha \longmapsto T(\alpha) > 0$ with $T(\alpha) \leq T$, which we will call time data, and an element

$$u \in C^1([0, T(\alpha^*)); \mathbb{B}_{\alpha^*})$$

satisfying $u(0) = u_0$ and for all $\alpha \in (\alpha_*, \alpha^*]$ we have

$$u_\alpha := u|_{[0,T(\alpha))} \in C^1([0, T(\alpha)); \mathbb{B}_\alpha) \tag{6}$$

and

$$\frac{\partial u_\alpha}{\partial t}(t) = L(t)u_\alpha(t)$$

in $\mathbb{B}_\alpha$.

Given a scale of Banach spaces of type 2 and $L(t) \in L(\mathbb{B}'_\alpha, \mathbb{B}'_{\alpha'})$ for $\alpha' < \alpha$. A solution in the scale $\mathbb{B}'_\alpha$ to the Cauchy problem

$$\frac{\partial u}{\partial t}(t) = L(t)u(t), \quad u(0) = u_0 \in \mathbb{B}'_{\alpha^*}, \quad t \in [0,T]$$

is given by a continuous, monotonically decreasing function $[\alpha_*, \alpha^*) \ni \alpha \longmapsto T(\alpha) > 0$ with $T(\alpha) \leq T$ and an element

$$u \in C^1([0, T(\alpha_*)); \mathbb{B}'_{\alpha_*})$$

satisfying $u(0) = u_0$ and for all $\alpha \in [\alpha_*, \alpha^*)$ we have

$$u_\alpha := u|_{[0,T(\alpha))} \in C^1([0, T(\alpha)); \mathbb{B}_\alpha)$$

and

$$\frac{\partial u_\alpha}{\partial t}(t) = L(t)u_\alpha(t)$$

in $\mathbb{B}'_\alpha$.

Remark 2.4.

1. The time data $T(\alpha)$ may depend on the initial condition. Nevertheless in our approach this will not be the case.

2. If we start with some given $T(\alpha) > 0$ and unique elements u_α as in (6) satisfying the corresponding equations one can show that $u := u_{\alpha^*}$ is a solution in the scale $\mathbb{B}_\alpha$.

3. The continuity and monotonicity of $T(\alpha)$ implies that for $t \in [0, T(\alpha))$ there exists some $\alpha' < \alpha$ such that $0 \leq t \leq T(\alpha') \leq T(\alpha)$ holds. Thus one has $u_\alpha(t) \in \mathbb{B}_{\alpha'}$ and hence $L(t)u_\alpha(t)$ is well-defined as an element in $\mathbb{B}_\alpha$.

It is possible to rewrite the problem (5) in the integral form

$$u(t) = u_0 + \int_0^t L(\tau)u(\tau)\mathrm{d}\tau,$$

which proof shall be omitted.

Lemma 2.4. *Assume that* $[0, T] \ni t \longmapsto L(t) \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$ *is strongly continuous for any* α, α' *with* $\alpha_* \leq \alpha' < \alpha \leq \alpha^*$*, then the following statements are equivalent:*

1. *u is the solution to* (5) *in the scale* $\mathbb{B}_\alpha$ *with the time data* $T(\alpha)$
2. $u \in C([0, T(\alpha)); \mathbb{B}_\alpha)$ *for all* $\alpha \in (\alpha_*, \alpha^*]$ *and solves*

$$u(t) = u_0 + \int_0^t L(\tau)u(\tau)\mathrm{d}\tau \in \mathbb{B}_\alpha, \quad u_0 \in \mathbb{B}_{\alpha_*} \tag{7}$$

for $t \in [0, T(\alpha))$*, where* $T(\alpha) \leq T$ *is continuous and monotonically increasing.*

With the help of Lemma 2.8 it is easy to show the existence of a solution to equation (5) on a bounded time interval. Assume $\|L(t)\|_{\alpha'\alpha} \leq \dfrac{M}{\alpha - \alpha'}$ for $\alpha' < \alpha$ and that $[0, T] \ni t \longmapsto L(t) \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$ is strongly continuous, where $\| \cdot \|_{\alpha'\alpha}$ denotes the operator norm on $L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$. We will show this only for a scale of type 1, since the other case can be shown analogous. Let $u_0 \in \mathbb{B}_{\alpha_*}$ and define the sequence

$$u_0(t) = u_0, \quad u_{n+1}(t) = u_0 + \int_0^t L(\tau)u_n(\tau)\mathrm{d}\tau, \quad n \in \mathbb{N}_0, \tag{8}$$

which satisfies

$$u_n(t) = u_0 + \sum_{k=1}^n \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} L(t_1)\dots L(t_k)u_0\mathrm{d}t_k \dots \mathrm{d}t_1 \in C([0, T(\alpha)); \mathbb{B}_\alpha).$$

For $n \in \mathbb{N}$ and $\alpha_* < \alpha < \alpha^*$ define

$$\varepsilon = \frac{\alpha - \alpha_*}{n} \text{ and } \alpha_j = \alpha_* + j\varepsilon \text{ for } j = 0, \dots, n, \tag{9}$$

so we have $\alpha_0 = \alpha_*$, $\alpha_n = \alpha$ and $\alpha_{j+1} - \alpha_j = \varepsilon$ and hence

$$\begin{aligned}\|u_n(t) - u_{n-1}(t)\|_\alpha &\leq \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \left(\frac{M}{\varepsilon}\right)^n \|u_0\|_{\alpha_*} \mathrm{d}t_n \dots \mathrm{d}t_1 \\ &= \frac{1}{n!} \left(\frac{tnM}{\alpha - \alpha_*}\right)^n \|u_0\|_{\alpha_*}. \end{aligned} \tag{10}$$

Using Stirlings formula we see that the right hand side is summable in $n \in \mathbb{N}$ for $|t| < T(\alpha)$ with

$$T(\alpha) = \frac{\alpha - \alpha_*}{eM}. \tag{11}$$

Hence, $(u_n(t))_{n\in\mathbb{N}} \subset \mathbb{B}_\alpha$ is a fundamental sequence and therefore has a limit $u(t) = \lim\limits_{n\to\infty} u_n(t) \in \mathbb{B}_\alpha$ for $t \in [0, T(\alpha))$. Moreover, the convergence is uniform on each interval $[0, s] \subset [0, T(\alpha))$. To show this, consider for $n < m$

$$\|u_m(t) - u_n(t)\|_\alpha \leq \sum_{k=n}^{m-1} \|u_{k+1}(t) - u_k\|_\alpha \leq \text{Const.} \sum_{k=n}^{\infty} \left(\frac{t}{T(\alpha)}\right)^k$$

and obtain by passing to the limit $m \to \infty$

$$\|u_n(t) - u(t)\|_\alpha \leq \text{Const.} \sum_{k=n}^{\infty} \left(\frac{t}{T(\alpha)}\right)^k.$$

Therefore $u \in C([0, T(\alpha)); \mathbb{B}_\alpha)$ and by

$$\begin{aligned}\|L(t)u_n(t) - L(t)u_{n-1}(t)\|_\alpha &\leq \left(\frac{M}{\varepsilon}\right)^{n+1} \|u_0\|_{\alpha_*} \frac{t^n}{n!} \\ &= \frac{nM}{\alpha - \alpha_*} \|u_0\|_{\alpha_*} \frac{1}{n!} \left(\frac{tnM}{\alpha - \alpha_*}\right)^n \end{aligned}$$

the convergence $L(t)u_n(t) \to L(t)u(t)$ holds uniformly on compact intervals $t \in [0, s] \subset [0, T(\alpha))$. Consequently taking the limit in (8) we obtain equation (7).

Remark 2.5.

1. In the same way one can show the existence for arbitrary initial times t_0. In this case we would have the condition $|t - t_0| < T(\alpha)$ for convergence.

2. The difficulty is to show that the solution above is unique. Our assumptions on $L(t)$ do not allow to apply the Gronwall Lemma. To overcome this difficulty we will solve the corresponding integral equation (7) in some Banach space S^β, which reflects the properties of a solution in a scale $\mathbb{B}_\alpha$.

The general result for a quasilinear Cauchy problem in a scale of type 2 was published by Safonov in 1995 in [26]. Here we will only present a proof for the linear equation in a scale of type 1. The last result suggests that $T(\alpha) = \dfrac{\alpha - \alpha_*}{\lambda}$ for $\lambda > 0$ is a natural candidate for the time data. This motivates the following definition.

Definition 2.5. For $\lambda > 0$ and $\beta \geq 0$ let

$$S_1^\beta(\alpha_*, \alpha^*, \lambda) \equiv S_1^\beta = \left\{ u \in \bigcap_{\alpha_* \leq \alpha \leq \alpha^*} C\left(\left[0, \frac{\alpha - \alpha_*}{\lambda}\right); \mathbb{B}_\alpha\right) \,\middle|\, \|u\|_1^{(\beta)} < \infty \right\}$$

for the type 1 scale and

$$S_2^\beta(\alpha_*, \alpha^*, \lambda) \equiv S_2^\beta = \left\{ u \in \bigcap_{\alpha_* \leq \alpha \leq \alpha^*} C\left(\left[0, \frac{\alpha^* - \alpha}{\lambda}\right); \mathbb{B}'_\alpha\right) \,\middle|\, \|u\|_2^{(\beta)} < \infty \right\}$$

for the type 2 scale. The norms are given by

$$\|u\|_1^{(\beta)} = \sup_{\alpha \in [\alpha_*, \alpha^*],\ t \in [0, T_1(\alpha))} (\alpha - \alpha_* - \lambda t)^\beta \|u(t)\|_\alpha$$

$$\|u\|_2^{(\beta)} = \sup_{\alpha \in [\alpha_*, \alpha^*],\ t \in [0, T_2(\alpha))} (\alpha^* - \alpha - \lambda t)^\beta \|u(t)\|_\alpha$$

with $T_1(\alpha) = \dfrac{\alpha - \alpha_*}{\lambda}$ and $T_2(\alpha) = \dfrac{\alpha^* - \alpha}{\lambda}$.

Here we use the notation $C([0,0); \mathbb{B}_{\alpha_*}) = \mathbb{B}_{\alpha_*}$ and $C([0,0); \mathbb{B}_{\alpha^*}) = \mathbb{B}_{\alpha^*}$.

Clearly this spaces are complete and therefore Banach spaces.

2.4 Existence of local solutions and properties

In the main part of this section we will discuss two possibilities to show existence of solutions to (5). The first existence result is a simplified version of the general result from [26].

Theorem 2.5. *Consider a scale $(\mathbb{B}_\alpha, \|\cdot\|_\alpha)_{\alpha_* \leq \alpha \leq \alpha^*}$ of type 1 and assume that there exist $\lambda_a > 0$ and $M \geq 0$ such that*

1. $\left[0, \dfrac{\alpha^* - \alpha_*}{\lambda_a}\right) \ni t \longmapsto L(t) \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$ *is strongly continuous for any* $\alpha' < \alpha$
2. $\|L(t)\|_{\alpha'\alpha} \leq \dfrac{M}{\alpha - \alpha'}$ *for any* $\alpha' < \alpha$ *and* $t \in \left[0, \dfrac{\alpha^* - \alpha_*}{\lambda_a}\right)$.

Then there exists $\lambda_0 > \lambda_a > 0$ and $T_\lambda : (\alpha_*, \alpha^*] \longrightarrow \mathbb{R}_+$ continuous and monotonically increasing given by

$$T_\lambda(\alpha) = \frac{\alpha - \alpha_*}{\lambda}, \quad \text{with } \lambda > \lambda_0$$

such that for each initial condition $u_0 \in \mathbb{B}_{\alpha_*}$ there exists a unique solution $u \in S_1^\beta(\lambda)$ to the Cauchy problem

$$\frac{\partial u}{\partial t}(t) = L(t)u(t), \quad u(0) = u_0$$

in the scale $\mathbb{B}_\alpha$.

Assuming we have proved this theorem, we can also state the following version.

Theorem 2.6. *Consider the type 2 scale* $(\mathbb{B}'_\alpha, \|\cdot\|_\alpha)_{\alpha_* \le \alpha \le \alpha^*}$ *and assume that there exist* $\lambda_a > 0$ *and* $M \ge 0$ *such that*

1. $\left[0, \frac{\alpha^* - \alpha_*}{\lambda_a}\right) \ni t \longmapsto L(t) \in L(\mathbb{B}'_\alpha, \mathbb{B}'_{\alpha'})$ *is strongly continuous for any* $\alpha' < \alpha$
2. $\|L(t)\|_{\alpha\alpha'} \le \frac{M}{\alpha - \alpha'}$ *for any* $\alpha' < \alpha$ *and* $t \in \left[0, \frac{\alpha^* - \alpha_*}{\lambda_a}\right)$.

Then there exists $\lambda_0 > \lambda_a > 0$ *and* $T_\lambda : [\alpha_*, \alpha^*) \longrightarrow \mathbb{R}_+$ *continuous and monotonically decreasing given by*

$$T_\lambda(\alpha) = \frac{\alpha^* - \alpha}{\lambda}, \quad \textit{with } \lambda > \lambda_0$$

such that for each initial condition $u_0 \in \mathbb{B}'_{\alpha^*}$ *there exists a unique solution* $u \in S_2^\beta(\lambda)$ *to the Cauchy problem*

$$\frac{\partial u}{\partial t}(t) = L(t)u(t), \quad u(0) = u_0$$

in the scale $\mathbb{B}_\alpha$.

Proof. Define the spaces $\widetilde{\mathbb{B}}_\alpha = \mathbb{B}'_{\alpha_* + \alpha^* - \alpha}$ with the norm $\|\cdot\|'_\alpha = \|\cdot\|_{\alpha_* + \alpha^* - \alpha}$ for $\alpha_* \le \alpha \le \alpha^*$ and apply the first result. □

Now we will prove the first stated version, namely Theorem 2.11.

Proof. By Lemma 2.8 it is enough to solve the equation

$$u(t) = u_0 + \int_0^t L(\tau)u(\tau)\mathrm{d}\tau =: u_0 + (Tu)(t)$$

in the space S^γ. So let $\lambda \ge \lambda_a$ and $\beta > 0$. To abuse notation, we will write in this proof $\|\cdot\|^{(\beta)}$ for the norm $\|\cdot\|_1^{(\beta)}$.

1. For $u \in S^\beta$ we have: $\|L(\cdot)u(\cdot)\|^{(\beta+1)} \le M 2^{\beta+1}\|u\|^{(\beta)}$.

   Indeed, let $0 \le t < \frac{\alpha - \alpha_*}{\lambda}$ and take $\alpha' < \alpha$ so close to α that we have $0 \le t < \frac{\alpha' - \alpha_*}{\lambda} < \frac{\alpha - \alpha_*}{\lambda}$. Thus $u(t) \in \mathbb{B}_{\alpha'}$ implies $L(t)u(t) \in \mathbb{B}_\alpha$ and since α and t were arbitrary we obtain

   $$L(t)u(t) \in \mathbb{B}_\alpha, \quad 0 \le t < \frac{\alpha - \alpha_*}{\lambda}.$$

   Now let $\alpha \in (\alpha_*, \alpha^*]$ and $t \in [0, T_\lambda(\alpha))$ be arbitrary and define

   $$\rho = \alpha - \alpha_* - \lambda t, \quad \alpha' = \alpha - \frac{\rho}{2}.$$

   For such ρ and α the following holds

(a) $\alpha_* \le \alpha' < \alpha \le \alpha^*$

(b) $\alpha - \alpha' = \frac{\rho}{2} = \alpha - \alpha_* - \lambda t - \frac{\rho}{2} = \alpha' - \alpha_* - \lambda t$

(c) $\alpha - \alpha_* - \lambda t = \rho = 2(\alpha' - \alpha_* - \lambda t)$

and hence we obtain

$$\begin{aligned}(\alpha - \alpha_* - \lambda t)^{\beta+1} \|L(t)u(t)\|_\alpha &\le M \frac{(\alpha - \alpha_* - \lambda t)^{\beta+1}}{\alpha - \alpha'} \|u(t)\|_{\alpha'} \\ &= M 2^{\beta+1} (\alpha' - \alpha_* - \lambda t)^\beta \|u(t)\|_{\alpha'} \\ &\le M 2^{\beta+1} \|u\|^{(\beta)}\end{aligned}$$

which implies $\|L(\cdot)u(\cdot)\|^{(\beta+1)} \le M2^{\beta+1}\|u\|^{(\beta)}$.

2. For $u \in S^{\beta+1}$ we have: $\|Tu\|^{(\beta)} \le \frac{M2^{\beta+1}}{\beta\lambda}\|u\|^{(\beta+1)}$.

   Indeed, let $\alpha_* \le \alpha \le \alpha^*$ and $t \in [0, T_\lambda(\alpha))$, then we have

$$\begin{aligned}\left\| \int_0^t u(\tau) \mathrm{d}\tau \right\|_\alpha &\le \int_0^t \|u(\tau)\|_\alpha \mathrm{d}\tau \le \int_0^t (\alpha - \alpha_* - \lambda\tau)^{-\beta-1} \mathrm{d}\tau \|u\|^{(\beta+1)} \\ &\le \frac{\|u\|^{(\beta+1)}}{\beta\lambda} (\alpha - \alpha_* - \lambda t)^{-\beta}\end{aligned}$$

   and so $\left\| \int_0^{\bullet} u(\tau)\mathrm{d}\tau \right\|^{(\beta)} \le \frac{\|u\|^{(\beta+1)}}{\beta\lambda}$. Now the statement follows from

$$\|Tu\|^{(\beta)} = \left\| \int_0^{\bullet} L(\tau)u(\tau)\mathrm{d}\tau \right\|^{(\beta)} \le \frac{1}{\beta\lambda}\|L(\cdot)u(\cdot)\|^{(\beta+1)} \le \frac{M2^{\beta+1}}{\beta\lambda}\|u\|^{(\beta)}.$$

3. We saw that for all $\lambda > \max\{\lambda_a, \frac{M2^{\beta+1}}{\beta}\} =: \lambda_0 \ge \lambda_a$

$$\|Tu\|^{(\beta)} \le \frac{\lambda_0}{\lambda}\|u\|^{(\beta)} < \|u\|^{(\beta)}$$

   holds. Let $u_0 \in \mathbb{B}_{\alpha_*}$ be arbitrary. Using

$$\begin{aligned}\|u_0\|^{(\beta)} &= \sup_{\alpha \in [\alpha_*, \alpha^*],\ t \in [0, T_\lambda(\alpha))} (\alpha - \alpha_* - \lambda t)^\beta \|u_0\|_\alpha \\ &\le \left(\sup_{\alpha \in [\alpha_*, \alpha^*],\ t \in [0, T_\lambda(\alpha))} (\alpha - \alpha_* - \lambda t)^\beta \right) \|u_0\|_{\alpha_*}\end{aligned}$$

   one sees that $u_0 \in S^\beta$ and hence the sequence $(u^{(k)})_{k\in\mathbb{N}}$ given by $u^{(0)} = u_0$ and $u^{(k+1)} = u_0 + Tu^{(k)}$ satisfies $u^{(k)} \in S^\beta$, cf. Definition 5. Due to

$$\|u^{(k+1)} - u^{(k)}\|^{(\beta)} \le \left(\frac{\lambda_0}{\lambda}\right)^k \|u^{(1)} - u^{(0)}\|^{(\beta)}$$

this sequence has a limit $u \in S^\beta$. By definition this limit satisfies

$$u(t) = u_0 + \int_0^t L(\tau)u(\tau)\mathrm{d}\tau, \quad \forall t \in [0, T_\lambda(\alpha))$$

with the time data $T_\lambda(\alpha) = \dfrac{\alpha - \alpha_*}{\lambda}$.

4. For uniqueness let $v \in S^\beta$ solves the Cauchy problem with the zero initial data or equivalently

$$v(t) = \int_0^t L(\tau)v(\tau)\mathrm{d}\tau = (Tv)(t).$$

So v is a fix-point of T and because of $\|T\|_{L(S^\beta)} \leq \dfrac{\lambda_0}{\lambda} < 1$ we have that $v = 0$. □

Remark 2.6.

1. Minimizing the expression $\dfrac{2^\beta}{\beta}$ we obtain for $\beta = \dfrac{1}{\log(2)}$ and λ_a small

$$\lambda_0 = 2eM\log(2) = eM\log(4).$$

So up to the factor log(4) this is the same time data as in the first existence result, cf. (11).

2. Note $u(t) \in \bigcap_{\alpha > \alpha_t} \mathbb{B}_\alpha$, where α_t is given by

$$0 \leq t < \frac{\alpha_t - \alpha_*}{\lambda} \Longleftrightarrow 0 \leq \alpha_* + \lambda t < \alpha_t.$$

Thus we have $u(t) \in \bigcap_{\alpha > \alpha_* + \lambda t} \mathbb{B}_\alpha$.

3. Now we have solved the Cauchy problem for each $\beta > 0$ and $\lambda > \lambda_0$, so there are solutions $u = u_{\beta,\lambda}$. For each $\lambda > \lambda_0$ and $\beta' < \beta$ the inequality

$$(\alpha - \alpha_* - \lambda t)^\beta = (\alpha - \alpha_* - \lambda t)^{\beta'}(\alpha - \alpha_* - \lambda t)^{\beta - \beta'} \leq (\alpha^* - \alpha_*)^{\beta - \beta'}(\alpha - \alpha_* - \lambda t)^{\beta'}$$

implies $\|\cdot\|_1^\beta \leq (\alpha^* - \alpha_*)^{\beta - \beta'}\|\cdot\|_1^{\beta'}$. The same holds for a scale of type 2. Consequently we obtain $S^{\beta'} \subset S^\beta$ for each $\beta' < \beta$. Since λ_0 depends on β we use Remark 2.12.1 and chose $\beta = \dfrac{1}{\log(2)}$ to obtain solutions on the biggest possible time interval. But in the same way

$$(\alpha - \alpha_* - \lambda t)^\beta \leq (\alpha - \alpha_* - \lambda' t)^\beta$$

for $\lambda' < \lambda$ implies that the solutions satisfy

$$u_{\beta,\lambda}(t) = u_{\beta,\lambda'}(t) \quad \text{for } t \in [0, T_\lambda(\alpha)) \subset [0, T_{\lambda'}(\alpha)).$$

So taking $\beta = \dfrac{1}{\log(2)}$ and $T(\alpha) = \dfrac{\alpha - \alpha_*}{\lambda_0}$ we obtain the existence of a unique solution $u : \left[0, \dfrac{\alpha^* - \alpha_*}{\lambda_0}\right) \longrightarrow \mathbb{B}_{\alpha^*}$. By construction each restriction to $[0, T_\lambda(\alpha)]$ corresponds to some element

$$u|_{[0,T_\lambda(\alpha)]} \in C^1\left([0, T_\lambda(\alpha)]; \mathbb{B}_\alpha\right)$$

for $\lambda > \lambda_0$ and $\sup\limits_{\alpha\in[\alpha_*,\alpha^*], t\in[0,T_\lambda(\alpha))} (\alpha - \alpha_* - \lambda t)^\beta \|u\|_\alpha < \infty$ solving the Cauchy problem in $\mathbb{B}_\alpha$.

Now let $\lambda > \lambda_0$, $u_0, v_0 \in \mathbb{B}_{\alpha_*}$ be two initial conditions and u respectively v the corresponding solutions. Then we have

$$\|u - v\|^{(\beta)} \le \|u_0 - v_0\|^{(\beta)} + \|T(u - v)\|^{(\beta)} \le \|u_0 - v_0\|^{(\beta)} + \frac{\lambda_0}{\lambda}\|u - v\|^{(\beta)}$$

and hence

$$\|u - v\|^{(\beta)} \le \frac{\lambda}{\lambda - \lambda_0}\|u_0 - v_0\|^{(\beta)}.$$

Taking into account that

$$\begin{aligned} \|u_0 - v_0\|^{(\beta)} &= \sup_{\alpha\in[\alpha_*,\alpha^*],\ \ t\in[0,T_\lambda(\alpha))} (\alpha - \alpha_* - \lambda t)^\beta \|u_0 - v_0\|_\alpha \\ &\le (\alpha^* - \alpha_*)^\beta \|u_0 - v_0\|_{\alpha_*} \end{aligned}$$

we can rewrite

$$\|u - v\|^{(\beta)} \le \frac{\lambda}{\lambda - \lambda_0}(\alpha^* - \alpha_*)^\beta \|u_0 - v_0\|_{\alpha_*}$$

or using $\alpha \in (\alpha_*, \alpha^*]$ and $t \in [0, T_\lambda(\alpha))$

$$\|u(t) - v(t)\|_\alpha \le \frac{\lambda}{\lambda - \lambda_0}\left(\frac{\alpha^* - \alpha_*}{\alpha - \alpha_* - \lambda t}\right)^\beta \|u_0 - v_0\|_{\alpha_*}.$$

This shows, that the solutions depend continuously on the initial data u_0, v_0. It is possible to show a stronger result, but this part shall be omitted. Now we would like to handle the situation, where $L(t)$ does not satisfy an estimate $\|L(t)\|_{\alpha'\alpha} \le \dfrac{M}{\alpha - \alpha'}$. In applications effects like pair interaction lead to operators, which do not satisfy above estimate. Nevertheless the following approach may be still applicable. Assume $L(t)$ can be decomposed into $L(t) = A(t) + B(t)$, where $B(t)$ still satisfies this assumption. If we can solve the Cauchy problem for $A(t)$ with an evolution family, one can try to solve the Cauchy problem for $L(t)$ using similar arguments like the ones before. This approach is realized in the next theorem.

Theorem 2.7. *Let $(\mathbb{B}_\alpha, \|\cdot\|_\alpha)_{\alpha_*\le\alpha\le\alpha^*}$ be a scale of type 1 and $\lambda_a > 0$ such that $A(t)$ satisfies the following assumptions*

1. *For all α', α with $\alpha_* \le \alpha' < \alpha \le \alpha^*$ the mapping $[0, \dfrac{\alpha^* - \alpha_*}{\lambda_a}] \ni t \longmapsto A(t) \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$ is strongly continuous,*

2. *For all* $\alpha \in [\alpha_*, \alpha^*]$ *there exists an evolution family* $U : \Delta \longrightarrow L(\mathbb{B}_\alpha)$ *such that* $\|U(t,s)\|_{L(\mathbb{B}_\alpha)} \leq 1$ *for* $(t,s) \in \Delta = \{(t,s) \in [0, \frac{\alpha^* - \alpha_*}{\lambda_a}]^2 : \ s \leq t\}$,

3. *For all* $\alpha' < \alpha$ *and* $u \in \mathbb{B}_{\alpha'}$

$$\Delta \ni (t,s) \longmapsto U(t,s)u \in \mathbb{B}_\alpha$$

is differentiable with derivatives

$$\frac{\partial U}{\partial t}(t,s)u = A(t)U(t,s)u, \quad 0 \leq s \leq t \leq \frac{\alpha^* - \alpha_*}{\lambda_a}$$

and

$$\frac{\partial U}{\partial s}(t,s)u = -U(t,s)A(s)u, \quad 0 \leq s \leq t \leq \frac{\alpha^* - \alpha_*}{\lambda_a}.$$

In the case of $s = t$ *the derivative* $\frac{\partial U}{\partial t}(t,s)u$ *is to be understood as a right-sided derivative.*

Further assume that $[0, \frac{\alpha^* - \alpha_*}{\lambda_a}] \ni t \longmapsto B(t) \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$ *is strongly continuous for all* $\alpha' < \alpha$ *satisfying* $\|B(t)\|_{\alpha'\alpha} \leq \frac{M}{\alpha - \alpha'}$. *Then there exists* $\lambda_0 > \lambda_a > 0$ *and* $T_\lambda : (\alpha_*, \alpha^*] \longrightarrow \mathbb{R}_+$ *continuous and monotonically increasing given by*

$$T_\lambda(\alpha) = \frac{\alpha - \alpha_*}{\lambda} \quad \text{with } \lambda > \lambda_0$$

such that for each initial condition $u_0 \in \mathbb{B}_{\alpha_*}$ *there exists a unique solution* u *in* $S^\beta(\lambda)$ *to the Cauchy problem*

$$\frac{\partial u}{\partial t}(t) = (A(t) + B(t))u(t), \quad u(0) = u_0 \tag{12}$$

in the scale $\mathbb{B}_\alpha$.

Analogous to the previous result the first step is to reformulate the Cauchy problem in the integral form. This will be the content of the next lemma, for which (4) is needed.

Lemma 2.8. *Let* $A(t), B(t), U(t,s)$ *be like in Theorem 2.13. Then the following statements are equivalent:*

1. u *is a solution to* (12) *in the scale* $(\mathbb{B}_\alpha, \|\cdot\|_\alpha)_{\alpha_* \leq \alpha \leq \alpha^*}$ *with a time data* $T(\alpha) > 0$

2. $u \in \bigcap_{\alpha_* \leq \alpha \leq \alpha^*} C([0, T(\alpha)); \mathbb{B}_\alpha)$ *solves the equation*

$$u(t) = U(t,0)u_0 + \int_0^t U(t,\tau)B(\tau)u(\tau)\mathrm{d}\tau \tag{13}$$

in $\mathbb{B}_\alpha$ *for* $t \in [0, T(\alpha))$ *and* $\alpha \in (\alpha_*, \alpha^*]$.

Using Lemma 2.14 we are now in a position to prove Theorem 2.13 in a scale of Banach spaces of type 1.

Proof. For $\lambda > \lambda_a$ we will solve the equation (13)

$$u(t) = U(t,0)u_0 + \int_0^t U(t,\tau)B(\tau)u(\tau)\mathrm{d}\tau = U(t,0)u_0 + (Tu)(t).$$

Write $\|\cdot\|^{(\beta)}$ and T_λ as before. Using $\|U(t,s)\|_{L(\mathbb{B}_\alpha)} \le 1$ we obtain for $u \in S^{\beta+1}$, $\alpha \in [\alpha_*, \alpha^*]$ and $t \in [0, T_\lambda(\alpha))$ by the proof of Theorem 2.10

$$\left\| \int_0^t U(t,\tau)u(\tau)\mathrm{d}\tau \right\|_\alpha \le \int_0^t \|u(\tau)\|_\alpha \mathrm{d}\tau \le \frac{(\alpha - \alpha_* - \lambda t)^{-\beta}}{\beta\lambda} \|u\|^{(\beta+1)}.$$

As a result we have shown $\left\| \int_0^{\bullet} U(\cdot,\tau)u(\tau)\mathrm{d}\tau \right\|^{(\beta)} \le \frac{1}{\beta\lambda}\|u\|^{(\beta+1)}$ and therefore

$$\|Tu\|^{(\beta)} = \left\| \int_0^{\bullet} U(\cdot,\tau)B(\tau)u(\tau)\mathrm{d}\tau \right\|^{(\beta)} \le \frac{1}{\beta\lambda}\|B(\cdot)u(\cdot)\|^{(\beta+1)} \le \frac{M2^{\beta+1}}{\beta\lambda}\|u\|^{(\beta)}.$$

For the same λ as in the previous proof and $\beta > 0$ we have $\|Tu\|^{(\beta)} \le \frac{\lambda_0}{\lambda}\|u\|^{(\beta)}$. Now define a sequence by $u^{(0)}(t) = U(t,0)u_0$ and $u^{(k+1)}(t) = U(t,0)u_0 + (Tu^{(k)})(t)$. From

$$\begin{aligned} \|u^{(0)}\|^{(\beta)} &= \sup_{\alpha \in [\alpha_*, \alpha^*],\ t \in [0, T_\lambda(\alpha))} (\alpha - \alpha_* - \lambda t)^\beta \|U(t,0)u_0\|_\alpha \\ &\le \sup_{\alpha \in [\alpha_*, \alpha^*],\ t \in [0, T_\lambda(\alpha))} (\alpha - \alpha_* - \lambda t)^\beta \|u_0\|_\alpha \\ &\le (\alpha^* - \alpha_*)^\beta \|u_0\|_{\alpha_*} < \infty \end{aligned}$$

one easily sees $(u^{(k)})_{k\in\mathbb{N}} \subset S^\beta$. Therefore, $(u^{(k)})_{k\in\mathbb{N}}$ is a fundamental sequence for $\lambda > \lambda_0 = \max\left\{ \frac{M2^{\beta+1}}{\beta}, \lambda_a \right\}$ and hence there exists a limit $\lim_{k\to\infty} u^{(k)} = u \in S^\beta$, which solves the equation

$$u = U(\cdot,0)u_0 + Tu$$

by definition, which shows (12). This shows the existence of a solution. For uniqueness let $v \in S^\beta$ be another solution, then $w = u - v$ solves $w = Tw$ and therefore $w = 0$, since T is a contraction. □

Remark 2.7.

1. Under some modifications it is clear that a similar result can be stated for a scale of Banach spaces of type 2.

2. A similar result for the time independent case was stated in [5]. The authors have shown the existence of solutions directly by using (9). To establish uniqueness they have used analyticity at 0 and the formula $\frac{\mathrm{d}^n u}{\mathrm{d}t^n}(0) = L^n u(0)$. Unfortunately such a formula does not hold for the time dependent case and due to the properties of the operators it is not possible to apply the Gronwall Lemma, which is the reason for this approach.

3. The same considerations as in Remark 2.12.3 hold also here. If we weaken the assumption $\|U(t,s)\|_{L(\mathbb{B}_\alpha)} \leq 1$ to
$$\sup_{(t,s)\in\Delta} \|U(t,s)\|_{L(\mathbb{B}_\alpha)} \leq C < \infty$$
with $\Delta = \left\{(t,s) \in \left[0, \frac{\alpha^* - \alpha_*}{\lambda_a}\right]^2 : \quad s \leq t\right\}$ and for some constant $C > 0$ independent of α, then a similar result holds. More precisely one has
$$\lambda_0 = \min\left\{\lambda_a, \frac{MC2^{\beta+1}}{\beta}\right\}$$
and consequently Remark 2.12.3 still holds. Note that the supremum always exists, but in general might be not bounded with respect to α.

Similar to the first version one can show $\|u - v\|^{(\beta)} \leq C\|u_0 - v_0\|^{(\beta)}$ for some constant $C > 0$. Likewise it is possible to show a stronger result concerning continuous dependence of the solutions on parameters. To summarize we have shown the existence of solutions in scales of Banach spaces under the condition that either $\|L(t)\|_{\alpha'\alpha} \leq \frac{M}{\alpha - \alpha'}$ holds or $L(t) = A(t) + B(t)$ satisfies $\|B(t)\|_{\alpha'\alpha} \leq \frac{M}{\alpha - \alpha'}$ and $A(t)$ generates an evolution family. For many applications in interacting particle systems or partial differential equations such results can be used, cf. [26]. For further developments it is useful to construct evolution families under more general assumptions or even using the properties of scales of Banach spaces.

3 Evolutions of interacting particle systems

For motivation we start with an explicit model of interacting particle systems. Consider a habitat with living individuals, e.g. humans, located in $\mathbb{R}^d$. For such individuals we would like to model natural birth and death as elementary events. Now assume that the habitat is contaminated due to some mechanism, i.e. an atomic catastrophy. Hence the individuals will become sick and die according to specific rates. For applications one would like to know how this system will behave in the time evolution. Important questions are concerned with the possibility of whether the individuals would survive this catastrophy or not. To model such a system mathematically we will not distinguish between individuals, meaning that the only important information is the position of the individual. Therefore a population can be described as a subset $\gamma \subset \mathbb{R}^d$. Since we will describe this system in a probabilistic way via Markov evolutions it is enough to give the formal Markov pre-generator. In this case such generator has the form

$$\begin{aligned}(L(t)F)(\gamma) &= \sum_{x\in\gamma}(m(t)+P_t(x))(F(\gamma\backslash x)-F(\gamma)) \\ &\quad + \int_{\mathbb{R}^d}\left(\sum_{y\in\gamma}a_t(x-y)\right)(F(\gamma\cup x)-F(\gamma))\mathrm{d}x.\end{aligned}$$

Here and in the further chapters we will just write $\gamma\cup x$ and $\gamma\backslash x$ instead of $\gamma\cup\{x\}$ and $\gamma\backslash\{x\}$ for brevity. The interpretation is that each individual $x\in\gamma$ might die due to a space independent mortality rate $m(t)\geq 0$ and additionally to a space dependent rate $P_t(x)\geq 0$, which describes the habitat. Further each individual located at some point $y\in\mathbb{R}^d$ may produce another individual located at $x\in\mathbb{R}^d$ depending on the time dependent birth rate a_t. In this model the new individual at point $x\in\mathbb{R}^d$ immediately may produce new individuals by themselves. Note that the birth is modeled translation invariant. More generally one can consider a general birth-and-death process given by

$$(L(t)F)(\gamma)=\sum_{x\in\gamma}d_t(x,\gamma\backslash x)(F(\gamma\backslash x)-F(\gamma))+\int_{\mathbb{R}^d}b_t(x,\gamma)(F(\gamma\cup x)-F(\gamma))\mathrm{d}x.$$

A general approach to dynamics on configuration spaces was given in [10] and references therein and [14] contains all necessary technical details for this approach via correlation functions. In the next section we will give a brief outline on general birth and death dynamics on configuration spaces. Afterwards we will use the Sourgailis and continuous Contact model to answer the given questions above. Further sections are devoted to Glauber-type dynamics, Bolkmann-Dieckmann-Law-Pacala model and general birth and death models.

3.1 General Dynamics on Configuration Spaces

The configuration space Γ over $\mathbb{R}^d$ for $d\in\mathbb{N}$ is defined as the set of all locally finite subsets of $\mathbb{R}^d$, i.e.

$$\Gamma=\{\gamma\subset\mathbb{R}^d:\ |\gamma\cap\Lambda|<\infty,\quad\forall\Lambda\subset\mathbb{R}^d\text{ compact }\}.$$

We will use the notation $\gamma\cap\Lambda=\gamma_\Lambda$ and $|\gamma_\Lambda|$ denotes the cardinality of the set γ_Λ. Denote by $\Gamma_0^{(n)}=\{\gamma\subset\mathbb{R}^d:\ |\gamma|=n\}$ the space of n-point configurations and by

$$\Gamma_0=\bigsqcup_{n=0}^{\infty}\Gamma_0^{(n)}$$

the space of all finite configurations. Via the identification

$$\Gamma\ni\gamma\longmapsto\mathrm{d}\gamma=\sum_{x\in\gamma}\delta_x\in\mathscr{M}(\mathbb{R}^d)$$

one can endow Γ with a topological structure. Here $\mathscr{M}(\mathbb{R}^d)$ stands for the space of all Radon measures on $\mathbb{R}^d$. The topology on Γ is the weakest where all mappings

$$\Gamma \ni \gamma \longmapsto \langle \varphi, \gamma \rangle = \int\limits_{\mathbb{R}^d} \varphi(x) \mathrm{d}\gamma(x) = \sum_{x \in \gamma} \varphi(x) \in \mathbb{R}$$

are continuous for $\varphi \in C_c(\mathbb{R}^d)$. In [25] the author showed that Γ is a polish space and gave a characterization of compact subsets of Γ. It is also possible to define a differentiable structure on Γ and on Γ_0, for further aspects see [1]. Using this differential structure it is possible to prove an integration by parts formula and characterize Gibbs measures, which are the equilibrium states for the Glauber dynamics. The Poisson measure π_z for $z > 0$ is defined as in [1], i.e. as the unique probability measure on $(\Gamma, \mathscr{B}(\Gamma))$ with the Laplace transform

$$\int\limits_{\Gamma} \exp(\langle \varphi, \gamma \rangle) \mathrm{d}\pi_z(\gamma) = \exp\left(z \int\limits_{\mathbb{R}^d} (e^{\varphi(x)} - 1) \mathrm{d}x \right)$$

for $\varphi \in C_c(\mathbb{R}^d)$. It is also possible to define this measure as a projective limit using the Kolmogorov theorem for projective limits. The Lebesgue-Poisson measure λ_z is defined by

$$\lambda_z = \sum_{n=0}^{\infty} \frac{z^n}{n!} m^{(n)} = \delta_{\{\emptyset\}} + \sum_{n=1}^{\infty} \frac{z^n}{n!} m^{(n)},$$

where $m^{(n)}$ is the image measure of the Lebesgue measure $m^{\otimes n}$ on $(\mathbb{R}^d)^n$ under the symmetrization-mapping

$$sym^n : \widetilde{(\mathbb{R}^d)}^n \longrightarrow \Gamma_0^{(n)}, \quad (x_1, \ldots, x_n) \longmapsto \{x_1, \ldots, x_n\}$$

with $\widetilde{(\mathbb{R}^d)^n} = \{(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n : \quad x_j \neq x_k, \text{ with } j \neq k\}$. For $z = 1$ we will write $\lambda = \lambda_1$. We call functions $F : \Gamma \longrightarrow \mathbb{R}$ observables and functions $G : \Gamma_0 \longrightarrow \mathbb{R}$ quasi-observables. The $K-$Transform, given by

$$(KG)(\gamma) = \sum_{\eta \subset \gamma, \ |\eta| < \infty} G(\eta)$$

defines a new function $KG : \Gamma \longrightarrow \mathbb{R}$ for appropriate $G : \Gamma_0 \longrightarrow \mathbb{R}$. The inverse mapping is given by

$$(K^{-1}F)(\eta) = \sum_{\xi \subset \eta} (-1)^{|\eta \backslash \xi|} F(\xi).$$

$\mathscr{B}_c(\mathbb{R}^d)$ denotes the set of all Borel sets with compact closure. In [14] it was shown that the $K-$transform is bijective between the space of all polynomially bounded cylindrical functions F, i.e. $F(\gamma) = F(\gamma_\Lambda)$ for some $\Lambda \in \mathscr{B}_c(\mathbb{R}^d)$, and $B_{bs}(\Gamma_0)$. Where $G \in B_{bs}(\Gamma_0)$ is a bounded function with bounded support, so there exists $N \in \mathbb{N}$ and $\Lambda \in \mathscr{B}_c(\mathbb{R}^d)$ such that

$$G(\eta) = 0, \quad \forall \eta \notin \bigsqcup_{n=0}^{N} \Gamma_{0,\Lambda}^{(n)}$$

with

$$\Gamma_{0,\Lambda}^{(n)} = \{\eta \in \Gamma_0 : \quad \eta \subset \Lambda, \quad |\eta| = n\}.$$

Also further properties of the $K-$transform were studied in [14]. For a measurable function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ denote by

$$e_\lambda(f,\eta) = \prod_{x\in\eta} f(x), \quad e_\lambda(f,\emptyset) = 1, \quad \eta \in \Gamma_0\backslash\{\emptyset\}$$

the Lebesgue exponential $e_\lambda(f)$. The general scheme and all necessary calculations for dynamics on configuration spaces can be found in [10] and references therein. Given a Markov pre-generator L the dynamics are described by the Kolmogorov equation

$$\frac{\partial F_t}{\partial t} = LF_t.$$

The pairing $\langle F,\mu\rangle = \int\limits_\Gamma F(\gamma)\mathrm{d}\mu(\gamma)$ for $F : \Gamma \longrightarrow \mathbb{R}$ and a probability measure $\mu \in \mathscr{M}^1(\Gamma)$ allows to consider the dual equation for measures

$$\frac{\partial \mu_t}{\partial t} = L^*\mu_t.$$

We construe each probability measure μ_t as a state of the system at time t. So the time evolution is given by $(\mu_t)_{t\geq 0}$. Unfortunately this equation is difficult to handle. Using the K-Transform it is possible to look at the evolutional equation for quasi-observables

$$\frac{\partial G_t}{\partial t} = \hat{L}G_t \tag{14}$$

with $\hat{L} = K^{-1}LK$ on some set of functions $G : \Gamma_0 \longrightarrow \mathbb{R}$, i.e. $B_{bs}(\Gamma_0)$. Given a probability measure μ on Γ the $K-$transform allows to define the correlation measure ρ_μ on Γ_0 via the identity

$$\int\limits_\Gamma (KG)(\gamma)\mathrm{d}\mu(\gamma) = \int\limits_{\Gamma_0} G(\eta)\mathrm{d}\rho_\mu(\eta), \quad G \in B_{bs}(\Gamma_0).$$

Under some general conditions there exist a one to one correspondence between measures on Γ and correlation measures, cf. [14]. If ρ_μ is absolutely continuous with respect to the Lebesgue-Poisson measure $\mathrm{d}\lambda$ then one defines the correlation function as the Radon-Nikodym derivative $k_\mu = \frac{\mathrm{d}\rho_\mu}{\mathrm{d}\lambda}$. Assuming that the evolution μ_t has this property $\rho_{\mu_t} = k_{\mu_t}\mathrm{d}\lambda$ then rewriting equation (14) with the use of

$$\int\limits_{\Gamma_0} (\hat{L}G)(\eta)k(\eta)\mathrm{d}\lambda(\eta) = \int\limits_{\Gamma_0} G(\eta)(L^\Delta k)(\eta)\mathrm{d}\lambda(\eta)$$

we arrive at a strong equation for correlation functions $k_t = k_{\mu_t}$

$$\frac{\partial k_t}{\partial t} = L^\Delta k_t. \tag{15}$$

One great simplification is that in the last two equations the functions depend only on finite configurations. Note that (15) is formulated and will be solved in the strong sense. Since it was originally obtained as a dual equation it is possible to consider the weak form and dual evolutions k_t^D, obtained by the strong solution of the equation for quasi-observables (14). This analysis was done, e.g. in [5], but is not the main goal of this work. The first model will give a brief outline on how to realize this approach. But even having the solution to (15) it is not clear whether this k_t is a correlation function, i.e. corresponds to an evolution of states. Some further analysis is required. For calculations the following two formulas will be essential.

Lemma 3.1. *For $H : \Gamma_0 \times \Gamma_0 \times \Gamma_0 \longrightarrow \mathbb{R}$ and $G : \Gamma_0 \times \mathbb{R}^d \longrightarrow \mathbb{R}$ such that the right-hand sides exist for $|G|$ and $|H|$, the following formulas hold:*

$$\int_{\Gamma_0} \sum_{\xi \subset \eta} H(\xi, \eta \backslash \xi, \eta) \mathrm{d}\lambda(\eta) = \int_{\Gamma_0} \int_{\Gamma_0} H(\xi, \eta, \eta \cup \xi) \mathrm{d}\lambda(\xi) \mathrm{d}\lambda(\eta)$$

and

$$\int_{\Gamma_0} \sum_{x \in \eta} G(\eta, x) \mathrm{d}\lambda(\eta) = \int_{\Gamma_0} \int_{\mathbb{R}^d} G(\eta \cup x, x) \mathrm{d}x \mathrm{d}\lambda(\eta).$$

There is another technique which can be used to analyze the time evolution of such continuous interacting particle systems. This approach is based on generating functionals. All details and proofs for this approach can be found in [15] and [11]. For a given state μ on Γ one can define the so-called Bogoliubov generating functional by

$$B_\mu(\Theta) = \int_{\Gamma} \prod_{x \in \gamma} (1 + \Theta(x)) \mathrm{d}\mu(\gamma),$$

provided that the right-hand side exists. Of course the domain of those Θ for which $B_\mu(\Theta)$ is well-defined depends on μ itself. The Bogoliubov generating functional allows to study properties of μ or even the time evolution via functional analytic methods. Assuming μ has finite local exponential moments, i.e.

$$\int_{\Gamma} e^{\alpha |\gamma_\Lambda|} \mathrm{d}\mu(\gamma) < \infty, \quad \forall \alpha > 0, \quad \forall \Lambda \in \mathscr{B}_c(\mathbb{R}^d)$$

then the generating functional exists for all bounded functions Θ with compact support. According to general results on configuration spaces there is a connection to the correlation measure ρ_μ given by

$$B_\mu(\Theta) = \int_{\Gamma} (K e_\lambda(\Theta))(\gamma) \mathrm{d}\mu(\gamma) = \int_{\Gamma_0} e_\lambda(\Theta, \eta) \mathrm{d}\rho_\mu(\eta).$$

If the correlation measure is absolutely continuous with respect to the Lebesgue-

Poisson measure we can write

$$\begin{aligned} B_\mu(\Theta) &= \int_{\Gamma_0} e_\lambda(\Theta,\eta) k_\mu(\eta) \mathrm{d}\lambda(\eta) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} \Theta(x_1)\cdots\Theta(x_n) k^{(n)}(x_1,\ldots,x_n) \mathrm{d}x_1 \ldots \mathrm{d}x_n \end{aligned}$$

with symmetric functions $k^{(n)} : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}_+$ given by

$$k^{(n)}(x_1,\ldots,x_n) = \begin{cases} k_\mu(\{x_1,\ldots,x_n\}) & ,|\{x_1,\ldots,x_n\}| = n \\ 0 & ,|\{x_1,\ldots,x_n\}| < n \end{cases}.$$

For $\mu = \pi_z$ one has $k_\mu(\eta) = z^{|\eta|}$ and hence

$$\begin{aligned} B_\mu(\Theta) &= \int_{\Gamma_0} e_\lambda(z\Theta,\eta) \mathrm{d}\lambda(\eta) = \exp\Big(z \int_{\mathbb{R}^d} \Theta(x) \mathrm{d}x \Big) \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{(\mathbb{R}^d)^n} \Theta(x_1)\ldots\Theta(x_n) \mathrm{d}x_1 \ldots \mathrm{d}x_n \end{aligned}$$

for $z \geq 0$. If a functional B admits an such a series expansion it is called entire. In this approach we will be dealing entire generating functionals. As a reminder we give the exact definition of an entire functional.

Definition 3.1. A functional $B : L^1(\mathbb{R}^d, \mathbb{C}) \longrightarrow \mathbb{C}$ is called entire if B is locally bounded and for all $\Theta_0, \Theta \in L^1$ the mapping

$$\mathbb{C} \ni z \longmapsto B(\Theta_0 + z\Theta)$$

is entire. Consequently for each $\Theta_0 \in L^1$ it admits a representation

$$B(\Theta_0 + z\Theta) = \sum_{n=0}^{\infty} \frac{z^n}{n!} d^n B(\Theta_0; \Theta, \ldots, \Theta)$$

for $z \in \mathbb{C}$ and $\Theta \in L^1$, where $d^n B(\Theta_0, \cdot)$ is a symmetric bounded n-linear form.

In L^1 spaces it is possible to represent the differentials $d^n B$ by symmetric kernels $\delta^n B \in L^\infty$. Note that a similar result does not hold for L^p spaces with $p > 1$. The following result was shown in [15].

Theorem 3.2. *Let B be an entire functional on L^1. Then each differential $d^n B(\Theta_0; \cdot)$ can be represented by a symmetric kernel $\delta^n B(\Theta_0, \cdot) \in L^\infty((\mathbb{R}^d)^n)$ via*

$$d^n B(\Theta_0, \Theta_1, \ldots, \Theta_n) = \int_{(\mathbb{R}^d)^n} \delta^n B(\Theta_0, x_1, \ldots, x_n) \Theta_1(x_1) \cdots \Theta_n(x_n) \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

for $\Theta_1, \ldots, \Theta_n \in L^1$. Moreover. the operatornorm of $d^n B(\Theta_0, \cdot)$ coincides with the norm of $\delta^n B(\Theta_0, \cdot)$ and

$$\|\delta^n B(\Theta_0, \cdot)\|_{L^\infty((\mathbb{R}^d)^n)} \leq n! \left(\frac{e}{r}\right)^n \sup_{\|\Theta'\| \leq r} |B(\Theta_0 + \Theta')|$$

holds. We call an entire functional of bounded type if the right-hand side is finite for each $r > 0$ *and* $\Theta_0 \in L^1$.

Applying this to configuration spaces and Bogoliubov generating functionals in [15] the authors showed that the correlation measure is absolutely continuous with respect to the Lebesgue-Poisson measure λ.

Theorem 3.3. *Let* μ *be a probability measure on* Γ *and* B_μ *an entire Bogoliubov generating functional (short GF) on* L^1. *Then the correlation functions* k_μ *exists and are given for* λ*-a.a.* $\eta \in \Gamma_0$ *by*

$$k_\mu(\eta) = \delta^{|\eta|} B_\mu(0; \eta).$$

For an entire GF thus the correlation functions can be interpreted as the Taylor coefficients. Assuming

$$|B_\mu(\Theta)| \leq C \exp\left(\frac{e}{r} \|\Theta\|_{L^1}\right) \tag{16}$$

for $C \geq 0$ and $r > 0$ it follows

$$k_\mu(\eta) \leq C \left(\frac{e}{r}\right)^{|\eta|}$$

for λ-a.a $\eta \in \Gamma_0$. Therefore condition (16) implies the so-called generalized Ruelle bound, which can be used to show the existence of an evolution of states. As it was shown in [15], one can rewrite the equation for correlation functions to a Cauchy Problem

$$\frac{\partial B_t}{\partial t} = \widetilde{L} B_t, \quad B_t|_{t=0} = B_0,$$

which may be solved in some scale of Banach spaces. (16) suggests to consider a scale of Banach spaces of the form

$$\mathbb{B}'_\alpha = \{B : L^1 \longrightarrow \mathbb{C} : \quad B \text{ is entire and } \|B\|_\alpha < \infty \text{ holds}\}, \tag{17}$$

where the norm is given by $\|B\|_\alpha = \sup\limits_{\Theta \in L^1} |B(\Theta)| e^{-\frac{1}{\alpha}\|\Theta\|_{L^1}}$ for $\alpha > 0$. To show how this general approach can be realized we will analyse the Sourgailis and continuous Contact model as one of the simplest birth and death models in the next section.

3.2 Continuous Sourgailis and Contact Model

The continuous Sourgailis model is the simplest model without interaction. It can be described heuristically by two elementary events birth and death. Both events can be described by spaces homogeneous rates $m = m(t)$ and $\kappa = \kappa(t) \geq 0$. Therefore each particle can die with rate m and at each free site a new particle can be born with rate κ. The Markov pre-generator for such model is given by

$$(L(t)F)(\gamma) = m(t) \sum_{x \in \gamma} (F(\gamma \backslash x) - F(\gamma)) + \kappa(t) \int\limits_{\mathbb{R}^d} (F(\gamma \cup x) - F(\gamma)) \mathrm{d}x.$$

The corresponding expression for $\hat{L}(t)$ on quasi-observables is given by

$$(\hat{L}(t)G)(\eta) = -m(t)|\eta|G(\eta) + \kappa(t)\int_{\mathbb{R}^d} G(\eta \cup x)\mathrm{d}x$$

for $G \in B_{bs}(\Gamma_0)$. For correlation functions we likewise achieve

$$(L^{\Delta}(t)k)(\eta) = -m(t)|\eta|k(\eta) + \kappa(t)\sum_{x\in\eta} k(\eta\backslash x)$$

for appropriate k. The case of time independent coefficients was studied in [3]. The author gave an explicit formula for the solution of (15) and studied the long time behavior. More precisely, he has proved that the correlation functions converge to the correlation functions of the invariant state in some proper Banach space. We will now give a short analysis of the corresponding model with the time dependent coefficients $m = m(t) \geq 0$ and $\kappa = \kappa(t) \geq 0$. For this purpose we will always assume that $\overline{m} = \sup\limits_{t\geq 0} m(t)$ is finite and m, κ are continuous on $\mathbb{R}_+ = [0, \infty)$.

Lemma 3.4. *The unique point wise solution of the equation*

$$\frac{\partial k_t}{\partial t} = L^{\Delta}(t)k_t, \quad k_t|_{t=0} = k_0$$

is given by

$$k_t(\eta) = e^{-|\eta|M(t)}\sum_{\xi\subset\eta} H(t)^{|\xi|}k_0(\eta\backslash\xi), \quad \eta \in \Gamma_0 \tag{18}$$

where $M(t) = \int\limits_0^t m(s)\mathrm{d}s$ *and*

$$H(t) = \int_0^t \kappa(s)e^{M(s)}\mathrm{d}s.$$

Define $h_0 = 1$ and h_n recursively by the formula

$$h_n(t) = n\int_0^t \kappa(s)e^{M(s)}h_{n-1}(s)\mathrm{d}s, \quad n \geq 1.$$

Then, using

$$\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}} f'(t_1)\cdots f'(t_n)\mathrm{d}t_n\cdots\mathrm{d}t_1 = \frac{(f(t)-f(0))^n}{n!}$$

for a continuously differentiable function f, one can show that $h_n(t) = H(t)^n$ holds. Taking into account the definition of the convolution $(k_1 * k_2)(\eta) = \sum\limits_{\xi\subset\eta} k_1(\xi)k_2(\eta\backslash\xi)$, formula (18) takes the form

$$k_t(\eta) = e^{-|\eta|M(t)}\left(H(t)^{|\cdot|} * k_0\right)(\eta) = \left(e_\lambda(H(t)e^{-M(t)}) * e_\lambda(e^{-M(t)})k_0\right)(\eta).$$

Uniqueness follows from general results on ordinary differential equations and to show the validity of formula (18) a simple calculation is required, which shall be omitted.

Let $\mathbb{B}_\alpha = L^\infty(\Gamma_0, e^{-\alpha|\cdot|}\mathrm{d}\lambda)$ with the norm $\|k\|_\alpha = \operatorname*{ess\,sup}_{\eta\in\Gamma_0} |k(\eta)|e^{-\alpha|\eta|}$, which means that each $k \in \mathbb{B}_\alpha$ is sub-poissonian, i.e. $|k(\eta)| \le \|k\|_\alpha e^{\alpha|\eta|}$. Since $e_\lambda\left(H(t)e^{-M(t)}\right)$ is a correlation function corresponding to $\pi_{H(t)e^{-M(t)}}$ and by Lemma 3.9 from [3] also $e_\lambda\left(e^{-M(t)}\right)k_0$ is a correlation function for $k_0 \in \mathbb{B}_\alpha$ for $k_0 \in \mathbb{B}_\alpha$ we obtain that the convolution k_t is a correlation function, so formula (18) defines an evolution of states μ_t. Fix some $k_0 \in \mathbb{B}_\alpha$ and assume for this section $\kappa(t) \le zm(t)$ for $t \ge 0$ and some constant $z \ge 0$. Then we have

$$h_n(t) = H(t)^n = \left(\int_0^n \kappa(s)e^{M(s)}\mathrm{d}s\right)^n \le z^n\left(\int_0^t m(s)e^{M(s)}\mathrm{d}s\right)^n = z^n(e^{M(t)}-1)^n.$$

Hence

$$\begin{aligned} |k_t(\eta)| &\le e^{-|\eta|M(t)}\sum_{\xi\subset\eta}(z(e^{M(t)}-1))^{|\xi|}|k_0(\eta\backslash\xi)| \\ &\le \|k_0\|_\alpha e^{-|\eta|M(t)}\sum_{\xi\subset\eta}(z(e^{M(t)}-1))^{|\xi|}e^{\alpha|\eta\backslash\xi|} \\ &= \|k_0\|_\alpha e^{-|\eta|M(t)}(z(e^{M(t)}-1)+e^\alpha)^{|\eta|} \\ &= \|k_0\|_\alpha(z(1-e^{-M(t)})+e^\alpha e^{-M(t)})^{|\eta|} \\ &\le \max\{z, e^\alpha\}^{|\eta|}\|k_0\|_\alpha. \end{aligned}$$

For $e^\alpha \ge z$ we obtain $|k_t(\eta)| \le \|k_0\|_\alpha e^{\alpha|\eta|}$ and so $k_t \in \mathbb{B}_\alpha$ with $\|k_t\|_\alpha \le \|k_0\|_\alpha$. Therefore we have shown that for large α the evolution stays in one Banach space. In the next step we will show the continuity of $t \longmapsto k_t \in \mathbb{B}_\alpha$.

Lemma 3.5. *Let α' be arbitrary and fixed. Suppose, that*

$$z \le e^{\alpha'}. \tag{19}$$

Then for any $\alpha \in \mathbb{R}$ such that

$$\log(2) + \alpha' < \alpha. \tag{20}$$

the mapping

$$\mathbb{R}_+ \ni t \longmapsto k_t \in \mathbb{B}_{\alpha'} \subset \mathbb{B}_\alpha$$

is continuous on $\mathbb{B}_\alpha$ for $k_0 \in \mathbb{B}_{\alpha'}$.

Proof. Let $t, t_0 \in \mathbb{R}_+$. Denote by $t^* = \max\{t, t_0\}$ and $t_* = \min\{t, t_0\}$. Then, for $\xi \subset \eta$ using

$$H(t)^n = h_n(t) \le z^n(e^{M(t)}-1)^n \le z^n e^{nM(t)}$$

the following holds

$$\begin{aligned} |e^{-|\eta|M(t)} - e^{-|\eta|M(t_0)}|h_{|\xi|}(t) &\le z^{|\xi|}e^{|\xi|M(t)}|\eta|e^{-|\eta|M(t_*)}|M(t)-M(t_0)| \\ &\le z^{|\xi|}|\eta||M(t)-M(t_0)|e^{\overline{m}|\eta||t-t_0|}. \end{aligned}$$

Hence, for $n \in \mathbb{N}$

$$|h_n(t) - h_n(t_0)| = n \int_{t_*}^{t^*} \kappa(s) e^{M(s)} h_{n-1}(s) \mathrm{d}s \leq n \int_{t_*}^{t^*} \kappa(s) e^{M(s)} z^{n-1} e^{(n-1)M(s)} \mathrm{d}s.$$

Using $\kappa(s) \leq zm(s)$ the latter expression can be estimated by

$$\begin{aligned}
& nz^n \int_{t_*}^{t^*} \left(\frac{\mathrm{d}}{\mathrm{d}s} e^{M(s)} \right) e^{(n-1)M(s)} \mathrm{d}s \\
= \;& nz^n \left(e^{nM(t^*)} - e^{nM(t_*)} - \frac{n-1}{n} \int_{t_*}^{t^*} nm(s) e^{nM(s)} \mathrm{d}s \right) \\
= \;& nz^n \left(e^{nM(t^*)} - e^{nM(t_*)} - \frac{n-1}{n} \left(e^{nM(t^*)} - e^{nM(t_*)} \right) \right) \\
= \;& z^n (e^{nM(t^*)} - e^{nM(t_*)}) = z^n |e^{nM(t)} - e^{nM(t_0)}|.
\end{aligned}$$

For $a, b > 0$ we use the inequality

$$|b^n - a^n| \leq n|b-a| \max\{a,b\}^{n-1}$$

to obtain

$$\begin{aligned}
& e^{-|\eta| M(t_0)} |h_{|\xi|}(t) - h_{|\xi|}(t_0)| \\
\leq \;& z^{|\xi|} e^{-|\eta| M(t_0)} |e^{|\xi| M(t)} - e^{|\xi| M(t_0)}| \\
\leq \;& z^{|\xi|} e^{-|\eta| M(t_0)} |\xi| \left| e^{M(t)} - e^{M(t_0)} \right| \max \left\{ e^{M(t)}, e^{M(t_0)} \right\}^{|\xi|-1} \\
\leq \;& z^{|\xi|} |\eta| \left| e^{M(t)} - e^{M(t_0)} \right| e^{|\eta|(M(t^*) - M(t_0))} \\
\leq \;& z^{|\xi|} |\eta| \left| e^{M(t)} - e^{M(t_0)} \right| e^{\overline{m}|\eta||t-t_0|}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
& |k_t(\eta) - k_{t_0}(\eta)| \\
\leq \;& \sum_{\xi \subset \eta} \left| e^{-|\eta| M(t)} h_{|\xi|}(t) - e^{-|\eta| M(t_0)} h_{|\xi|}(t_0) \right| |k_0(\eta \backslash \xi)| \\
\leq \;& \sum_{\xi \subset \eta} h_{|\xi|}(t) |e^{-|\eta| M(t)} - e^{-|\eta| M(t_0)}| |k_0(\eta \backslash \xi)| \\
& + \sum_{\xi \subset \eta} e^{-|\eta| M(t_0)} |h_{|\xi|}(t) - h_{|\xi|}(t_0)| |k_0(\eta \backslash \xi)| \\
\leq \;& \|k_0\|_{\alpha'} |\eta| e^{\overline{m}|\eta||t-t_0|} |M(t) - M(t_0)| \sum_{\xi \subset \eta} z^{|\xi|} e^{\alpha' |\eta \backslash \xi|} \\
& + \|k_0\|_{\alpha'} |\eta| \left| e^{M(t)} - e^{M(t_0)} \right| e^{\overline{m}|\eta||t-t_0|} \sum_{\xi \subset \eta} z^{|\xi|} e^{\alpha' |\eta \backslash \xi|} \\
\leq \;& \|k_0\|_{\alpha'} \left(|M(t) - M(t_0)| + \left| e^{M(t)} - e^{M(t_0)} \right| \right) |\eta| e^{\overline{m}|\eta||t-t_0|} \left(z + e^{\alpha'} \right)^{|\eta|}.
\end{aligned}$$

Now let $\varepsilon > 0$ and take $\delta > 0$ such that for $|t - t_0| < \delta$

$$|M(t) - M(t_0)| + \left| e^{M(t)} - e^{M(t_0)} \right| < \varepsilon$$

and

$$\log(2) + \overline{m}\delta + \alpha' < \alpha$$

holds. According to (19) and (20) we have

$$e^{\overline{m}\delta - \alpha}(z + e^{\alpha'}) \leq 2e^{\overline{m}\delta\alpha' - \alpha} < 2e^{\alpha - \log(2) - \alpha} = 1, \tag{21}$$

which implies $\|k_t - k_{t_0}\|_\alpha \leq Const \cdot \varepsilon \|k_0\|_{\alpha'}$ and thus the desired result. □

Remark 3.1.

1. It is enough to have the strict inequality for either (19) or (20), cf. (21).
2. This proof also shows that for $\xi \subset \eta \in \Gamma_0$

$$e^{-|\eta|M(t)}|h_{|\xi|}(t) - h_{|\xi|}(s)| \leq z^{|\xi|}|\eta| \left| e^{M(t)} - e^{M(s)} \right| e^{|\eta|\overline{m}|t-s|}. \tag{22}$$

We saw that continuity of the solution requires additional regularity, which is reflected by the condition $\alpha - \alpha' > \log(2)$. The reason for such difficulties is due to the fact that we deal with L^∞ spaces. In more general models similar conditions were already used, cf. [5, 8]. To show differentiability we will likewise require regularity of initial date, i.e. $\alpha - \alpha' > \log(2) + \overline{m}$. The precise formulation is the content of the next lemma.

Lemma 3.6. *For $k_0 \in \mathbb{B}_{\alpha'}$ and (19) the mapping*

$$\mathbb{R}_+ \ni t \longmapsto k_t \in \mathbb{B}_\alpha$$

is continuously differentiable under the condition

$$\overline{m} + \log(2) + \alpha' < \alpha \tag{23}$$

for $t \geq 0$.

Proof. Using the notation $h_{-1}(t) = 0$ we have for each $\eta \in \Gamma_0$

$$\begin{aligned}
& L^\Delta(t)k_t(\eta) \\
= \ & -|\eta|m(t)k_t(\eta) + \kappa(t)\sum_{x\in\eta} k_t(\eta\backslash x) \\
= \ & -|\eta|m(t)e^{-|\eta|M(t)}\sum_{\xi\subset\eta} h_{|\xi|}(t)k_0(\eta\backslash\xi) \\
& + \kappa(t)\sum_{x\in\eta}\sum_{\xi\subset(\eta\backslash x)} e^{-|\eta|M(t)}e^{M(t)}h_{|\xi|}(t)k_0(\eta\backslash(\xi\cup x)) \\
= \ & -|\eta|m(t)e^{-|\eta|M(t)}\sum_{\xi\subset\eta} h_{|\xi|}(t)k_0(\eta\backslash\xi) \\
& + \kappa(t)e^{M(t)}\sum_{\xi\subset\eta}\sum_{x\in\xi} e^{-|\eta|M(t)}h_{|\xi|-1}(t)k_0(\eta\backslash\xi) \\
= \ & \sum_{\xi\subset\eta} k_0(\eta\backslash\xi)\left(-|\eta|m(t)e^{-|\eta|M(t)}h_{|\xi|}(t) + \kappa(t)e^{M(t)}|\xi|e^{-|\eta|M(t)}h_{|\xi|-1}(t)\right).
\end{aligned}$$

Similar calculations show for $h \in \mathbb{R}$ such that $t+h, t \geq 0$

$$\begin{aligned}
&\frac{k_{t+h}(\eta) - k_t(\eta)}{h} \\
=& \frac{1}{h}\left(e^{-|\eta|M(t+h)} \sum_{\xi \subset \eta} h_{|\xi|}(t+h) k_0(\eta \backslash \xi) - e^{-|\eta|M(t)} \sum_{\xi \subset \eta} h_{|\xi|}(t) k_0(\eta \backslash \xi) \right) \\
=& \sum_{\xi \subset \eta} k_0(\eta \backslash \xi) \left(\frac{e^{-|\eta|M(t+h)} h_{|\xi|}(t+h) - e^{-|\eta|M(t)} h_{|\xi|}(t)}{h} \right) \\
=& \sum_{\xi \subset \eta} k_0(\eta \backslash \xi) \left(h_{|\xi|}(t+h) \frac{e^{-|\eta|M(t+h)} - e^{-|\eta|M(t)}}{h} + e^{-|\eta|M(t)} \frac{h_{|\xi|}(t+h) - h_{|\xi|}(t)}{h} \right).
\end{aligned}$$

The difference $\frac{k_{t+h}(\eta) - k_t(\eta)}{h} - L^{\Delta}(t) k_t(\eta)$ has now the form

$$\begin{aligned}
& \sum_{\xi \subset \eta} k_0(\eta \backslash \xi) \left(h_{|\xi|}(t+h) \frac{e^{-|\eta|M(t+h)} - e^{-|\eta|M(t)}}{h} + |\eta| m(t) e^{-|\eta|M(t)} h_{|\xi|}(t) \right) \\
+ & \sum_{\xi \subset \eta} k_0(\eta \backslash \xi) \left(e^{-|\eta|M(t)} \frac{h_{|\xi|}(t+h) - h_{|\xi|}(t)}{h} - \kappa(t) e^{M(t)} |\xi| e^{-|\eta|M(t)} h_{|\xi|-1}(t) \right)
\end{aligned}$$

and the multiplicant in the first summand can be rewritten to

$$\begin{aligned}
& h_{|\xi|}(t+h) \frac{e^{-|\eta|M(t+h)} - e^{-|\eta|M(t)}}{h} + |\eta| m(t) e^{-|\eta|M(t)} h_{|\xi|}(t) \\
=& h_{|\xi|}(t+h) \left(\frac{e^{-|\eta|M(t+h)} - e^{-|\eta|M(t)}}{h} + |\eta| m(t) e^{-|\eta|M(t)} \right) \\
& + |\eta| m(t) e^{-|\eta|M(t)} (h_{|\xi|}(t) - h_{|\xi|}(t+h)).
\end{aligned}$$

Now let $\varepsilon > 0$ and take $\min\{\varepsilon, 1\} > \delta > 0$ such that

1. $\left| m(t) - \frac{M(t+h) - M(t)}{h} \right| < \varepsilon$
2. $\left| \kappa(s) e^{M(s)} - \kappa(t) e^{M(t)} \right| < \varepsilon$
3. $\left| e^{M(s)} - e^{M(t)} \right| < \varepsilon$
4. $(1+\delta)\overline{m} + \log(2) + \alpha' < \alpha$

holds for $|t-s| < |h| < \delta$. Then we obtain by (22) for such h

$$|\eta| m(t) e^{-|\eta|M(t)} |h_{|\xi|}(t) - h_{|\xi|}(t+h)| \leq |\eta|^2 m(t) z^{|\xi|} \varepsilon e^{|\eta| \overline{m} \delta}.$$

The first part can be estimated by

$$\begin{aligned}
& h_{|\xi|}(t+h)\left|\frac{e^{-|\eta|M(t+h)}-e^{-|\eta|M(t)}}{h}+|\eta|m(t)e^{-|\eta|M(t)}\right| \\
\leq\ & z^{|\xi|}e^{|\xi|M(t+h)}e^{-|\eta|M(t)}\left|m(t)|\eta|+\frac{e^{-|\eta|M(t+h)+|\eta|M(t)}-1}{h}\right| \\
\leq\ & z^{|\xi|}e^{|\eta|\overline{m}\delta}|\eta|\left|m(t)-\frac{M(t+h)-M(t)}{h}\right| \\
& +\frac{z^{|\xi|}e^{|\eta|\overline{m}\delta}}{|h|}\sum_{k=2}^{\infty}\frac{|\eta|^k}{k!}|M(t+h)-M(t)|^k \\
\leq\ & z^{|\xi|}e^{|\eta|\overline{m}\delta}|\eta|\varepsilon+z^{|\xi|}e^{|\eta|\overline{m}\delta}|h|\sum_{k=2}^{\infty}\frac{|\eta|^k|h|^{k-2}\overline{m}^k}{k!} \\
\leq\ & z^{|\xi|}e^{|\eta|\overline{m}\delta}\left(|\eta|+\sum_{k=2}^{\infty}\frac{|\eta|^k\overline{m}^k}{k!}\right)\varepsilon \\
\leq\ & z^{|\xi|}e^{|\eta|\overline{m}\delta}\left(|\eta|+e^{|\eta|\overline{m}}\right)\varepsilon.
\end{aligned}$$

Altogether we have shown

$$h_{|\xi|}(t+h)\left|\frac{e^{-|\eta|M(t+h)}-e^{-|\eta|M(t)}}{h}+m(t)|\eta|e^{-|\eta|M(t)}\right|\leq z^{|\xi|}e^{|\eta|\overline{m}\delta}\left(|\eta|+e^{|\eta|\overline{m}}\right)\varepsilon.$$

Taking now the sum the first part of the difference $\frac{k_{t+h}(\eta)-k_t(\eta)}{h}-L^{\Delta}(t)k_t(\eta)$ can be estimated by

$$\begin{aligned}
& \sum_{\xi\subset\eta}|k_0(\eta\backslash\xi)|\left|h_{|\xi|}(t+h)\frac{e^{-|\eta|M(t+h)}-e^{-|\eta|M(t)}}{h}+|\eta|m(t)e^{-|\eta|M(t)}h_{|\xi|}(t)\right| \\
\leq\ & \|k_0\|_{\alpha'}\varepsilon\left(|\eta|^2m(t)e^{|\eta|\overline{m}\delta}+e^{|\eta|\overline{m}\delta}(|\eta|+e^{\overline{m}|\eta|})\right)\sum_{\xi\subset\eta}z^{|\xi|}e^{\alpha'|\eta\backslash\xi|} \\
=\ & \|k_0\|_{\alpha'}\varepsilon\left(|\eta|^2m(t)+|\eta|+e^{\overline{m}|\eta|}\right)e^{|\eta|\overline{m}\delta}\left(z+e^{\alpha'}\right)^{|\eta|} \\
\leq\ & \|k_0\|_{\alpha'}e^{\alpha|\eta|}\varepsilon\left(|\eta|^2m(t)+|\eta|+1\right)e^{(1+\delta)\overline{m}|\eta|-\alpha|\eta|}\left(z+e^{\alpha'}\right)^{|\eta|}.
\end{aligned}$$

Using $(1+\delta)\overline{m}+\log(2)+\alpha'<\alpha$ we consequently obtain

$$e^{(1+\delta)\overline{m}-\alpha}\left(z+e^{\alpha'}\right)\leq 2e^{(1+\delta)\overline{m}+\alpha'-\alpha}<1$$

and thus it implies for $\beta\geq 0$

$$|\eta|^{\beta}e^{((1+\delta)\overline{m}-\alpha)|\eta|}\left(z+e^{\alpha'}\right)^{|\eta|}\leq Const$$

pointwise, which gives the desired result. In the same way we estimate the second difference with $t_* = \min(t, t+h)$ and $t^* = \max(t, t+h)$

$$\begin{aligned} & \left| e^{-|\eta|M(t)} \frac{h_{|\xi|}(t+h) - h_{|\xi|}(t)}{h} - \kappa(t)e^{M(t)}|\xi|e^{-|\eta|M(t)}h_{|\xi|-1}(t) \right| \\ = \quad & |\xi|e^{-|\eta|M(t)} \left| \frac{1}{h} \int\limits_t^{t+h} \kappa(s)e^{M(s)}h_{|\xi|-1}(s)\mathrm{d}s - \kappa(t)e^{M(t)}h_{|\xi|-1}(t) \right| \\ \leq \quad & e^{-|\eta|M(t)} \frac{|\xi|}{|h|} \int\limits_{t_*}^{t^*} |\kappa(s)e^{M(s)}h_{|\xi|-1}(s) - \kappa(t)e^{M(t)}h_{|\xi|-1}(t)|\mathrm{d}s. \end{aligned}$$

The integrand can be estimated by

$$\begin{aligned} & \left|\kappa(s)e^{M(s)}h_{|\xi|-1}(s) - \kappa(t)e^{M(t)}h_{|\xi|-1}(t)\right| \\ \leq \quad & \kappa(s)e^{M(s)}\left|h_{|\xi|-1}(s) - h_{|\xi|-1}(t)\right| + h_{|\xi|-1}(t)\left|\kappa(s)e^{M(s)} - \kappa(t)e^{M(t)}\right| \\ \leq \quad & \overline{\kappa}e^{\overline{m}(t+\delta)}|\xi|z^{|\xi|-1}\left|e^{M(t)} - e^{M(s)}\right|e^{|\eta|\overline{m}\delta} + \varepsilon z^{|\xi|-1}e^{|\xi|M(t)} \\ \leq \quad & z^{|\xi|}e^{|\xi|M(t)}\varepsilon\left(\frac{\overline{\kappa}e^{\overline{m}(t+1)}}{z}|\eta|e^{|\eta|\overline{m}\delta} + z^{-1}\right) \\ \leq \quad & z^{|\xi|}e^{|\eta|M(t)}\varepsilon\left(\frac{\overline{\kappa}e^{\overline{m}(t+1)}}{z}|\eta|e^{|\eta|\overline{m}\delta} + z^{-1}\right) \end{aligned}$$

with $\overline{\kappa} = \sup\limits_{t\geq 0} \kappa(t)$ and thus

$$\begin{aligned} & e^{-|\eta|M(t)} \frac{|\xi|}{|h|} \int\limits_{t_*}^{t^*} |\kappa(s)e^{M(s)}h_{|\xi|-1}(s) - \kappa(t)e^{M(t)}h_{|\xi|-1}(t)|\mathrm{d}s \\ \leq \quad & e^{-|\eta|M(t)}|\xi|z^{|\xi|}e^{|\eta|M(t)}\varepsilon\left(\frac{\overline{\kappa}e^{\overline{m}(t+1)}}{z}|\eta|e^{|\eta|\overline{m}\delta} + z^{-1}\right) \\ \leq \quad & z^{|\xi|}\varepsilon\left(\frac{\overline{\kappa}e^{\overline{m}(t+1)}}{z}|\eta|^2e^{|\eta|\overline{m}\delta} + z^{-1}|\eta|\right) \end{aligned}$$

Now taking the sum $\sum\limits_{\xi\subset\eta}$ we obtain the assertion analogous to the previous difference. □

We are interested in solutions on some Banach spaces $\mathbb{B}_\alpha$. Right now we have a pointwise solution formula, and under some restrictions, continuity and differentiability properties for some initial values. It still remains to find some Banach space such that the solution formula defines a continuous operator, which is differentiable in some norm on some subspace. From Lemma 3.7 it is natural to consider this in the norm $\|\cdot\|_\alpha$ together with some closed subspace.

Theorem 3.7. *For each $\alpha' < \alpha$ with $z \leq e^{\alpha'}$ and $\overline{m} + \log(2) + \alpha' < \alpha$ there exists a family of contraction operators $(T^\Delta_{\alpha'\alpha}(t))_{t\geq 0}$ on $\mathbb{B} := \overline{\mathbb{B}_{\alpha'}}^{\|\cdot\|_\alpha}$ with the properties*

1. *$T^{\Delta}_{\alpha'\alpha}(t)$ is strongly continuous on $\mathbb{B}$*

2. *$[0, T(\alpha, \alpha')) \ni t \longmapsto T^{\Delta}_{\alpha'\alpha}(t)k \in \mathbb{B}$ is continuously differentiable for $k \in \mathbb{B}_{\alpha'}$ with the derivative*

$$\frac{\mathrm{d}T^{\Delta}_{\alpha'\alpha}(t)k}{\mathrm{d}t} = L^{\Delta}(t)T^{\Delta}_{\alpha'\alpha}(t)k$$

on $\mathbb{B}$.

Hence for $k_0 \in \mathbb{B}_{\alpha'}$ the unique solution of the Cauchy problem

$$\frac{\partial k_t}{\partial t} = L^{\Delta}(t)k_t, \quad k_t|_{t=0} = k_0$$

on $\mathbb{B}$ is given by $k_t = T^{\Delta}_{\alpha'\alpha}(t)k_0$ and moreover $k_t \in \mathbb{B}_{\alpha'} \subset \mathbb{B}_{\alpha}$.

Note that the family $(T^{\Delta}_{\alpha'\alpha}(t))_{t \geq 0}$ is not a semigroup. Under slight changes it is possible to give, at least, a heuristic formula for an evolution family $U^{\Delta}_{\alpha'\alpha}(t, s)$.

Proof. We have shown $\|k_t\|_{\alpha} \leq \|k_0\|_{\alpha}$ for $k_0 \in \mathbb{B}_{\alpha'} \subset \mathbb{B}_{\alpha}$. Hence the densely defined operator $T^{\Delta}_{\alpha'\alpha}(t)k_0 = k_t$ has a unique extension on $\mathbb{B}$, which we denote again by $T^{\Delta}_{\alpha'\alpha}(t)$. Strong continuity follows from the contraction property and Lemma 3.5. Strong differentiability was shown in Lemma 3.7 and therefore for each $k_0 \in \mathbb{B}_{\alpha'}$ there exists a solution given by $k_t = T^{\Delta}_{\alpha'\alpha}(t)k_0 \in \mathbb{B}_{\alpha'} \subset \mathbb{B}$. The uniqueness follows from the uniqueness of the pointwise solution formula. □

Having the existence of an evolution we will discuss some conditions for invariant states and convergence to invariant states. One special case is the time independent dynamics.

Remark 3.2. Assume that m is not integrable, i.e. $M(t) \to \infty$ for $t \to \infty$, e.g. if m is periodic.

1. For some initial condition $k_0 \in \mathbb{B}_{\alpha'}$ one has the solution

$$k_t(\eta) = e^{-|\eta|M(t)} \sum_{\xi \subset \eta} H(t)^{|\xi|} k_0(\eta \backslash \xi).$$

In the special case $k_0(\eta) = e^{\alpha'|\eta|}$ we obtain

$$k_t(\eta) = e^{-|\eta|M(t)} \left(H(t) + e^{\alpha'}\right)^{|\eta|} = \left(H(t)e^{-M(t)} + e^{\alpha'}e^{-M(t)}\right)^{|\eta|}.$$

Using $\underline{\kappa} = \min\limits_{t \geq 0} \kappa(t)$ we obtain

$$\underline{\kappa}t \leq H(t) \leq z\left(e^{M(t)} - 1\right)$$

and hence

$$\left(\underline{\kappa}te^{-M(t)} + e^{\alpha'}e^{-M(t)}\right)^{|\eta|} \leq k_t(\eta) \leq \left(z + (e^{\alpha'} - z)e^{-M(t)}\right)^{|\eta|}.$$

For a general initial condition $k_0 \in \mathbb{B}_{\alpha'}$ we obtain by $H(t)e^{-M(t)} \leq z$ using the decomposition

$$\begin{aligned} k_t(\eta) &= e^{-|\eta|M(t)}k_0(\eta) + H(t)^{|\eta|}e^{-|\eta|M(t)} \\ &+ \sum_{\xi\subset\eta,\xi\neq\emptyset,\xi\neq\eta} H(t)^{|\xi|}e^{-|\xi|M(t)}k_0(\eta\backslash\xi)e^{-|\eta\backslash\xi|M(t)} \end{aligned}$$

that the existence of the limit $\lim\limits_{t\to\infty} k_t(\eta) = k(\eta)$ is equivalent to the existence of the limit $\lim\limits_{t\to\infty} H(t)e^{-M(t)} = a$ and we have

$$k(\eta) = \lim_{t\to\infty} k_t(\eta) = a^{|\eta|}$$

for which $0 \leq a \leq z$ holds. So the condition $\kappa(t) \leq zm(t)$ and $k_0 \in \mathbb{B}_{\alpha'}$ for some $\alpha' \in \mathbb{R}$ imply that the limiting state will be always Poissonian, i.e. π_a.

2. Now take $k_0(\eta) = e^{\alpha'|\eta|}$ for some $\alpha' \in \mathbb{R}$ and assume that the limit $\lim\limits_{t\to\infty} H(t)e^{-M(t)} = a$ exists. Then for each $\alpha \in \mathbb{R}$, which satisfies $a < e^{\alpha}$, we have $k_t \to e_\lambda(a)$ for $t \to \infty$ in $\mathbb{B}_\alpha$. To show this, let $\varepsilon > 0$ with $a \neq \frac{\varepsilon}{2}$, $a + \frac{\varepsilon}{2} < e^{\alpha}$ and take $t_0 > 0$ such that for each $t \geq t_0$

   (a) $a - \frac{\varepsilon}{2} \leq H(t)e^{-M(t)} \leq a + \frac{\varepsilon}{2}$

   (b) $e^{\alpha'}e^{-M(t)} \leq \frac{\varepsilon}{2}$

   (c) $\left|H(t)e^{-M(t)} + e^{\alpha'}e^{-M(t)} - a\right| \leq \varepsilon$

   holds. Then the assertion follows from

$$\begin{aligned} \left|k_t(\eta) - a^{|\eta|}\right| &\leq |\eta|\frac{\left|H(t)e^{-M(t)} + e^{\alpha'}e^{-M(t)} - a\right|}{\max\left\{a, H(t)e^{-M(t)} + e^{\alpha'}e^{-M(t)}\right\}} \\ &\quad \times \max\left\{a, H(t)e^{-M(t)} + e^{\alpha'}e^{-M(t)}\right\}^{|\eta|} \\ &\leq \varepsilon|\eta|\frac{\max\left\{a, a+\varepsilon\right\}^{|\eta|}}{\max\left\{a, a - \frac{\varepsilon}{2}\right\}} \\ &= \varepsilon\frac{1}{a - \frac{\varepsilon}{2}}e^{\alpha|\eta|}|\eta|\left(e^{-\alpha|\eta|}(a+\varepsilon)\right)^{|\eta|} \\ &\leq Const \cdot \varepsilon e^{\alpha|\eta|} \end{aligned}$$

   for $a \neq 0$. The case $a = 0$ can be shown analogously.

3. The condition $t \longmapsto \frac{\kappa(t)}{m(t)}$ is mononically increasing implies

$$H(t) = \int\limits_0^t \frac{\kappa(s)}{m(s)}m(s)e^{M(s)}\mathrm{d}s \leq \frac{\kappa(t)}{m(t)}e^{M(t)}.$$

Hence $\lim_{t\to\infty} \frac{\kappa(t)}{m(t)} = z$ and moreover

$$\frac{\mathrm{d}}{\mathrm{d}t} H(t)e^{-M(t)} = \kappa(t) - m(t)H(t)e^{-M(t)} \geq 0$$

implies $\lim_{t\to\infty} H(t)e^{-M(t)} = z$. Consequently we have shown

$$\lim_{t\to\infty} k_t(\eta) = \lim_{t\to\infty} \left(H(t)e^{-M(t)}\right)^{|\eta|} = z^{|\eta|}$$

pointwise for all $\eta \in \Gamma_0$.

4. Now take $z = \frac{\kappa}{m}$ time independent. Then π_z is an invariant state and for $k_0(\eta) = e^{\alpha'|\eta|}$ we obtain

$$k_t(\eta) = e^{-|\eta|M(t)} z^{|\xi|}(e^{M(t)} - 1)^{|\xi|} e^{\alpha'|\eta\backslash\xi|} = \left(z + (e^{\alpha'} - z)e^{-M(t)}\right)^{|\eta|}.$$

Therefore the time evolution is Poissonian and converges to the invariant state π_z. We obtain with $\max\{z, e^{\alpha'}\} > 0$

$$|k_t(\eta) - k_{inv}(\eta)| \leq e^{-M(t)} \frac{|z - e^{\alpha'}|}{\max\{z, e^{\alpha'}\}} |\eta| \max\{z, e^{\alpha'}\}^{|\eta|}$$

and hence $k_t \to k_{inv}$ in $\mathbb{B}_\alpha$ with $e^\alpha > \max\{z, e^{\alpha'}\}$.

5. Now consider $m(t) = a > 0$ and $\kappa(t) = e^{-bt}$, then we obtain

$$H(t) = \begin{cases} \frac{e^{(a-b)t} - 1}{a - b} & , a \neq b \\ t & , a = b \end{cases}.$$

The expression $H(t)e^{-M(t)} = \frac{e^{-bt} - e^{-at}}{a - b}$ converges for $b > 0$ to 0 and hence $k_t(\eta) \to 0^{|\eta|}$, so all particles will die. In the case $b < 0$ the expression k_t does not have a limit for $t \to \infty$.

More generally now let the death rate be space dependent and introduce some branching, meaning that each particle may produce another new particle. This model was already described in the introduction and the Markov pre-generator has for quasi-observables the form

$$\begin{aligned} (\hat{L}(t)G)(\eta) &= -m(t)|\eta|G(\eta) - \left(\sum_{x\in\eta} P_t(x)\right)G(\eta) \\ &+ \int_{\mathbb{R}^d} \sum_{y\in\eta} a_t(x-y)G((\eta\backslash y)\cup x)\mathrm{d}x + \int_{\mathbb{R}^d} \sum_{y\in\eta} a_t(x-y)G(\eta\cup x)\mathrm{d}x. \end{aligned}$$

for $G \in B_{bs}(\Gamma_0)$. We consider this model under the assumptions

1. $m \geq 0$ is a continuous function on $[0, T]$ for some $T > 0$

2. $P_t : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ with $P_t(x) = P_t(-x)$ satisfies
$$P_\bullet \in C([0,T]; L^\infty(\mathbb{R}^d))$$

3. $0 \leq a_t \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with $a_t(x) = a_t(-x)$ for $t \in [0,T]$

4. $[0,T] \ni t \longmapsto a_t \in L^p(\mathbb{R}^d)$ is continuous for $p = 1, \infty$.

In such case $\hat{L}(t)$ can be realised as a bounded linear operator $\hat{L}(t) \in L(\mathbb{B}'_\alpha, \mathbb{B}'_{\alpha'})$ for all $\alpha' < \alpha$, where $\mathbb{B}'_\alpha = L^1(\Gamma_0, e^{\alpha|\cdot|}\mathrm{d}\lambda)$ with the norm

$$\|G\|_\alpha = \int\limits_{\Gamma_0} |G(\eta)|e^{\alpha|\eta|}\mathrm{d}\lambda(\eta) = \sum_{n=0}^{\infty} \frac{e^{\alpha n}}{n!} \int\limits_{(\mathbb{R}^d)^n} |G^{(n)}(x_1, \dots, x_n)|\mathrm{d}x_1 \dots \mathrm{d}x_n$$

is a scale of Banach spaces of type 2.

Lemma 3.8. *The expression given for $\hat{L}(t)$ defines a bounded linear operator $\hat{L}(t) \in L(\mathbb{B}'_\alpha, \mathbb{B}'_{\alpha'})$ such that the mapping*

$$[0,T] \ni t \longmapsto \hat{L}(t) \in L(\mathbb{B}'_\alpha, \mathbb{B}'_{\alpha'})$$

is continuous in the uniform operator topology for $\alpha' < \alpha$.

Proof. For $\alpha' < \alpha$ it is simple to show

$$\|\hat{L}(t)\|_{\alpha\alpha'} \leq \frac{m(t) + \|P_t\|_{L^\infty} + \|a_t\|_{L^1}}{e(\alpha - \alpha')} + \frac{4\|a_t\|_\infty e^{-\alpha'}}{e^2(\alpha - \alpha')^2}, \tag{24}$$

which shows the first assertion. Since the operator $\hat{L}$ depends linearly on the parameters m, P, a the continuity follows immediately from (24). □

In order to solve the equation for quasi-observables it would be sufficient to show that

$$(A(t)G)(\eta) = \int\limits_{\mathbb{R}^d} \sum_{x \in \eta} a_t(x-y)G(\eta \cup x)\mathrm{d}y$$

generates for each $t \in [0,T]$ a C_0-semigroup such that Theorem 2.1 and 2.3 are applicable. The existence of a C_0-semigroup was proved in [5] for more general dynamics. Therefore we will realise this approach in the section 3.3. Instead we will turn to correlation functions and solve the corresponding equation for the particle densities. For correlation functions the following representation of $L^\Delta(t)$ holds for appropriate G and correlation functions k satisfying $k(\eta) \leq |\eta|C^{|\eta|}$, cf. [18],

$$\begin{aligned}(L^\Delta(t)k)(\eta) &= -|\eta|m(t)k(\eta) - \sum_{x\in\eta} P_t(x)k(\eta) \\ &\quad + \sum_{x\in\eta} \sum_{y\in(\eta\setminus x)} k(\eta\setminus x)a_t(x-y) + \sum_{x\in\eta} \int\limits_{\mathbb{R}^d} a_t(x-y)k((\eta\setminus x)\cup y)\mathrm{d}y.\end{aligned}$$

Analogous to previous calculations one can show that $L^{\Delta}(t)$ satisfies the same bound as in (24) and continuity. To analyse the long time behavior of this system we will consider only the first correlation function, which can be construed as a density. For $\eta = \{x\}$ the corresponding equation takes the form

$$\begin{aligned}\frac{\partial k_t^{(1)}(x)}{\partial t} &= -m(t)k_t^{(1)}(x) - P_t(x)k_t(x) + \int_{\mathbb{R}^d} a_t(x-y)k_t^{(1)}(y)\mathrm{d}y \\ &\leq -(\underline{m} + \underline{P}(x))k_t^{(1)}(x) + z\int_{\mathbb{R}^d} a_t(x-y)\mathrm{d}y \\ &= -M(x)k_t^{(1)}(x) + z\kappa(t)\end{aligned}$$

with $\kappa(t) = \int_{\mathbb{R}^d} a_t(y)\mathrm{d}y \geq 0$, $M(x) = \underline{m} + \underline{P}(x)$, $\underline{m} = \inf_{t\geq 0} m(t) \geq 0$, $\underline{P}(x) = \inf_{t\geq 0} P_t(x) \geq 0$ and the assumption $k_t(x) \leq z$. This leads to the bound

$$k_t^{(1)}(x) \leq e^{-M(x)t}k_0(x) + ze^{-M(x)t}\int_0^t \kappa(s)e^{M(x)s}\mathrm{d}s$$

for the solution $k_t^{(1)}(x)$. If κ asymptotically has exponential decay, then clearly $k_t^{(1)}(x) \to 0, \quad t \to \infty$ holds for $M(x) > 0$. Of course our approach and our assumptions have simplified the situation a lot. For more specific properties more detailed analysis is required. In applications one would use computer simulations instead of solving the equations explicitly or at least asymptotically. To show the existence of a solution we will work in the space

$$X_T = C([0,T]; L^\infty(\mathbb{R}^d)), \quad \|v\|_T = \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |v_t(x)| = \sup_{t\in[0,T]} \|v_t\|_{L^\infty} \quad (25)$$

and denote the closed cone of all non-negative functions $v \in X_T$ by X_T^+ . For $T' < T$ one has the natural embedding $X_{T'} \subset X_T$, where $X_{T'}$ is a closed subspace.

Lemma 3.9. *Let $A \in X_T^+$, $0 \leq a_t \in L^1(\mathbb{R}^d)$ for $t \in [0,T]$ and assume $(t,x) \longmapsto a_t(x) \geq 0$ is measurable with*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} a_t(x)\mathrm{d}x = \overline{a} < \infty.$$

Then the equation

$$\frac{\partial k_t(x)}{\partial t} = -A(t,x)k_t(x) + (a_t * k_t)(x), \quad k_t|_{t=0} = k_0 \in L^\infty(\mathbb{R}^d) \qquad (26)$$

has a unique non-negative solution $k_t \in L^\infty(\mathbb{R}^d)$ for $k_0 \geq 0$ and $t \in [0, \widetilde{T})$ with

$$\widetilde{T} = \begin{cases} T & ,\overline{a} = 0, \\ \min\left\{T, \frac{1}{\overline{a}}\right\} & ,\overline{a} > 0 \end{cases}.$$

This solution satisfies $0 \leq k_\bullet \in C^1([0,T']; L^\infty(\mathbb{R}^d))$ for each $T' < \widetilde{T}$.

Proof. Define the mapping $\Phi : X_{T'} \longrightarrow X_{T'}$ given by

$$(\Phi v)_t(x) = \exp\Big(-\int_0^t A(s,x)\mathrm{d}s\Big)k_0(x) + \int_0^t \exp\Big(-\int_s^t A(\tau,x)\mathrm{d}\tau\Big)(a_s * v_s)(x)\mathrm{d}s$$

for $T' < \widetilde{T}$. Clearly Φ is positivity preserving and by

$$|(a_s * v_s)(x)| \leq (a_s * |v_s|)(x) \leq \|a_s\|_{L^1}\|v_s\|_{L^\infty} \leq \overline{a}\|v\|_{T'}$$

we obtain

$$\begin{aligned} |(\Phi v)_t(x)| &\leq k_0(x) + \int_0^t \exp\Big(-\int_s^t A(\tau,x)\mathrm{d}\tau\Big)|a_s * v_s|(x)\mathrm{d}s \\ &\leq \|k_0\|_{L^\infty} + \int_0^t \overline{a}\|v\|_{T'}\mathrm{d}s \\ &\leq \|k_0\|_{L^\infty} + T'\overline{a}\|v\|_{T'} \end{aligned}$$

and hence $\Phi v \in X_{T'}$ for $v \in X_{T'}$, note that $t \longmapsto (\Phi v)_t \in L^\infty(\mathbb{R}^d)$ is continuous. In the same way

$$|(\Phi v)_t(x) - (\Phi w)_t(x)| \leq \int_0^t \exp\Big(-\int_s^t A(\tau,x)\mathrm{d}\tau\Big)(a_s * |v_s - w_s|)(x)\mathrm{d}s \leq T'\overline{a}\|v-w\|_{T'}$$

implies that Φ has the contraction property. Thus the sequence $(v^{(n)})_{n\in\mathbb{N}} \subset X_{T'}^+$ given by $v^{(0)} = k_0$ and $v^{(n+1)} = \Phi v^{(n)}$ is a fundamental sequence and hence has a limit $v = \lim\limits_{n\to\infty} v^{(n)} \in X_{T'}^+$. Consequently $v = \Phi v$, i.e.

$$v_t(x) = \exp\Big(-\int_0^t A(s,x)\mathrm{d}s\Big)k_0(x) + \int_0^t \exp\Big(\int_s^t A(\tau,x)\mathrm{d}\tau\Big)(a_s * v_s)(x)\mathrm{d}s \quad (27)$$

for a.a. $x \in \mathbb{R}^d$ holds, which shows the existence of a solution to (26). Since every solution of (26) solves (27) the uniqueness follows for $t \in [0, T']$ and hence on $[0, \widetilde{T})$. □

Corollary 3.10. *Let $A \in X_T^+$ for each $T > 0$ and $0 \leq a_t \in L^1(\mathbb{R}^d)$ for $t \geq 0$, $(t,x) \longmapsto a_t(x) \geq 0$ be measurable and assume*

$$\sup_{t\geq 0} \int_{\mathbb{R}^d} a_t(x)\mathrm{d}x = \overline{a} < \infty$$

Then the equation

$$\frac{\partial k_t(x)}{\partial t} = -A(t,x)k_t(x) + (a_t * k_t)(x), \quad k_t|_{t=0} = k_0 \in L^\infty(\mathbb{R}^d)$$

has a unique non-negative solution $k_t \in L^\infty(\mathbb{R}^d)$ for $k_0 \geq 0$ and $t \geq 0$. Moreover $k_\bullet \in C^1([0,T]; L^\infty(\mathbb{R}^d))$ holds for each $T > 0$.

Proof. Under this assumption one can take $\widetilde{T} = \frac{1}{\overline{a}}$ and hence consider iteratively the same Cauchy problem with initial conditions $k_t|_{t=0} = k_{lT'}$ with $l \in \mathbb{N}$ and $T' < \frac{1}{\overline{a}}$. □

In order to apply Lemma 3.11 we need

$$\operatorname*{ess\,sup}_{(t,x)\in[0,T]\times\mathbb{R}^d} m(t) + P_t(x) < \infty$$

and

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} a_t(x)\mathrm{d}x = \overline{a} < \infty.$$

Both conditions are satisfyied since $0 \le m \in C([0,T])$, $P_\bullet \in C([0,T]; L^\infty(\mathbb{R}^d))$ and $a_\bullet \in C([0,T]; L^p(\mathbb{R}^d))$ for $p = 1, \infty$. Hence there exists a unique solution to the equation for densities.

3.3 Bolker-Dieckman-Law-Pacala Model

In this section we will discuss an ecological birth and death model. Each individual may die due to a space independent mortality rate m and due to competition of individuals. This competition is described translation invariant by a competition kernel a^-, i.e $a^-(x,y) \equiv a^-(x-y) = a^-(y-x)$. High values for a^- lead to high probabilities of death. Analogously each individual can produce another individual, where the probability distribution of this elementary event is given by the dispersion kernel a^+. Therefore we can describe this model by the following Markov pre-generator

$$(LF)(\gamma) = \sum_{x\in\gamma}(m+E^-(x,\gamma\backslash x))(F(\gamma\backslash x)-F(\gamma)) + \int_{\mathbb{R}^d} E^+(y,\gamma)(F(\gamma\cup y)-F(\gamma))\mathrm{d}y$$

with $m > 0$ and $E^\pm(x,\gamma) = \sum_{y\in\gamma} a^\pm(x-y)$. This model was discussed in [5], where the authors proved local existence of solutions for quasi-observables, and correlation functions. Moreover the existence of evolution of states was shown. In this section we will prove the existence of solutions for quasi-observables in the time dependent case, i.e.

$$\begin{aligned}(L(t)F)(\gamma) \quad &= \quad \sum_{x\in\gamma}(m(t) + E_t^-(x,\gamma\backslash x))(F(\gamma\backslash x) - F(\gamma)) \\ &\quad + \int_{\mathbb{R}^d} E_t^+(y,\gamma)(F(\gamma\cup y) - F(\gamma))\mathrm{d}y\end{aligned}$$

under the following assumptions for $T > 0$

1. m is a continuous non-negative function in $t \in [0,T]$
2. The dispersion and competition kernels $a_t^\pm(x) = a_t^\pm(-x) \ge 0$ are continuous as mappings

$$[0,T] \ni t \longmapsto a_t^\pm \in L^\infty(\mathbb{R}^d), \quad [0,T] \ni t \longmapsto a_t^\pm \in L^1(\mathbb{R}^d).$$

3. There exists a $\Theta > 0$ such that
$$a_t^+(x) \leq \Theta a_t^-(x) \tag{28}$$
holds for all $t \in [0,T]$ and almost all $x \in \mathbb{R}^d$.

The last condition (28) means that the dispersion kernel is dominated by the competition kernel uniformly in the time $t \in [0,T]$. The corresponding operator for quasi-observables is formally given by the expressions

$$\hat{L}(t) = A(t) + B(t)$$

with

$$\begin{aligned} A(t) &= A_1(t) + A_2(t) \\ (A_1(t)G)(\eta) &= -E_t(\eta)G(\eta) \\ (A_2(t)G)(\eta) &= \int_{\mathbb{R}^d} E_t^+(y,\eta)G(\eta \cup y)\mathrm{d}y \end{aligned}$$

and

$$\begin{aligned} B(t) &= B_1(t) + B_2(t) \\ (B_1(t)G)(\eta) &= -\sum_{x\in\eta} E_t^-(x,\eta\backslash x)G(\eta\backslash x) \\ (B_2(t)G)(\eta) &= \int_{\mathbb{R}^d} \sum_{x\in\eta} a_t^+(x-y)G(\eta\backslash x \cup y)\mathrm{d}y, \end{aligned}$$

where $E_t(\eta) = \sum\limits_{x\in\eta}(m(t) + E_t^-(x,\eta\backslash x)) = m(t)|\eta| + E_t^-(\eta)$ and $E_t^{\pm}(\eta) = \sum\limits_{x\in\eta} E_t^{\pm}(x,\eta\backslash x)$. As usual we will work in the scale $\mathbb{B}'_\alpha = L^1(\Gamma_0, e^{\alpha|\cdot|}\mathrm{d}\lambda)$, then a simple calculation shows the following result.

Lemma 3.11. *The above expressions define linear bounded operators $A, B \in L(\mathbb{B}'_\alpha, \mathbb{B}'_{\alpha'})$ for $\alpha' < \alpha$ with norm estimates*

$$\|A(t)\|_{\alpha\alpha'} \leq \frac{m(t)}{e(\alpha-\alpha')} + \frac{4(\|a_t^-\|_{L^\infty} + \|a_t^+\|_{L^\infty}e^{-\alpha'})}{e^2(\alpha-\alpha')^2} \tag{29}$$

and

$$\|B(t)\|_{\alpha\alpha'} \leq \frac{\|a_t^-\|_{L^1}e^{\alpha'} + \|a_t^+\|_{L^1}}{e(\alpha-\alpha')}. \tag{30}$$

In view of Theorem 2.10 we have as a consequence of (30) that $\|B(t)\|_{\alpha\alpha'} \leq \dfrac{M}{\alpha-\alpha'}$ for some constant $M = M(\alpha_*, \alpha^*)$ if we fix $\alpha_* < \alpha^*$, cf. Definition 5. Since we cannot apply Theorem 2.10 for the operator A, c.f. (29), the next step for us will be to prove existence of an evolution family corresponding to A in order to apply Theorem 2.13. But first we need to show the continuity of $t \longmapsto A(t)$ and $t \longmapsto B(t)$ in the uniform operator topology. For $\alpha < \alpha'$ consider the mappings

$$\mathbb{R}_+ \times X_+ \times X_+ \longrightarrow L(\mathbb{B}'_\alpha, \mathbb{B}'_{\alpha'}), \quad (m, a^+, a^-) \longmapsto \hat{L}(m, a^+, a^-). \tag{31}$$

with

$$X = \{f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) : \quad f(x) = f(-x), \quad \text{for a.a. } x \in \mathbb{R}^d\}.$$

Here X_+ denotes the positive cone of X consisting of all elements $0 \leq f \in X$. The previous Lemma shows, that this map is well-defined. Endow X with the norm

$$\|f\|_X = \max\{\|f\|_{L^1}, \|f\|_{L^\infty}\}$$

so $(X, \|\cdot\|_X)$ is a closed subspace of the Banach space $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and thus a Banach space itself. If we define on the parameter space $\mathbb{R}_+ \times X_+ \times X_+$ the metric

$$d((m, a^+, a^-), (m', b^+, b^-)) = |m - m'| + \|a^+ - b^+\|_X + \|a^- - b^-\|_X$$

the following result holds.

Lemma 3.12. *For $\alpha' < \alpha$ the mapping* (31) *is continuous, where $L(\mathbb{B}'_\alpha, \mathbb{B}'_{\alpha'})$ has the topology induced by the operator norm.*

Proof. Since $\hat{L}$ depend linearly on the coefficients m, a^+, a^- we obtain from Lemma 3.13

$$\begin{aligned} &\|\hat{L}(m, a^+, a^-) - \hat{L}(m', b^+, b^-)\|_{\alpha\alpha'} \\ \leq \quad &\frac{4\|a^- - b^-\|_{L^\infty} + 4\|a^+ - b^+\|_{L^\infty} e^{-\alpha'}}{e^2(\alpha - \alpha')^2} \\ &+ \frac{|m - m'| + \|a^- - b^-\|_{L^1} e^{\alpha'} + \|a^+ - b^+\|_{L^1}}{e(\alpha - \alpha')}. \quad \square \end{aligned}$$

The continuity of m, a^+, a^- imply the continuity of

$$[0, T] \ni t \longmapsto (m(t), a_t^+, a_t^-) \in \mathbb{R}_+ \times X_+ \times X_+$$

and as a consequence we obtain the desired continuity of

$$[0, T] \ni t \longmapsto A(t), \quad [0, T] \ni t \longmapsto B(t)$$

in the uniform operator topology on $L(\mathbb{B}'_\alpha, \mathbb{B}'_{\alpha'})$. Now we are prepared to prove the existence of an evolution family corresponding to $A(t)$.

Theorem 3.13. *Let α_* be such that $\Theta e^{-\alpha_*} < 1$ holds. Then for all $\alpha_* \leq \alpha' < \alpha$ there exists a unique evolution family $(\hat{U}(t, s))_{0 \leq s \leq t \leq T}$ on $\mathbb{B}'_{\alpha'}$ satisfying*

1. $\dfrac{\partial \hat{U}}{\partial t}(t, s)G = A(t)\hat{U}(t, s)G$ *on $\mathbb{B}_{\alpha'}$ for $G \in \mathbb{B}'_\alpha$, in the case of $t = s$ the derivative is meant to be a right-sided derivative.*

2. $\dfrac{\partial \hat{U}}{\partial s}(t, s)G = -\hat{U}(t, s)A(s)G$ *on $\mathbb{B}_{\alpha'}$ for $G \in \mathbb{B}'_\alpha$.*

Proof. By [5] for each $\alpha_* \leq \alpha'$ there exists a sub stochastic analytic C_0-semigroup $S_t^{\alpha'}(\tau) = e^{\tau A(t)}$ on $\mathbb{B}'_{\alpha'}$. The generator is given by $(A(t), D_{\alpha'}(A(t)))$ with

$$D_{\alpha'}(A(t)) = \{G \in \mathbb{B}'_{\alpha'} : \quad E_t(\cdot)G(\cdot) \in \mathbb{B}'_{\alpha'}\}.$$

For $\alpha_* \leq \alpha' < \alpha$ the part $\widetilde{A}(t)$ of $(A(t), D_{\alpha'}(A(t)))$ on $\mathbb{B}'_\alpha$ is given by

$$\begin{aligned} D(\widetilde{A}(t)) &= \{G \in \mathbb{B}'_\alpha \cap D_{\alpha'}(A(t)) : \quad E_t(\cdot)G(\cdot) \in \mathbb{B}'_\alpha\} \\ &= \{G \in \mathbb{B}'_\alpha : \quad E_t(\cdot)G(\cdot) \in \mathbb{B}'_\alpha\} = D_\alpha(A(t)) \end{aligned}$$

and hence is a generator of a substochastic analytic semigroup, which shows the assumptions of Theorem 2.1. Therefore for $\alpha_* \leq \alpha' < \alpha$ the semigroups satisfy

$$S_t^\alpha(\tau) = S_t^{\alpha'}(\tau)|_{\mathbb{B}'_\alpha}, \quad \forall t \in [0,T] \quad \text{and } \tau \geq 0.$$

Concerning the proof of Theorem 2.1, cf. [24], the evolution families are obtained as limits $\hat{U}^\alpha(t,s) = \lim_{n\to\infty} \hat{U}_n^\alpha(t,s)$ in $\mathbb{B}'_\alpha$. Since $U_n^\alpha(t,s)$ is a composition of $S_t^\alpha(\tau)$

$$\hat{U}_n^\alpha(t,s) = \hat{U}_n^{\alpha'}(t,s)|_{\mathbb{B}'_\alpha}, \quad (t,s) \in \Delta$$

for $\alpha_* \leq \alpha' < \alpha$ follows. To show the property

$$\hat{U}^\alpha(t,s) = \hat{U}^{\alpha'}(t,s)|_{\mathbb{B}'_\alpha} \tag{32}$$

consider for $G \in \mathbb{B}'_\alpha$

$$\begin{aligned} & \|\hat{U}^\alpha(t,s)G - \hat{U}^{\alpha'}(t,s)G\|_{\alpha'} \\ \leq \; & \|\hat{U}^\alpha(t,s)G - \hat{U}_n^\alpha(t,s)G\|_{\alpha'} + \|\hat{U}_n^\alpha(t,s)G - \hat{U}^{\alpha'}(t,s)G\|_{\alpha'} \\ \leq \; & \|\hat{U}^\alpha(t,s)G - \hat{U}_n^\alpha(t,s)G\|_{\alpha} + \|\hat{U}_n^{\alpha'}(t,s)G - \hat{U}^{\alpha'}(t,s)G\|_{\alpha'} \end{aligned}$$

and take $n \to \infty$. Hence $\hat{U}^\alpha(t,s)G = \hat{U}^{\alpha'}(t,s)G$ in $\mathbb{B}'_{\alpha'}$ and therefore by definition of the norm also pointwise for a.a. $\eta \in \Gamma_0$, which implies (32) in $\mathbb{B}'_\alpha$. Now (32) implies the conditions for Theorem 2.3 and hence the desired result. □

Corollary 3.14. *Let α_* be such that $\Theta e^{-\alpha_*} < 1$ and fix some $\alpha^* > \alpha_*$. Then there exists a continuous function $T(\alpha)$ monotonically decreasing and for each $G_0 \in \mathbb{B}'_{\alpha^*}$ a unique solution G_t of*

$$\frac{\mathrm{d}G_t}{\mathrm{d}t} = \hat{L}(t)G_t, \quad G_t|_{t=0} = G_0$$

in the scale $\mathbb{B}'_\alpha$ given by Remark 2.15.3.

3.4 Glauber-type Dynamics in Continuum

The non-equilibrium Glauber-type dynamics can be described by the heuristic Markov pre-generator

$$(LF)(\gamma) = m \sum_{x\in\gamma} (F(\gamma\backslash x) - F(\gamma)) + z \int_{\mathbb{R}^d} (F(\gamma \cup x) - F(\gamma)) \exp(-E(x,\gamma))\mathrm{d}x.$$

Let $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ be an even non-negative function. For any $\gamma \in \Gamma$, $x \in \mathbb{R}^d \backslash \gamma$ we set $E(x,\gamma) = \sum_{y\in\gamma} \phi(x-y) \in [0,\infty]$. Here $z > 0$ is an activity parameter and $m > 0$ is a mortality rate. As before each particle may die according to the rate m. New particles are influenced by existing particles, which is described by the potential ϕ. Big values of ϕ lead to a small factor $e^{-E(x,\gamma)}$ and hence to smaller probabilities for new particles to appear in the regions where $E(x,\gamma)$ is big. The operator for quasi-observables is given by

$$(\hat{L}G)(\eta) = -|\eta| m G(\eta) + z \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} e^{-E(x,\xi)} G(\xi \cup x) e_\lambda (e^{-\phi(x-\cdot)} - 1, \eta \backslash \xi) \mathrm{d}x.$$

The existence of a C_0-semigroup associated to $\hat{L}$ was shown in [17]. In [6], it was proven that this semigroup can be approximated uniformly on compact time intervals using discretization of time. Solutions in scales of Banach spaces were studied in [4] and [11]. This part will partially generalize the results to time dependent coefficients. Likewise the evolution of correlation functions and states will be studied. The evolution equation for correlation functions is given by the operator

$$(L^\Delta k)(\eta) = -|\eta| m k(\eta) + z \sum_{x \in \eta} e^{-E(x,\eta\backslash x)} \int_{\Gamma_0} e_\lambda(t_x, \xi) k((\eta \backslash x) \cup \xi) \mathrm{d}\lambda(\xi) \quad (33)$$

with $t_x(y) = e^{-\phi(x-y)} - 1$. In [6] the existence of correlation function evolution was proven by discretization and further ergodicity properties were studied. We will be concerned with the time dependent case $z = z(t), m = m(t)$ and $\phi = \phi_t$. Starting again with the equation for quasi-observables in the scale $\mathbb{B}'_\alpha = L^1(\Gamma_0, e^{\alpha|\cdot|})$ of type 2 we will impose the following conditions to hold for some $T > 0$

1. $[0,T] \ni t \longmapsto z(t) \geq 0$, $[0,T] \ni t \longmapsto m(t) \geq 0$ are continuous;
2. $\phi_t(x) = \phi_t(-x) \geq 0$ is a continuous mapping in the sense that

   $$[0,T] \ni t \longmapsto \phi_t \in L^\infty(\mathbb{R}^d), \quad [0,T] \ni t \longmapsto \phi_t \in L^1(\mathbb{R}^d)$$

   is continuous;
3. there exists a potential $\phi(x) = \phi(-x) \geq 0$ such that $\phi_t(x) \leq \phi(x)$ and

   $$\beta = \int_{\mathbb{R}^d} (1 - e^{-\phi(x)}) \mathrm{d}x < \infty.$$

Note that 3. implies $1 - e^{-\phi_t(x-\cdot)} \leq 1 - e^{-\phi(x-\cdot)}$ and hence $\int_{\mathbb{R}^d} 1 - e^{-\phi_t(x)} \mathrm{d}x = \beta_t \leq \beta < \infty$. The last condition is important to have uniform bounds in the time variable t. As a first step we will show continuity properties of

$$\begin{aligned} (\hat{L}(t)G)(\eta) &= -|\eta| m(t) G(\eta) \\ &\quad + z(t) \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} e^{-E_t(x,\xi)} G(\xi \cup x) e_\lambda(e^{-\phi_t(x-\cdot)} - 1, \eta \backslash \xi) \mathrm{d}x. \end{aligned} \quad (34)$$

Lemma 3.15. *Under conditions 1-3. expression* (34) *defines a bounded linear operator* $\hat{L}(t) \in L(\mathbb{B}'_\alpha, \mathbb{B}'_{\alpha'})$ *for* $\alpha' < \alpha$ *satisfying*

$$\|\hat{L}(t)\|_{\alpha\alpha'} \leq \frac{m(t) + z(t)\exp\left(e^{\alpha'}\beta_t\right)e^{-\alpha'}}{e(\alpha-\alpha')}.$$

Further the mapping $[0,T] \ni t \longmapsto \hat{L}(t) \in L(\mathbb{B}'_\alpha, \mathbb{B}'_{\alpha'})$ *is continuous in the uniform operator topology.*

Proof. For $\alpha' < \alpha$ and $G \in \mathbb{B}'_\alpha$ we obtain

$$\begin{aligned}
& \int\limits_{\Gamma_0} \sum_{\xi\subset\eta} \int\limits_{\mathbb{R}^d} e^{-E_t(x,\xi)} e_\lambda(|t_x|, \eta\backslash\xi)|G(\xi\cup x)e^{\alpha'|\eta|}\mathrm{d}x\mathrm{d}\lambda(\eta) \\
= & \int\limits_{\Gamma_0}\int\limits_{\Gamma_0}\int\limits_{\mathbb{R}^d} e^{-E_t(x,\xi)} e_\lambda(|t_x|;\eta)|G(\xi\cup x)|e^{\alpha'|\eta|}e^{\alpha'|\xi|}\mathrm{d}x\mathrm{d}\lambda(\xi)\mathrm{d}\lambda(\eta) \\
\leq & \exp\left(e^{\alpha'}\beta_t\right)\int\limits_{\Gamma_0}\int\limits_{\mathbb{R}^d} |G(\xi\cup x)|e^{\alpha'|\xi|}\mathrm{d}x\mathrm{d}\lambda(\xi) \\
\leq & \frac{\exp\left(e^{\alpha'}\beta_t\right)e^{-\alpha'}}{e(\alpha-\alpha')}\|G\|_\alpha
\end{aligned}$$

which shows the first assertion. For the second part of the assertion of the lemma take $G \in \mathbb{B}'_\alpha$ and $t, s \in [0,T]$, then we have for the death part

$$|m(t)-m(s)|\int\limits_{\Gamma_0}|\eta||G(\eta)|e^{\alpha'|\eta|}\mathrm{d}\lambda(\eta) \leq \frac{|m(t)-m(s)|}{e(\alpha-\alpha')}\|G\|_\alpha,$$

which has the desired property. The birth part can be estimated by (+)

$$\begin{aligned}
& z(t)\int\limits_{\Gamma_0}\int\limits_{\Gamma_0}\int\limits_{\mathbb{R}^d}\left|e_\lambda(1-e^{-\phi_t(x-\cdot)},\eta) - e_\lambda(1-e^{-\phi_s(x-\cdot)},\eta)\right| \\
& \qquad\qquad \times|G(\xi\cup x)|e^{\alpha'|\eta\cup\xi|}\mathrm{d}x\mathrm{d}\lambda(\xi)\mathrm{d}\lambda(\eta) \\
+ \quad & z(t)\int\limits_{\Gamma_0}\int\limits_{\Gamma_0}\int\limits_{\mathbb{R}^d} e_\lambda(1-e^{-\phi_s(x-\cdot)},\eta)\left|e^{-E_t(x,\xi)} - e^{-E_s(x,\xi)}\right| \\
& \qquad\qquad \times|G(\xi\cup x)|e^{\alpha'|\eta\cup\xi|}\mathrm{d}x\mathrm{d}\lambda(\xi)\mathrm{d}\lambda(\eta) \\
+ \quad & |z(t)-z(s)|\int\limits_{\Gamma_0}\int\limits_{\Gamma_0}\int\limits_{\mathbb{R}^d} e_\lambda(1-e^{-\phi_s(x-\cdot)},\eta)|G(\xi\cup x)|e^{\alpha'|\eta\cup\xi|}\mathrm{d}x\mathrm{d}\lambda(\xi)\mathrm{d}\lambda(\eta).
\end{aligned}$$

Using

$$\begin{aligned}
& \left|e_\lambda(e^{-\phi_t(x-\cdot)}-1,\eta) - e_\lambda(e^{-\phi_s(x-\cdot)}-1,\eta)\right| \\
\leq & \sum_{y\in\eta}|e^{-\phi_t(x-y)} - e^{-\phi_s(x-y)}|e_\lambda(1-e^{-\phi(x-\cdot)},\eta\backslash y) \\
\leq & \sum_{y\in\eta}|\phi_t(x-y) - \phi_s(x-y)|e_\lambda(1-e^{-\phi(x-\cdot)},\eta\backslash y)
\end{aligned}$$

we estimate the first part of (+) by

$$
\begin{aligned}
& z(t) \int_{\Gamma_0} \int_{\mathbb{R}^d} \int_{\Gamma_0} \sum_{y\in\eta} |\phi_t(x-y) - \phi_s(x-y)| e_\lambda(1-e^{-\phi(x-\cdot)}, \eta\backslash y) \\
& \qquad \times |G(\xi\cup x)| e^{\alpha'|\eta\cup\xi|} \mathrm{d}\lambda(\eta)\mathrm{d}x\mathrm{d}\lambda(\xi) \\
= \; & z(t) e^{\alpha'} \|\phi_t - \phi_s\|_{L^1} \int_{\Gamma_0} \int_{\mathbb{R}^d} \int_{\Gamma_0} e_\lambda(1-e^{-\phi(x-\cdot)}, \eta) e^{\alpha'|\eta|} \mathrm{d}\lambda(\eta) \\
& \qquad \times |G(\xi\cup x)| e^{\alpha'|\xi|} \mathrm{d}x\mathrm{d}\lambda(\xi) \\
= \; & z(t) e^{\alpha'} \|\phi_t - \phi_s\|_{L^1} \exp(e^{\alpha'}\beta) \int_{\Gamma_0} \int_{\mathbb{R}^d} |G(\xi\cup x)| e^{\alpha'|\xi|} \mathrm{d}x\mathrm{d}\lambda(\xi) \\
\leq \; & \frac{z(t)\|\phi_t - \phi_s\|_{L^1} \exp(e^{\alpha'}\beta)}{e(\alpha-\alpha')} \|G\|_\alpha .
\end{aligned}
$$

Because of

$$
\left| e^{-E_t(x,\xi)} - e^{-E_s(x,\xi)} \right| \leq |E_s(x,\xi) - E_t(x,\xi)| \leq |\xi| \|\phi_t - \phi_s\|_{L^\infty}
$$

we obtain for the second part of (+)

$$
\begin{aligned}
& z(t)\|\phi_t - \phi_s\|_\infty \int_{\Gamma_0} \int_{\Gamma_0} \int_{\mathbb{R}^d} e_\lambda(1-e^{-\phi_s(x-\cdot)}, \eta) e^{\alpha'|\eta|} \\
& \qquad \times |G(\xi\cup x)| |\xi| e^{\alpha'|\xi|} \mathrm{d}x\mathrm{d}\lambda(\xi)\mathrm{d}\lambda(\eta) \\
\leq \; & z(t)\|\phi_t - \phi_s\|_\infty \exp(e^{\alpha'}\beta_s) e^{-\alpha'} \int_{\Gamma_0} |\xi|^2 |G(\xi)| e^{\alpha'|\xi|} \mathrm{d}\lambda(\xi) \\
\leq \; & \frac{4z(t)\|\phi_t - \phi_s\|_\infty e^{-\alpha'}}{e^2(\alpha-\alpha')^2} \exp(e^{\alpha'}\beta_s) \|G\|_\alpha .
\end{aligned}
$$

For the last part of (+) we get

$$
\begin{aligned}
& |z(t) - z(s)| \int_{\Gamma_0} \int_{\Gamma_0} \int_{\mathbb{R}^d} e_\lambda(1-e^{-\phi_s(x-\cdot)}, \eta) |G(\xi\cup x)| e^{\alpha'|\xi|} e^{\alpha'|\eta|} \mathrm{d}x\mathrm{d}\lambda(\xi)\mathrm{d}\lambda(\eta) \\
\leq \; & \frac{|z(t) - z(s)| e^{-\alpha'}}{e(\alpha-\alpha')} \exp(e^{\alpha'}\beta_s) \|G\|_\alpha
\end{aligned}
$$

which proves the assertion. □

Since $\|\hat{L}(t)\|_{\alpha'\alpha} \leq \dfrac{M}{\alpha-\alpha'}$, with

$$
M = \frac{\overline{m} + \overline{z} \exp\left(e^{\alpha'}\beta\right) e^{-\alpha'}}{e}
$$

$\overline{m} = \sup\limits_{t\geq 0} m(t)$ and $\overline{z} = \sup\limits_{t\geq 0} z(t)$ we can apply Theorem 2.10 and prove the existence of solutions in the scale $\mathbb{B}'_\alpha$. For the time independent parameters the existence was proved directly in [4].

Theorem 3.16. *Under conditions 1-3. and for fixed $\alpha_* < \alpha^*$ there exists $T : [\alpha_*, \alpha^*) \longrightarrow [0, T]$ continuous and monotonically decreasing, such that for each $G_0 \in \mathbb{B}'_{\alpha^*} = L^1(\Gamma_0, e^{\alpha|\cdot|}\mathrm{d}\lambda)$ there exists a unique solution G_t to the Cauchy problem*

$$\frac{\partial G_t}{\partial t} = \hat{L}(t)G_t, \quad G_t|_{t=0} = G_0 \tag{35}$$

in the scale $\mathbb{B}'_\alpha$ given by Remark 2.12.3

Proof. Lemma 3.18 implies

$$\begin{aligned} \|\hat{L}(t)\|_{\alpha\alpha'} &\leq \frac{m(t) + z(t)\exp\left(e^{\alpha'}\beta_t\right)e^{-\alpha'}}{e(\alpha-\alpha')} \\ &\leq \frac{\overline{m} + \overline{z}\exp\left(e^{\alpha^*}\beta\right)e^{-\alpha_*}}{e(\alpha-\alpha')} \end{aligned}$$

with $\overline{m} = \sup\limits_{t\in[0,T]} m(t)$ and $\overline{z} = \sup\limits_{t\in[0,T]} z(t)$, which shows the first assumption of Theorem 2.10. Since continuity in the uniform operator topology implies strong continuity Theorem 2.10 is applicable and shows the existence of unique solutions to (35). □

Likewise using the same techniques we can prove existence of solutions for the corresponding equations for correlation functions, c.f. (33). First we show general properties of operators $L^\Delta(t)$ in the scale of Banach spaces $\mathbb{B}_\alpha = L^\infty(\Gamma_0, e^{-\alpha|\cdot|}\mathrm{d}\lambda)$.

Lemma 3.17. *Under conditions 1-3. the expression*

$$\begin{aligned} (L^\Delta(t)k)(\eta) &= -|\eta|m(t)k(\eta) \\ &\quad + z(t)\sum_{x\in\eta} e^{-E_t(x,\eta\backslash x)} \int_{\Gamma_0} e_\lambda(e^{-\phi_t(x-\cdot)}-1,\xi)k((\eta\backslash x)\cup\xi)\mathrm{d}\lambda(\xi) \end{aligned}$$

defines an operator $L^\Delta(t) \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$ for $\alpha' < \alpha$ such that

$$\|L^\Delta(t)\|_{\alpha'\alpha} \leq \frac{m(t) + z(t)e^{-\alpha'}\exp(e^{\alpha'}\beta_t)}{e(\alpha-\alpha')}.$$

Moreover, the mapping $[0, T] \ni t \longmapsto L^\Delta(t) \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$ is continuous in the uniform operator topology.

Proof. Let $\alpha' < \alpha$ and $k \in \mathbb{B}_{\alpha'}$ be fixed, then the first summand gives

$$|\eta|m(t)|k(\eta)|e^{-\alpha|\eta|} \leq m(t)\|k\|_{\alpha'}|\eta|e^{-(\alpha-\alpha')|\eta|} \leq \frac{m(t)}{e(\alpha-\alpha')}\|k\|_{\alpha'}$$

and for the second part

$$\begin{aligned} & z(t)\sum_{x\in\eta} e^{-E_t(x,\eta\backslash x)} \int_{\Gamma_0} e_\lambda(1-e^{-\phi_t(x-\cdot)},\xi)|k(\eta\backslash x\cup\xi)|\mathrm{d}\lambda(\xi)e^{-\alpha|\eta|} \\ \leq\ & z(t)\|k\|_{\alpha'}e^{-\alpha'}e^{-(\alpha-\alpha')|\eta|}\sum_{x\in\eta}\int_{\Gamma_0} e_\lambda(1-e^{-\phi_t(x-\cdot)},\xi)e^{\alpha'|\xi|}\mathrm{d}\lambda(\xi) \\ =\ & z(t)\|k\|_{\alpha'}e^{-\alpha'}|\eta|e^{-(\alpha-\alpha')|\eta|}\exp(e^{\alpha'}\beta_t) \leq \frac{z(t)e^{-\alpha'}\exp(e^{\alpha'}\beta_t)}{e(\alpha-\alpha')}\|k\|_{\alpha'}. \end{aligned}$$

Thus the first claim is proved. For the second part let $t, s \in [0,T]$ be arbitrary, then the death part can be estimated by

$$|\eta||m(t)-m(s)|e^{-\alpha|\eta|}|k(\eta)| \leq |m(t)-m(s)||\eta|e^{-(\alpha-\alpha')|\eta|}\|k\|_{\alpha'} \leq \frac{|m(t)-m(s)|}{e(\alpha-\alpha')}\|k\|_{\alpha'}.$$

Analogously to Lemma 3.18 the birth part can be estimated by

$$\begin{aligned} & |z(t)-z(s)|\sum_{x\in\eta} e^{-E_t(x,\eta\backslash x)}\int_{\Gamma_0} e_\lambda(1-e^{-\phi_t(x-\cdot)},\xi)|k(\eta\backslash x\cup\xi)|\mathrm{d}\lambda(\xi)e^{-\alpha|\eta|} \\ & +z(s)\sum_{x\in\eta}\left|e^{-E_t(x,\eta\backslash x)}-e^{-E_s(x,\eta\backslash x)}\right| \\ & \quad\times\int_{\Gamma_0} e_\lambda(1-e^{-\phi_t(x-\cdot)},\xi)|k(\eta\backslash x\cup\xi)|\mathrm{d}\lambda(\xi)e^{-\alpha|\eta|} \\ & +z(s)\sum_{x\in\eta} e^{-E_s(x,\eta\backslash x)} \\ & \quad\times\int_{\Gamma_0}\left|e_\lambda(1-e^{-\phi_t(x-\cdot)},\xi)-e_\lambda(1-e^{-\phi_s(x-\cdot)},\xi)\right||k(\eta\backslash x\cup\xi)|\mathrm{d}\lambda(\xi)e^{-\alpha|\eta|}. \end{aligned}$$

The first summand can be bounded by

$$\begin{aligned} & |z(t)-z(s)|e^{-\alpha'}|\eta|e^{-(\alpha-\alpha')|\eta|}\exp(e^{\alpha'}\beta_t)\|k\|_{\alpha'} \\ \leq\ & \frac{|z(t)-z(s)|e^{-\alpha'}\exp(e^{\alpha'}\beta)}{e(\alpha-\alpha')}\|k\|_{\alpha'} \end{aligned}$$

and the second one by

$$\begin{aligned} & z(s)\|\phi_t-\phi_s\|_\infty e^{-\alpha'}|\eta|^2e^{-(\alpha-\alpha')|\eta|}\exp(e^{\alpha'}\beta_t)\|k\|_{\alpha'} \\ \leq\ & \frac{4z(s)\|\phi_t-\phi_s\|_\infty\exp(e^{\alpha'}\beta_t)}{e^2(\alpha-\alpha')^2}\|k\|_{\alpha'}. \end{aligned}$$

As a result they have desired property. For the last term we have the following estimate

$$
\begin{aligned}
&z(s)\sum_{x\in\eta}\int_{\Gamma_0}\sum_{y\in\xi}|\phi_t(x-y)-\phi_s(x-y)|e_\lambda(1-e^{-\phi(x-\cdot)},\xi\backslash y)|k(\eta\backslash x\cup\xi)|e^{-\alpha|\eta|}\mathrm{d}\lambda(\xi)\\
\leq&z(s)\|k\|_{\alpha'}e^{-(\alpha-\alpha')|\eta|}\|\phi_t-\phi_s\|_{L^1}\sum_{x\in\eta}\int_{\Gamma_0}e_\lambda(1-e^{-\phi(x-\cdot)},\xi)e^{\alpha'|\xi|}\mathrm{d}\lambda(\xi)\\
\leq&\frac{z(s)\|\phi_t-\phi_s\|_{L^1}\exp(e^{\alpha'}\beta)}{e(\alpha-\alpha')}\|k\|_{\alpha'}
\end{aligned}
$$

which shows the continuity. □

As a consequence, by Theorem 2.10 we obtain the existence of local solutions.

Theorem 3.18. *Fix some $\alpha_* < \alpha^*$, then there exists $T:(\alpha^*,\alpha^*]\longrightarrow[0,T]$ continuous and monotonically increasing such that for each $k_0\in\mathbb{B}_{\alpha_*}$ there exists a unique solution k_t to the Cauchy problem*

$$
\frac{\partial k_t}{\partial t}(t)=L^{\Delta}(t)k_t,\quad k_t|_{t=0}=k_0 \tag{36}
$$

in the scale $\mathbb{B}_\alpha$ given by Remark 2.12.3.

To have the existence of a solution via evolution families it is sufficient to show that the operators $L^{\Delta}(t)$ generate contraction semigroups $T_t^{\Delta}(s)$ for $t\in[0,T]$. Since the scale $\mathbb{B}_\alpha$ is of L^∞-type it is not straightforward. The general approach is to consider the dual semigroups and show the existence of appropriate invariant subspaces. This analysis could be done, but is not the purpose of this work. Instead we will consider the evolution of Bogoliubov generating functionals. The fact that

$$
\sup_{x\in\mathbb{R}^d}\left|\frac{e^{hx}-1}{h}-x\right|=\infty,\quad\forall h>0
$$

causes difficulties in many calculations. Therefore we will only consider the simplified model with the time independent potential ϕ. Let m and z be continuous functions on some interval $I=[0,T]$ and $\phi(x)=\phi(-x)\geq 0$ be integrable, i.e.,

$$
\beta=\int_{\mathbb{R}^d}1-e^{-\phi(x)}\mathrm{d}x\leq\int_{\mathbb{R}^d}\phi(x)\mathrm{d}x=\|\phi\|_{L^1}.
$$

In [11] it was shown, that the generator $\widetilde{L}(t)$ for fixed $t\in[0,T]$ is given by

$$
\begin{aligned}
(\widetilde{L}(t)B)(\Theta)&=-\int_{\mathbb{R}^d}\Theta(x)\left(m(t)\delta B(\Theta;x)-z(t)B(\Theta e^{-\phi(x-\cdot)}+e^{-\phi(x-\cdot)}-1)\right)\mathrm{d}x\\
&=m(t)(L_0B)(\Theta)+z(t)(L_1B)(\Theta)
\end{aligned}
$$

with

$$
(L_0B)(\Theta)=-\int_{\mathbb{R}^d}\Theta(x)\delta B(\Theta;x)\mathrm{d}x
$$

and

$$(L_1 B)(\Theta) = \int_{\mathbb{R}^d} \Theta(x) B\left(\Theta e^{-\phi(x-\cdot)} + e^{-\phi(x-\cdot)} - 1)\right) \mathrm{d}x.$$

It is a simple matter to show that

$$\|\widetilde{L}(t)\|_{\alpha\alpha'} \leq \frac{\alpha^* \left(m(t) + z(t)\alpha^* \exp\left(\frac{\|\phi\|_{L^1}}{\alpha_*} - 1 \right) \right)}{\alpha - \alpha'} \tag{37}$$

where the norm of the operator is taken in $L(\mathbb{B}'_\alpha, \mathbb{B}'_{\alpha'})$ with $\alpha_* \leq \alpha' < \alpha \leq \alpha^*$ and $\mathbb{B}'_\alpha$ is defined in (17). This bound was shown in [11] for the case $m \equiv 1$.

Theorem 3.19. *Let m, z be continuous on $[0, T]$ and $0 \leq \phi \in L^1(\mathbb{R}^d)$ be symmetric. Then for each fixed $0 < \alpha_* < \alpha^*$ there exists a continuous and monotonically decreasing function $T : [\alpha_*, \alpha^*) \longrightarrow [0, T]$ such that for each $B_0 \in \mathbb{B}'_{\alpha^*}$ there exists a unique solution B_t of the Cauchy problem*

$$\frac{\partial B_t}{\partial t} = \widetilde{L}(t) B_t, \quad B_t|_{t=0} = B_0$$

in the scale $\mathbb{B}'_\alpha$ given by Remark 2.12.3.

Proof. Previous results, cf. (37), show that $\|\tilde{L}(t)\|_{\alpha\alpha'} \leq \dfrac{M}{\alpha - \alpha'}$ for some constant $M > 0$ independent of $t \in [0, T]$. Strong continuity follows from the inequality

$$\|\widetilde{L}(t)B - \widetilde{L}(s)B\|_{\alpha'} \leq |m(t) - m(s)| \|L_0 B\|_{\alpha'} + |z(t) - z(s)| \|L_1 B\|_{\alpha'}$$

for $\alpha' < \alpha$, $t, s \in [0, T]$, $B \in \mathbb{B}'_\alpha$ and the fact $L_0, L_1 \in L(\mathbb{B}'_\alpha, \mathbb{B}'_{\alpha'})$, which was shown in [11]. An application of Theorem 2.10 shows the existence of a unique evolution B_t in the scale $\mathbb{B}'_\alpha$. □

3.5 General birth-and-death dynamics

The aim of the last section is to prove the existence of solutions for the evolution of quasi-observables for the general birth-and-death dynamics heuristically given by the Markov pre-generator

$$\begin{aligned}(L(t)F)(\gamma) &= m(t) \sum_{x \in \gamma} d(x, \gamma \backslash x)(F(\gamma \backslash x) - F(\gamma)) \\ &\quad + \kappa(t) \int_{\mathbb{R}^d} b(x, \gamma)(F(\gamma \cup x) - F(\gamma)) \mathrm{d}x.\end{aligned}$$

For time independent m and κ this model was discussed recently in [8]. Under some conditions the authors proved the existence of evolution for quasi-observables via semigroup techniques. We will use this result together with Theorem 2.3 to construct an evolution of quasi-observables for time dependent coefficients $m = m(t)$ and $\kappa = \kappa(t)$. The assumptions on the model are the following

1. m, κ are non-negative, continuous on $\mathbb{R}_+$ and bounded.
2. $d(x,\gamma) \geq 0$ and $b(x,\gamma) \geq 0$ are locally integrable in $\eta \in \Gamma_0$, i.e.,
$$\int_{\Gamma_{0,\Lambda}^{(n)}} d(x,\eta) + b(x,\eta)\mathrm{d}\lambda(\eta) < \infty$$
for all $n \in \mathbb{N}$, $\Lambda \in \mathscr{B}_c(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$
3. There exists $\alpha^* \in \mathbb{R}$ and $a_1 \geq 1$ such that for all $\xi \in \Gamma_0$ and $x \in \mathbb{R}^d$
$$\sum_{x\in\xi}\int_{\Gamma_0} |K^{-1}d(x,\cdot\cup\xi\backslash x)|(\eta)e^{\alpha^*|\eta|}\mathrm{d}\lambda(\eta) \leq a_1 D(\xi)$$
4. There exists $a_2 > 0$ such that for all $\xi \in \Gamma_0$ and $x \in \mathbb{R}^d$
$$\sum_{x\in\xi}\int_{\Gamma_0} |K^{-1}b(x,\cdot\cup\xi\backslash x)|(\eta)e^{\alpha^*|\eta|}\mathrm{d}\lambda(\eta) \leq a_2 D(\xi)$$
5. There exists a constant $\nu > 0$ and $A > 0$ for which
$$d(x,\eta\backslash x) \leq Ae^{\nu|\eta|}$$
holds for each $\eta \in \Gamma_0$ and $x \in \mathbb{R}^d$.

The bound on d implies the bound
$$D(\eta) = \sum_{x\in\eta} d(x,\eta\backslash x) \leq A|\eta|e^{\nu|\eta|} \tag{38}$$
on D. Of course 5. can be replaced by $d(x,\eta\backslash x) \leq P(|\eta|)e^{\nu|\eta|}$ with P a polynomial. The expressions for quasi-observables are given by
$$\begin{aligned} & (\hat{L}(t)G)(\eta) \\ = \; & -m(t)\sum_{\xi\subset\eta}G(\xi)\sum_{x\in\xi}(K^{-1}d(x,\cdot\cup\xi\backslash x))(\eta\backslash\xi) \\ & +\kappa(t)\sum_{\xi\subset\eta}\int_{\mathbb{R}^d}G(\xi\cup x)(K^{-1}b(x,\cdot\cup\xi))(\eta\backslash\xi)\mathrm{d}x \\ = \; & m(t)\hat{L}_0 + \hat{L}_1(t) \end{aligned}$$
with $\hat{L}_1(t) = \hat{L}(t) - m(t)\hat{L}_0$ and $(L_0G)(\eta) = -D(\eta)G(\eta)$, where
$$D(\eta) = \sum_{x\in\eta} d(x,\eta\backslash x).$$

As a first step we will show that $\hat{L}(t)$ can be realized as bounded linear operators on $L(\mathbb{B}'_\alpha, \mathbb{B}'_{\alpha'})$.

Lemma 3.20. *$\hat{L}$ defines a bounded linear operator $\hat{L}(t) \in L(\mathbb{B}'_\alpha, \mathbb{B}'_{\alpha'})$ for $\alpha' + \nu < \alpha \le \alpha^*$ with*

$$\|\hat{L}(t)\|_{\alpha\alpha'} \le A\frac{m(t)a_1 + \kappa(t)a_2 e^{-\alpha'}}{\alpha - \alpha' - \nu}. \tag{39}$$

Moreover $\mathbb{R}_+ \ni t \longmapsto \hat{L}(t)$ is continuous in the uniform operator topology.

Proof. In [8] the authors have shown that $\hat{L}_1$ is relatively bounded with respect to $\hat{L}_0$. Similar calculations show that

$$\|\hat{L}_1(t)G\|_{\alpha'} \le (m(t)a_1 + \kappa(t)a_2 e^{-\alpha'} - m(t))\|\hat{L}_0 G\|_{\alpha'}.$$

Using (38) we obtain for $G \in \mathbb{B}'_\alpha$ with $\alpha' < \alpha$

$$\begin{aligned}
\|\hat{L}_0 G\|_{\alpha'} &\le \int_{\Gamma_0} D(\eta)|G(\eta)|e^{\alpha'|\eta|}\mathrm{d}\lambda(\eta) \\
&\le A\int_{\Gamma_0} |G(\eta)|e^{\alpha|\eta|}|\eta|e^{-(\alpha-\alpha'-\nu)|\eta|}\mathrm{d}\lambda(\eta) \\
&\le \frac{A}{\alpha - \alpha' - \nu}\|G\|_\alpha
\end{aligned}$$

for $\alpha > \alpha' + \nu$. Therefore

$$\|\hat{L}(t)\|_{\alpha\alpha'} \le m(t)\|\hat{L}_0\|_{\alpha\alpha'} + \|\hat{L}_1(t)\|_{\alpha\alpha'} \le A\frac{m(t)a_1 + \kappa(t)a_2 e^{-\alpha'}}{\alpha - \alpha' - \nu}$$

shows $\hat{L}(t) \in L(\mathbb{B}'_\alpha, \mathbb{B}'_{\alpha'})$. Continuity follows from the continuity of m, κ and the linear dependence on the parameters. □

(38) shows that it is possible to realise $\hat{L}_0$ and $\hat{L}(t)$ as an operator with the domain

$$\mathrm{Dom}(\hat{L})_\alpha = \{G \in \mathbb{B}'_\alpha : \quad D(\cdot)G(\cdot) \in \mathbb{B}'_\alpha\}$$

for $\alpha \le \alpha^*$.

Theorem 3.21. *Assume there exists $\alpha_* < \alpha^*$ satisfying*

$$a_1\overline{m} + a_2\overline{\kappa}e^{-\alpha_*} < \frac{3}{2},$$

where $\overline{m} = \sup_{t\ge 0} m(t)$ and $\overline{\kappa} = \sup_{t\ge 0} \kappa(t)$. Then there exists for each $\alpha_* \le \alpha' < \alpha \le \alpha^*$; $\alpha - \alpha' > \nu$ a unique evolution family $((\hat{U}(t,s))_{0\le s\le t}$ on $\mathbb{B}'_{\alpha'}$. Consequently for each $G_s \in \mathbb{B}'_\alpha$ the equation

$$\frac{\partial G_t}{\partial t} = \hat{L}(t)G_t, \quad s \le t, \quad G_t|_{t=s} = G_s$$

has a unique $\mathbb{B}'_\alpha-$valued solution $G_t = \hat{U}(t,s)G_s$ on $\mathbb{B}'_{\alpha'}$.

Proof. Last lemma implies that by [8] for each $\alpha_* \leq \alpha \leq \alpha^*$ there exists a unique holomorphic C_0−semigroup $(\hat{S}_t^\alpha(s))_{s\geq 0}$ with the generator $(\hat{L}(t), \mathrm{Dom}_\alpha(\hat{L}))$. The same arguments as in the proof of Theorem 3.15 show $\mathbb{B}'_{\alpha''}$-admissiblility for $\alpha < \alpha''$. The proof in [8] shows that this semigroup is a contraction semigroup on $\mathbb{R}_+$ which implies Kato-stability. Theorem 2.1 implies the existence of a unique evolution family and using again the same arguments as in the proof of Theorem 3.15 one can show that Theorem 2.3 is applicable. □

Remark 3.3. The reason to consider this simple case for the time dependent birth and death coefficients is the continuity of $t \longmapsto \hat{L}(t)$. For more general coefficients d_t and b_t one needs different assumptions, especially for the continuity.

3.6 Conclusion

Concerning correlation functions the major part is to construct an evolution family corresponding to the operator $A(t)$, which does not satisfy the bound $\|A(t)\|_{\alpha'\alpha} \leq \dfrac{M}{\alpha - \alpha'}$. Since the embeddings $\mathbb{B}_{\alpha'} \subset \mathbb{B}_\alpha$ are not dense for $\alpha' < \alpha$ it is not possible to apply Theorem 2.1 or Theorem 2.3. To overcome this problem in the time independent case it is possible to show via perturbation techniques, that A generates a C_0-semigroup $S(t)$, cf. [6] and [5], and afterwards consider the dual semigroup $S^*(t)$. Since the Banach spaces we are dealing with are not reflexive, the semigroup $S^*(t)$ will be in general only weak*-continuous. As shown in [21], one can restrict $S^*(t)$ to some invariant subspace $D(S^\odot)$ and obtain again a C_0-semigroup, the so-called sun-dual $S^\odot(t)$. To tackle the problem in the time dependent case we would propose to realize a similar approach for evolution families $U(t,s)$. One difficulty is that $A(t)U(t,s) = U(t,s)A(t)$ does not hold in general. The major question is how to characterize some invariant subspace $D(U^\odot)$ such that $D(U^\odot) \subset \bigcap_{t\in I} D(L(t))$ holds.

To show existence of global solutions we use general results for evolution families. Since they are not applicable for correlation functions further analysis is required. Special properties of the Banach spaces $\mathbb{B}_\alpha$ and of the operators $\hat{L}(t)$ and $L^\Delta(t)$ might be useful to prove approximation formulas in the spirit of [13, 22] and [23]. Consequently, such formulas might allow us to show the existence of an evolution of states. We should stress that only sub-poissonian solutions were considered, but in many applications clustering may appear and therefore the time evolution should also be considered in other classes of functions. Further steps can be dealing with Vlasov-scaling and existence of solutions for the corresponding equations. A next step of generalization is to deal with randomness in this models, meaning that the coefficients $z, m, a^\pm, d$ and b should be random variables. One motivation is the fact that in applications it is not possible to precisely measure the corresponding rates, but also fluctuations could be taken into account. For applications it is important to understand the properties of the solutions of our equations. Like in [**?**] one could analyze properties of solutions to integro-differential equations of the form (26) or even non-linear versions. The case of periodic coefficients play a special role in the understanding of the behavior of the solutions. Also other spaces than (25) can be taken to show the existence of solutions, e.g. $X = C([0,T]; C_b(\mathbb{R}^d))$.

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References

[1] S. Albeverio, Y. G. Kondratiev, M. Röckner. *Analysis and geometry on configuration spaces*, J. Funct. Anal. (1998), 154(2): 444–500

[2] O. Caps. *Evolution Equations in Scales of Banach Spaces*, Teubner, Stuttgart/Leipzig/Wiesbaden (2002)

[3] D. Finkelshtein. *Functional evolutions for homogeneous stationary death-immigration spatial dynamics*, Methods Funct. Anal. Topology 17 (2011), no. 4: 300–318

[4] D. Finkelshtein, Y. Kondratiev, Y. Kozitsky. *Glauber Dynamics in Continuum: A Constructive Approach to Evolution of States*, Discrete and Cont. Dynam. Syst. - Ser. A 33 (2013), no. 4: 1431–1450.

[5] D. Finkelshtein, Y. Kondratiev, Y. Kozitsky, O. Kutoviy. *Markov Evolution of Continuous Particle Systems with Dispersion and Competition*, arXiv:1112.0895v2 [math-ph]

[6] D. Finkelshtein, Y. Kondratiev, O. Kutoviy. *Correlation functions evolution for the Glauber dynamics in continuum*, Semigroup Forum 85 (2012), no. 2: 289–306

[7] D. Finkelshtein, Y. Kondratiev, O. Kutoviy. *Individual based model with competition in spatial ecology*, J. MATH. ANAL. Vol. 41, No. 1, pp. 297–317

[8] D. Finkelshtein, Y. Kondratiev, O. Kutoviy. *Semigroup approach to birth-and-death stochastic dynamics in continuum*, J. Funct. Anal. 262 (2012), no. 3: 1274–1308

[9] D. Finkelshtein, Y. Kondratiev, O. Kutoviy. *Vlasov scaling for the Glauber dynamics in continuum*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 14 (2011), no. 4: 537–569

[10] D. Finkelshtein, Y. Kondratiev, M. Oliveira. *Markov evolutions and hierarchical equations in the continuum I. One-component systems*, J. Evol. Equ. 9 (2009), no. 2: 197–233

[11] D. Finkelshtein, Y. Kondratiev, M. Oliveira. *Glauber dynamics in the continuum via generating functional evolution*, Complex Anal. Oper. Theory 6 (2012), no. 4: 923–945

[12] D. Finkelshtein, Y. Kondratiev, O. Kutoviy, E. Zhizhina *An approximative approach for construction of the Glauber dynamics in continuum*, Mathematische Nachrichten vol. 285 (2012), no. 2: 223–225

[13] T. Kato. *Linear evolution equations of hyperbolic type, II*, J. Math. Soc. Japan Vol 25, No. 4 (1973)

[14] Y. Kondratiev, T. Kuna. *Harmonic Analysis on Configuration space I.General Theory*, Infinite Dim. Analysis, Quantum Prob. and Related Topics (2002), Vol. 5, No. 2, 201–232

[15] Y. Kondratiev, T. Kuna, M. Oliveira *Holomorphic Bogoliubov functionals for interacting particle systems in continuum*, Journal of Functional Analysis, 238 (2006): 375–404

[16] Y. Kondratiev, O. Kutoviy, R. Minlos. *On non-equilibrium stochastic dynamics for interacting particle systems in continuum*, J. Funct. Anal. 255 (2008): 200–227

[17] Y. Kondratiev, O. Kutoviy, E. Zhizhina. *Nonequilibrium Glauber-type dynamics in continuum*, J. Math. Phys. 47 (2006), 113501, 17

[18] Y. Kondratiev, O. Kutoviy, S. Pirogov. *Correlation functions and invariant measures in Continuous Contact Model*, Inf. Dim. Analysis, Quantum Prob. and Related Topics (2008), Vol. 11, No. 2: 231–258

[19] Y. Kondratiev, E. Lytvynov. *Glauber dynamics of continuous particle systems*, Ann. Inst. H: Poincare Probab. Statist. 41, no.4 (2005): 685–702

[20] Y. Kondratiev, A. Skorohod. *On Contact Processes in continuum*, Inf. Dim. Analysis, Quantum Prob. and Related Topics (2006), Vol. 9, No. 2: 187–198

[21] J. Neerven. *The Adjoint of a Semigroup of Linear Operators*, Springer-Verlag, Berlin-Heidelberg (1992), Lecture Notes in Mathematics 1529

[22] G. Nickel. *Evolutions Semigroups for Nonautonomous Cauchy Problems*, Mancorp Publishing, Inc. (1996), 73–95

[23] G. Nickel, R. Schnaubelt. *An extension of Kato's stability condition for nonatuonomous Cauchy Problems*, Taiwanese Journal of Mathematics, Vol. 2, No. 4 (1998): 483–496

[24] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer Verlag, New York (1983)

[25] O. Pugachev. *The Space of Simple Configurations is Polish*, Mathematical Notes, Vol. 71 no. 4 (2002): 530–537

[26] M. Safonov. *The Abstract Cauchy-Kovalevskaya Theorem in a Weighted Banach Space*, Communications on Pure and Applied Mathematics, Vol. XLVII (1995): 629–637

[27] D. Surgailis. *On Poisson multiple stochastic integrals and associated equilibrium Markov processes*, in Theory and application of random fields (Bangalore 1982), vol. 49 of Lecture Notes in Control and Inform. Sci., Springer, Berlin, 1983: 233–248

[28] D. Surgailis. *On multiple Poisson stochastic integrals and associated Markov semigroups*, Probab. Math. Statist., 4 (1984): 217–239