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Statistical approach for stochastic evolutions of complex systems in the continuum

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Abstract. We present a general background for the study of complex systems in the continuum and explain the mathematical tools to deal with stochastic evolutions in the continuum. The statistical description of Markov dynamics of complex systems in the continuum is described in details. The review of recent developments for birth-and-death evolutions is given.

1 Complex systems in the continuum

In recent decades, different brunches of natural and life sciences have been addressing to a unifying point of view on a number of phenomena occurring in systems composed of interacting subunits. This leads to formation of a interdisciplinary science which is referred to as the theory of complex systems. It provides reciprocation of concepts and tools involving wide spectrum of applications as well as various mathematical theories such that statistical mechanics, probability, nonlinear dynamics, chaos theory, numerical simulation and many others.

Nowadays complex systems theory is a quickly growing interdisciplinary area with a very broad spectrum of motivations and applications. For instance, having in mind biological applications, S. Levin [40] characterized complex adaptive systems by such properties as diversity and individuality of components, localized interactions among components, and the outcomes of interactions used for replication or enhancement of components. We will use a more general informal description of a complex system as a specific collection of interacting elements which has so-called collective behavior that means appearance of system properties which are not peculiar to inner nature of each element itself. The significant physical example of such properties is thermodynamical effects which were a background for creation by L. Boltzmann of statistical physics as a mathematical language for studying complex systems of molecules.

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We assume that all elements of a complex system are identical by properties and possibilities. Thus, one can model these elements as points in a proper space whereas the complex system will be modeled as a discrete set in this space. Mathematically this means that for the study of complex systems the proper language and techniques are delivered by the interacting particle models which form a rich and powerful direction in modern stochastic and infinite dimensional analysis. Interacting particle systems have a wide use as models in condensed matter physics, chemical kinetics, population biology, ecology (individual based models), sociology and economics (agent based models). For instance a population in biology or ecology may be represented by a configuration of organisms located in a proper habitat.

In spite of completely different orders of numbers of elements in real physical, biological, social, and other systems (typical numbers start from 10^{23} for molecules and, say, 10^5 for plants) their complexities have analogous phenomena and need similar mathematical methods. One of them consists in mathematical approximation of a huge but finite real-world system by an infinite system realized in an infinite space. This approach was successfully approved to the thermodynamic limit for models of statistical physics and appeared quite useful for the ecological modeling in the infinite habitat to avoid boundary effects in a population evolution.

Therefore, our phase space for the mathematical description should consist of countable sets from an underlying space. This space itself may have discrete or continuous nature that leads to segregation of the world of complex systems on two big classes. Discrete models correspond to systems whose elements can occupy some prescribing countable set of positions, for example, vertices of the lattice $\mathbb{Z}^d$ or, more generally, of some graph embedded to $\mathbb{R}^d$. These models are widely studied and the corresponding theories were realized in numerous publications, see e.g. [41, 42] and the references therein. Continuous models, or models in the continuum, were studied not so intensively and broadly. We concentrate our attention exactly on continuous models of systems whose elements may occupy any points in Eucledian space $\mathbb{R}^d$. Having in mind that real elements have physical sizes we will consider only the so-called locally finite subsets of the underlying space $\mathbb{R}^d$, that means that in any bounded region we assume to have a finite number of the elements. Another our restriction will be prohibition of multiple elements in the same position of the space.

We will consider systems of elements of the same type only. The mathematical realization of considered approaches may be successfully extended to multi-type systems, meanwhile such systems will have more rich qualitative properties and will be an object of interest for applications. Some particular results can be found e.g. in [12, 13, 23].

2 Mathematical description of complex systems

We proceed to the mathematical realization of complex systems. Let $\mathcal{B}(\mathbb{R}^d)$ be the family of all Borel sets in $\mathbb{R}^d$, $d \geq 1$; $\mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ denotes the system of all bounded sets from $\mathcal{B}(\mathbb{R}^d)$.

The configuration space over space $\mathbb{R}^d$ consists of all locally finite subsets (configurations) of $\mathbb{R}^d$. Namely,

$$\Gamma = \Gamma\big(\mathbb{R}^d\big) := \Big\{\gamma \subset \mathbb{R}^d \;\Big|\; |\gamma_\Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)\Big\}. \tag{1}$$

Here $|\cdot|$ means the cardinality of a set, and $\gamma_\Lambda := \gamma \cap \Lambda$. We may identify each $\gamma \in \Gamma$ with the non-negative Radon measure $\sum_{x\in\gamma} \delta_x \in \mathcal{M}(\mathbb{R}^d)$, where δ_x is the Dirac measure with unit mass at x, $\sum_{x\in\emptyset} \delta_x$ is, by definition, the zero measure, and $\mathcal{M}(\mathbb{R}^d)$ denotes the space of all non-negative Radon measures on $\mathcal{B}(\mathbb{R}^d)$. This identification allows to endow Γ with the topology induced by the vague topology on $\mathcal{M}(\mathbb{R}^d)$, i.e. the weakest topology on Γ with respect to which all mappings

$$\Gamma \ni \gamma \mapsto \sum_{x\in\gamma} f(x) \in \mathbb{R} \tag{2}$$

are continuous for any $f \in C_0(\mathbb{R}^d)$ that is the set of all continuous functions on $\mathbb{R}^d$ with compact supports. It is worth noting the vague topology may be metrizable in such a way that Γ becomes a Polish space (see e.g. [32] and references therein).

The corresponding (to the vague topology) Borel σ-algebra $\mathcal{B}(\Gamma)$ appears the smallest σ-algebra for which all mappings

$$\Gamma \ni \gamma \mapsto N_\Lambda(\gamma) := |\gamma_\Lambda| \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \tag{3}$$

are measurable for any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$, see e.g. [1]. This σ-algebra may be generated by the sets

$$Q(\Lambda, n) := \big\{\gamma \in \Gamma \;\big|\; N_\Lambda(\gamma) = |\gamma_\Lambda| = n\big\}, \qquad \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d), n \in \mathbb{N}_0. \tag{4}$$

Clearly, for any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$,

$$\Gamma = \bigsqcup_{n\in\mathbb{N}_0} Q(\Lambda, n). \tag{5}$$

Among all measurable functions $F : \Gamma \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ we mark out the set $\mathcal{F}_0(\Gamma)$ consisting of such of them for which $|F(\gamma)| < \infty$ at least for all $|\gamma| < \infty$. The important subset of $\mathcal{F}_0(\Gamma)$ formed by cylindric functions on Γ. Any such a function is characterized by a set $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ such that $F(\gamma) = F(\gamma_\Lambda)$ for all $\gamma \in \Gamma$. The class of cylindric functions we denote by $\mathcal{F}_{\mathrm{cyl}}(\Gamma) \subset \mathcal{F}_0(\Gamma)$.

Functions on Γ are usually called *observables*. This notion is borrowed from statistical physics and means that typically in course of empirical investigation we may estimate, check, see only some quantities of a whole system rather then look on the system itself.

Example 2.1. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ and consider the so-called *linear function* on Γ, cf. (2),

$$\langle \varphi, \gamma \rangle := \begin{cases} \displaystyle\sum_{x\in\gamma} \varphi(x), & \text{if } \displaystyle\sum_{x\in\gamma} |\varphi(x)| < \infty, \quad \gamma \in \Gamma, \\ +\infty, & \text{otherwise.} \end{cases} \tag{6}$$

Then, evidently, $\langle\varphi,\cdot\rangle \in \mathcal{F}_0(\Gamma)$. If, additionally, $\varphi \in C_0(\mathbb{R}^d)$ then $\langle\varphi,\cdot\rangle \in \mathcal{F}_{\mathrm{cyl}}(\Gamma)$. Not that for e.g. $\varphi(x) = \|x\|_{\mathbb{R}^d}$ (the Euclidean norm in $\mathbb{R}^d$) we have that $\langle\varphi,\gamma\rangle = \infty$ for any infinite $\gamma \in \Gamma$.

Example 2.2. Let $\phi : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ and ϕ is an even function, namely, $\phi(-x) = \phi(x)$, $x \in \mathbb{R}^d$. Then one can consider the so-called *energy function*

$$E^\phi(\gamma) := \begin{cases} \displaystyle\sum_{\{x,y\}\subset\gamma} \phi(x-y), & \text{if} \displaystyle\sum_{\{x,y\}\subset\gamma} |\phi(x-y)| < \infty, \quad \gamma \in \Gamma, \\ +\infty, & \text{otherwise.} \end{cases} \tag{7}$$

Clearly, $E^\phi \in \mathcal{F}_0(\Gamma)$. However, even for ϕ with a compact support, E^ϕ will not be a cylindric function.

As we discussed before, any configuration γ represents some system of elements in a real-world application. Typically, investigators are not able to take into account exact positions of all elements due to huge number of them. For quantitative and qualitative analysis of a system researchers mostly need some its statistical characteristics such as density, correlations, spatial structures and so on. This leads to the so-called statistical description of complex systems when people study distributions of countable sets in an underlying space instead of sets themselves. Moreover, the main idea in Boltzmann's approach to thermodynamics based on giving up the description in terms of evolution for groups of molecules and using statistical interpretation of molecules motion laws. Therefore, the crucial role for studying of complex systems plays distributions (probability measures) on the space of configurations. In statistical physics these measures usually called *states* that accentuates their role for description of systems under consideration.

We denote the class of all probability measures on $\bigl(\Gamma, \mathcal{B}(\Gamma)\bigr)$ by $\mathcal{M}^1(\Gamma)$. Given a distribution $\mu \in \mathcal{M}^1(\Gamma)$ one can consider a collection of random variables $N_\Lambda(\cdot)$, $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ defined in (3). They describe random numbers of elements inside bounded regions. The natural assumption is that these random variables should have finite moments. Thus, we consider the class $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ of all measures from $\mathcal{M}^1(\Gamma)$ such that

$$\int_\Gamma |\gamma_\Lambda|^n \, d\mu(\gamma) < \infty, \qquad \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d), n \in \mathbb{N}. \tag{8}$$

Example 2.3. Let σ be a Radon measure on $\bigl(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\bigr)$ which has not atoms. Then the *Poisson measure* π_σ with intensity measure σ is defined on $\mathcal{B}(\Gamma)$ by equality

$$\pi_\sigma\bigl(Q(\Lambda,n)\bigr) = \frac{\bigl(\sigma(\Lambda)\bigr)^n}{n!} \exp\bigl\{-\sigma(\Lambda)\bigr\}, \qquad \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d), n \in \mathbb{N}_0. \tag{9}$$

On the other words, the random variables N_Λ have Poissonian distribution with mean value $\sigma(\Lambda)$, $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$. Note that by the Rényi theorem [30, 52] a measure π_σ will be Poissonian if (9) holds for $n = 0$ only. In the case then $d\sigma(x) = \rho(x)\,dx$ one can say about nonhomogeneous Poisson measure π_ρ with density (or intensity) ρ. This notion goes back to the famous Campbell formula [8, 9] which states that

$$\int_\Gamma \langle\varphi,\gamma\rangle \, d\pi_\rho(\gamma) = \int_{\mathbb{R}^d} \varphi(x)\rho(x)\,dx, \tag{10}$$

if only the right hand side of (10) is well-defined. The generalization of (10) is the Mecke identity [44]

$$\int_\Gamma \sum_{x\in\gamma} h(x,\gamma)\, d\pi_\sigma(\gamma) = \int_\Gamma \int_{\mathbb{R}^d} h(x,\gamma\cup x)\, d\sigma(x)\, d\pi_\sigma(\gamma), \tag{11}$$

which holds for all measurable nonnegative functions $h : \mathbb{R}^d \times \Gamma \to \mathbb{R}$. Here and in the sequel we will omit brackets for the one-point set $\{x\}$. In [44], it was shown that the Mecke identity is a characterization identity for the Poisson measure. In the case $\rho(x) = z > 0$, $x \in \mathbb{R}^d$ one can say about the homogeneous Poisson distribution (measure) π_z with constant intensity z. We will omit subindex for the case $z = 1$, namely, $\pi := \pi_1 = \pi_{dx}$. Note that the property (8) is followed from (11) easily.

Example 2.4. Let ϕ be as in Example 2.2 and suppose that the energy given by (7) is *stable*: there exists $B \geq 0$ such that, for any $|\gamma| < \infty$, $E^\phi(\gamma) \geq -B|\gamma|$. An example of such ϕ my be given by the expansion

$$\phi(x) = \phi^+(x) + \phi^p(x), \quad x \in \mathbb{R}^d, \tag{12}$$

where $\phi^+ \geq 0$ whereas ϕ^p is a positive defined function on $\mathbb{R}^d$ (the Fourier transform of a measure on $\mathbb{R}^d$), see e.g. [24, 53]. Fix any $z > 0$ and define the *Gibbs measure* $\mu \in \mathcal{M}^1(\Gamma)$ with potential ϕ and activity parameter z as a measure which satisfies the following generalization of the Mecke identity:

$$\int_\Gamma \sum_{x\in\gamma} h(x,\gamma)\, d\mu(\gamma) = \int_\Gamma \int_{\mathbb{R}^d} h(x,\gamma\cup x) \exp\{-E^\phi(x,\gamma)\}\, z dx\, d\mu(\gamma), \tag{13}$$

where

$$E^\phi(x,\gamma) := \langle \phi(x-\cdot), \gamma\rangle = \sum_{y\in\gamma} \phi(x-y), \quad \gamma\in\Gamma, x\in\mathbb{R}^d\setminus\gamma. \tag{14}$$

The identity (13) is called the Georgii–Nguyen–Zessin identity, see [28, 46]. If potential ϕ is additionally satisfied the so-called integrability condition

$$\beta := \int_{\mathbb{R}^d} \left| e^{-\phi(x)} - 1 \right| dx < \infty, \tag{15}$$

then it can checked that the condition (8) for the Gibbs measure holds. Note that under conditions $z\beta \leq (2e)^{-1}$ there exists a unique measure on $\bigl(\Gamma, \mathcal{B}(\Gamma)\bigr)$ which satisfies (13). Heuristically, the measure μ may be given by the formula

$$d\mu(\gamma) = \frac{1}{Z} e^{-E^\phi(\gamma)}\, d\pi_z(\gamma), \tag{16}$$

where Z is a normalizing factor. To give rigorous meaning for (16) it is possible to use the so-called DLR-approach (named after R. L. Dobrushin, O. Lanford, D. Ruelle), see e.g. [2] and references therein. As was shown in [46], this approach gives the equivalent definition of the Gibbs measures which satisfies (13).

Note that (16) could have a rigorous sense if we restrict our attention on the space of configuration which belong to a bounded domain $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$. The space of such (finite) configurations will be denoted by $\Gamma(\Lambda)$. The σ-algebra $\mathcal{B}(\Gamma_\Lambda)$ may be generated by family of mappings $\Gamma(\Lambda) \ni \gamma \mapsto N_{\Lambda'}(\gamma) \in \mathbb{N}_0$, $\Lambda' \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$, $\Lambda' \subset \Lambda$. A measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is called locally absolutely continuous with respect to the Poisson measure π if for any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ the projection of μ onto $\Gamma(\Lambda)$ is absolutely continuous with respect to (w.r.t.) the projection of π onto $\Gamma(\Lambda)$. More precisely, if we consider the projection mapping $p_\Lambda : \Gamma \to \Gamma(\Lambda)$, $p_\Lambda(\gamma) := \gamma_\Lambda$ then $\mu^\Lambda := \mu \circ p_\Lambda^{-1}$ is absolutely continuous w.r.t. $\pi_\Lambda := \pi \circ p_\Lambda^{-1}$.

Remark 2.1. Having in mind (16), it is possible to derive from (13) that the Gibbs measure from Example 2.4 is locally absolutely continuous w.r.t. the Poisson measure, see e.g. [14] for the more general case.

By e.g. [31], for any $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ which is locally absolutely continuous w.r.t the Poisson measure there exists the family of (symmetric) *correlation functions* $k_\mu^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}_+ := [0,\infty)$ which defined as follows. For any symmetric function $f^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}$ with a finite support the following equality holds

$$\int_\Gamma \sum_{\{x_1,\dots,x_n\}\subset\gamma} f^{(n)}(x_1,\dots,x_n)\,d\mu(\gamma) = \frac{1}{n!}\int_{(\mathbb{R}^d)^n} f^{(n)}(x_1,\dots,x_n)k_\mu^{(n)}(x_1,\dots,x_n)\,dx_1\dots dx_n \quad (17)$$

for $n \in \mathbb{N}$, and $k_\mu^{(0)} := 1$.

The meaning of the notion of correlation functions is the following: the correlation function $k_\mu^{(n)}(x_1,\dots,x_n)$ describes the nonnormalized density of probability to have points of our systems in the positions $x_1,\dots,x_n$. More precisely,

Proposition 2.1. *Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be locally absolutely continuous w.r.t. the Poisson measure, and let $k_\mu^{(n)}$, $n \in \mathbb{N}$ be the corresponding correlation functions. Denote by $B_r(x)$ a ball in $\mathbb{R}^d$ with center at an $x \in \mathbb{R}^d$ and a radius $r > 0$. Then, for any $n \in \mathbb{N}$ and for a.a. $x_1,\dots,x_n \in \mathbb{R}^d$,*

$$k_\mu^{(n)}(x_1,\dots,x_n) = \lim_{r\to 0}\frac{\mu\bigl(\{\gamma\in\Gamma \bigm| |\gamma\cap B_r(x_1)| = \ldots = |\gamma\cap B_r(x_n)| = 1\}\bigr)}{\bigl(B_r(0)\bigr)^n}. \quad (18)$$

Correlation functions describe properties of a probability measure much more precisely than measure's moments (8). For instance, one can obtain measures of sets $Q(\Lambda,n)$, defined in (4), in terms of correlation functions or, in the other words, one can fully describe the distribution of random variables $N_\Lambda(\cdot)$ defined in (3). Namely,

Proposition 2.2. *For any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$, $n \in \mathbb{N}_0$,*

$$\mu\bigl(Q(\Lambda,n)\bigr) = \frac{1}{n!}\sum_{m=0}^\infty \frac{(-1)^m}{m!}\int_{\Lambda^{n+m}} k_\mu^{(n+m)}(x_1,\dots,x_{n+m})\,dx_1\dots dx_{n+m}. \quad (19)$$

Remark 2.2. Iterating the Mecke identity (11), it can be easily shown that

$$k_{\pi_\rho}^{(n)}(x_1, \dots, x_n) = \prod_{i=1}^{n} \rho(x_i), \tag{20}$$

in particular,

$$k_{\pi_z}^{(n)}(x_1, \dots, x_n) \equiv z^n. \tag{21}$$

Substituting the right hand side (r.h.s.) of (20) to (19) one can obtain (9).

Remark 2.3. Note that if potential ϕ from Example 2.4 satisfies to (12), (15) then, by [54], there exists $C = C(z, \phi) > 0$ such that for μ defined by (13)

$$k_\mu^{(n)}(x_1, \dots, x_n) \le C^n, \qquad x_1, \dots, x_n \in \mathbb{R}^d. \tag{22}$$

The inequality (22) is referred to as the Ruelle bound.

We dealt with symmetric function of n variables from $\mathbb{R}^d$, hence, they can be considered as functions on n-point subsets from $\mathbb{R}^d$. We proceed now to the exact constructions.

The space of n-point configurations in $Y \in \mathcal{B}(\mathbb{R}^d)$ is defined by

$$\Gamma^{(n)}(Y) := \bigl\{ \eta \subset Y \bigm| |\eta| = n \bigr\}, \qquad n \in \mathbb{N}.$$

We put $\Gamma^{(0)}(Y) := \{\emptyset\}$. As a set, $\Gamma^{(n)}(Y)$ may be identified with the symmetrization of

$$\widetilde{Y^n} = \bigl\{ (x_1, \dots, x_n) \in Y^n \bigm| x_k \ne x_l \text{ if } k \ne l \bigr\}.$$

Hence, one can introduce the corresponding Borel σ-algebra, which we denote by $\mathcal{B}\bigl(\Gamma^{(n)}(Y)\bigr)$. The space of finite configurations in $Y \in \mathcal{B}(\mathbb{R}^d)$ is defined as

$$\Gamma_0(Y) := \bigsqcup_{n \in \mathbb{N}_0} \Gamma^{(n)}(Y). \tag{23}$$

This space is equipped with the topology of the disjoint union. Let $\mathcal{B}\bigl(\Gamma_0(Y)\bigr)$ denote the corresponding Borel σ-algebra. In the case of $Y = \mathbb{R}^d$ we will omit the index Y in the previously defined notations. Namely,

$$\Gamma_0 := \Gamma_0(\mathbb{R}^d), \qquad \Gamma^{(n)} := \Gamma^{(n)}(\mathbb{R}^d), \quad n \in \mathbb{N}_0. \tag{24}$$

The restriction of the Lebesgue product measure $(dx)^n$ to $\bigl(\Gamma^{(n)}, \mathcal{B}(\Gamma^{(n)})\bigr)$ we denote by $m^{(n)}$. We set $m^{(0)} := \delta_{\{\emptyset\}}$. The Lebesgue–Poisson measure λ on Γ_0 is defined by

$$\lambda := \sum_{n=0}^{\infty} \frac{1}{n!} m^{(n)}. \tag{25}$$

For any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ the restriction of λ to $\Gamma_0(\Lambda) = \Gamma(\Lambda)$ will be also denoted by λ.

Remark 2.4. The space $\bigl(\Gamma, \mathcal{B}(\Gamma)\bigr)$ is the projective limit of the family of measurable spaces $\bigl\{ \bigl(\Gamma(\Lambda), \mathcal{B}(\Gamma(\Lambda))\bigr) \bigr\}_{\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)}$. The Poisson measure π on $\bigl(\Gamma, \mathcal{B}(\Gamma)\bigr)$ from Example 2.3 may be defined as the projective limit of the family of measures $\{\pi^\Lambda\}_{\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)}$, where $\pi^\Lambda := e^{-m(\Lambda)}\lambda$ is the probability measure on $\bigl(\Gamma(\Lambda), \mathcal{B}(\Gamma(\Lambda))\bigr)$ and $m(\Lambda)$ is the Lebesgue measure of $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ (see e.g. [1] for details).

Functions on Γ_0 will be called *quasi-observables*. Any $\mathcal{B}(\Gamma_0)$-measurable function G on Γ_0, in fact, is defined by a sequence of functions $\{G^{(n)}\}_{n\in\mathbb{N}_0}$ where $G^{(n)}$ is a $\mathcal{B}(\Gamma^{(n)})$-measurable function on $\Gamma^{(n)}$. We preserve the same notation for the function $G^{(n)}$ considered as a symmetric function on $(\mathbb{R}^d)^n$. Note that $G^{(0)} \in \mathbb{R}$.

A set $M \in \mathcal{B}(\Gamma_0)$ is called bounded if there exists $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that

$$M \subset \bigsqcup_{n=0}^{N} \Gamma^{(n)}(\Lambda). \tag{26}$$

The set of bounded measurable functions on Γ_0 with bounded support we denote by $B_{\mathrm{bs}}(\Gamma_0)$, i.e., $G \in B_{\mathrm{bs}}(\Gamma_0)$ iff $G \restriction_{\Gamma_0 \setminus M} = 0$ for some bounded $M \in \mathcal{B}(\Gamma_0)$. For any $G \in B_{\mathrm{bs}}(\Gamma_0)$ the functions $G^{(n)}$ have finite supports in $(\mathbb{R}^d)^n$ and may be substituted into (17). But, additionally, the sequence of $G^{(n)}$ vanishes for big n. Therefore, one can summarize equalities (17) by $n \in \mathbb{N}_0$. This leads to the following definition.

Let $G \in B_{\mathrm{bs}}(\Gamma_0)$, then we define the function $KG : \Gamma \to \mathbb{R}$ such that:

$$(KG)(\gamma) := \sum_{\eta \Subset \gamma} G(\eta) \tag{27}$$

$$= G^{(0)} + \sum_{n=1}^{\infty} \sum_{\{x_1,\dots,x_n\}\subset\gamma} G^{(n)}(x_1,\dots,x_n), \quad \gamma \in \Gamma, \tag{28}$$

see e.g. [31, 38, 39]. The summation in (27) is taken over all finite subconfigurations $\eta \in \Gamma_0$ of the (infinite) configuration $\gamma \in \Gamma$; we denote this by the symbol, $\eta \Subset \gamma$. The mapping K is linear, positivity preserving, and invertible, with

$$(K^{-1}F)(\eta) := \sum_{\xi\subset\eta} (-1)^{|\eta\setminus\xi|} F(\xi), \quad \eta \in \Gamma_0. \tag{29}$$

By [31], for any $G \in B_{\mathrm{bs}}(\Gamma_0)$, $KG \in \mathcal{F}_{\mathrm{cyl}}(\Gamma)$, moreover, there exists $C = C(G) > 0$, $\Lambda = \Lambda(G) \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$, and $N = N(G) \in \mathbb{N}$ such that

$$|KG(\gamma)| \le C\big(1 + |\gamma_\Lambda|\big)^N, \quad \gamma \in \Gamma. \tag{30}$$

The expression (27) can be extended to the class of all nonnegative measurable $G : \Gamma_0 \to \mathbb{R}_+$, in this case, evidently, $KG \in \mathcal{F}_0(\Gamma)$. Stress that the left hand side (l.h.s.) of (29) has a meaning for any $F \in \mathcal{F}_0(\Gamma)$, moreover, in this case $(KK^{-1}F)(\gamma) = F(\gamma)$ for any $\gamma \in \Gamma_0$.

For G as above we may summarize (17) by n and rewrite the result in a compact form:

$$\int_{\Gamma} (KG)(\gamma) d\mu(\gamma) = \int_{\Gamma_0} G(\eta) k_\mu(\eta) d\lambda(\eta). \tag{31}$$

As was shown in [31], the equality (27) may be extended on all functions G such that the l.h.s. of (31) is finite. In this case (27) holds for μ-a.a. $\gamma \in \Gamma$ and (31) holds too.

Remark 2.5. The equality (31) may be considered as definition of the correlation functional k_μ. In fact, the definition of correlation functions in statistical physics, given by N. N. Bogolyubov in [5], based on a similar relation. More precisely, consider for a $\mathcal{B}(\mathbb{R}^d)$-measurable function f the so-called coherent state, given as a function on Γ_0 by

$$e_\lambda(f,\eta) := \prod_{x\in\eta} f(x),\ \ \eta\in\Gamma_0\setminus\{\emptyset\}, \qquad e_\lambda(f,\emptyset) := 1.$$

Then for any $f \in C_0(\mathbb{R}^d)$ we have the point-wise equality

$$\big(Ke_\lambda(f)\big)(\gamma) = \prod_{x\in\gamma}\big(1+f(x)\big), \quad \eta\in\Gamma_0. \tag{32}$$

As a result, the correlation functions of different orders may be considered as kernels of a Taylor-type expansion

$$\begin{aligned}\int_\Gamma \prod_{x\in\gamma}\big(1+f(x)\big)\,d\mu(\gamma) &= 1+\sum_{n=1}^\infty \frac{1}{n!}\int_{(\mathbb{R}^d)^n}\prod_{i=1}^n f(x_i)k_\mu^{(n)}(x_1,\ldots,x_n)\,dx_1\ldots dx_n\\ &= \int_{\Gamma_0} e_\lambda(f,\eta)k_\mu(\eta)\,d\lambda(\eta).\end{aligned} \tag{33}$$

Remark 2.6. By (23)–(25), we have that for any $f\in L^1(\mathbb{R}^d,dx)$

$$\int_{\Gamma_0} e_\lambda(f,\eta)d\lambda(\eta) = \exp\Big\{\int_{\mathbb{R}^d} f(x)dx\Big\}. \tag{34}$$

As a result, taking into account (20), we obtain from (33) the expression for the Laplace transform of the Poisson measure

$$\begin{aligned}\int_\Gamma e^{-\langle\varphi,\gamma\rangle}\,d\pi_\rho(\gamma) &= \int_{\Gamma_0} e_\lambda\big(e^{-\varphi(x)}-1,\eta\big)e_\lambda(\rho,\eta)\,d\lambda(\eta)\\ &= \exp\Big\{-\int_{\mathbb{R}^d}\big(1-e^{-\varphi(x)}\big)\rho(x)dx\Big\}, \qquad \varphi\in C_0(\mathbb{R}^d).\end{aligned} \tag{35}$$

Remark 2.7. Of course, to obtain convergence of the expansion (33) for, say, $f\in L^1(\mathbb{R}^d,dx)$ we need some bounds for the correlation functions $k_\mu^{(n)}$. For example, if the generalized Ruelle bound holds, that is, cf. (22),

$$k_\mu^{(n)}(x_1,\ldots,x_n) \le AC^n(n!)^{1-\delta}, \quad x_1,\ldots,x_n\in\mathbb{R}^d \tag{36}$$

for some $A, C>0$, $\delta\in(0,1]$ independent on n, then the l.h.s. of (33) may be estimated by the expression

$$1+A\sum_{n=1}^\infty \frac{\big(C\|f\|_{L^1(\mathbb{R}^d)}\big)^n}{(n!)^\delta} < \infty.$$

For a given system of functions $k^{(n)}$ on $(\mathbb{R}^d)^n$ the question about existence and uniqueness of a probability measure μ on Γ which has correlation functions $k_\mu^{(n)} = k^{(n)}$ is an analog of the moment problem in classical analysis. One of the group of results in this area was obtained by A. Lenard.

Proposition 2.3 ([39], [37]). *Let* $k : \Gamma_0 \to \mathbb{R}$.

1. *Suppose that* k *is a positive definite function, that means that for any* $G \in B_{\mathrm{bs}}(\Gamma_0)$ *such that* $(KG)(\gamma) \geq 0$ *for all* $\gamma \in \Gamma$ *the following inequality holds*

$$\int_{\Gamma_0} G(\eta)k(\eta)\, d\lambda(\eta) \geq 0. \tag{37}$$

Suppose also that $k(\emptyset) = 1$. *Then there exists at least one measure* $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ *such that* $k = k_\mu$.

2. *For any* $n \in \mathbb{N}$, $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$, *we set*

$$s_n^\Lambda := \frac{1}{n!} \int_{\Lambda^n} k^{(n)}(x_1, \ldots, x_n)\, dx_1 \ldots dx_n.$$

Suppose that for all $m \in \mathbb{N}$, $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$

$$\sum_{n \in \mathbb{N}} \left(s_{n+m}^\Lambda\right)^{-\frac{1}{n}} = \infty. \tag{38}$$

Then there exists at most one measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ *such that* $k = k_\mu$.

Remark 2.8. 1. In [37, 39], the wider space of multiple configurations was considered. The adaptation for the space Γ was realized in [36].

2. It is worth noting also that the growth of correlation functions $k^{(n)}$ up to $(n!)^2$ is admissible to have (38).

3. Another conditions for existence and uniqueness for the moment problem on Γ were srudied in [31].

3 Statistical descriptions of Markov evolutions

Spatial Markov processes in $\mathbb{R}^d$ may be described as stochastic evolutions of configurations $\gamma \subset \mathbb{R}^d$. In course of such evolutions points of configurations may disappear (die), move (continuously or with jumps from one position to another), or new particles may appear in a configuration (that is birth). The rates of these random events may depend on whole configuration that reflect an interaction between elements of the our system.

The construction of a spatial Markov process in the continuum is highly difficult question which is not solved in a full generality at present, see e.g. a review [49] and more detail references about birth-and-death processes in [18] and in the next Section. Meanwhile, for the discrete systems the corresponding processes are constructed under quite general assumptions, see e.g. [41]. One of the main difficulties for continuous systems includes the necessity to control number of elements in a bounded region. Note that the construction of spatial processes on bounded sets from $\mathbb{R}^d$ are typically well solved, see e.g. [25].

The existing Markov process $\Gamma \ni \gamma \mapsto X_t^\gamma \in \Gamma$, $t > 0$ provides solution to the backward Kolmogorov equation for bounded continuous functions:

$$\frac{\partial}{\partial t} F_t = L F_t, \tag{39}$$

where L is the Markov generator of the process X_t. The question about existence and properties of solutions to (39) in proper spaces itself is also highly nontrivial problem of infinite-dimensional analysis. The Markov generator L should satisfies the following two (informal) properties: 1) to be conservative, that is $L1 = 0$, 2) maximum principle, namely, if there exists $\gamma_0 \in \Gamma$ such that $F(\gamma) \le F(\gamma_0)$ for all $\gamma \in \Gamma$, then $(LF)(\gamma_0) \le 0$. These properties might yield that the semigroup, related to (39) (provided it exists), will preserves constants and positive functions, correspondingly.

To consider an example of such L let us consider a general Markov evolution with appearing and disappearing of groups of points (giving up the case of continuous moving of particles). Namely, let $F \in \mathcal{F}_{\mathrm{cyl}}(\Gamma)$ and set

$$(LF)(\gamma) = \sum_{\eta \Subset \gamma} \int_{\Gamma_0} c(\eta, \xi, \gamma \setminus \eta) \big[F((\gamma \setminus \eta) \cup \xi) - F(\gamma)\big] \, d\lambda(\xi). \tag{40}$$

Heuristically, it means that any finite group η of points from the existing configuration γ may disappear and simultaneously a new group ξ of points may appear somewhere in the space $\mathbb{R}^d$. The rate of this random event is equal to $c(\eta, \xi, \gamma \setminus \eta) \ge 0$. We need some minimal conditions on the rate c to guarantee that at least

$$LF \in \mathcal{F}_0(\Gamma) \qquad \text{for all } F \in \mathcal{F}_{\mathrm{cyl}}(\Gamma). \tag{41}$$

The term in the sum in (40) with $\eta = \emptyset$ corresponds to a pure birth of a finite group ξ of points whereas the part of integral corresponding to $\xi = \emptyset$ (recall that $\lambda(\{\emptyset\}) = 1$) is related to pure death of a finite sub-configuration $\eta \subset \gamma$. The parts with $|\eta| = |\xi| \ne 0$ corresponds to jumps of one group of points into another positions in $\mathbb{R}^d$. The rest parts present splitting and merging effects. In the next section we consider the more traditional case of the one-point birth-and-death parts only, i.e. the cases $|\eta| = 0$, $|\xi| = 1$ and $|\eta| = 1$, $|\xi| = 0$, correspondingly.

As we noted before, for most cases appearing in applications, the existence problem for a corresponding Markov process with a generator L is still open. On the other hand, the evolution of a state in the course of a stochastic dynamics is an important question in its own right. A mathematical formulation of this question may be realized through the forward Kolmogorov equation for probability measures (states) on the configuration space Γ. Namely, we consider the pairing between functions and measures on Γ given by

$$\langle F, \mu \rangle := \int_\Gamma F(\gamma) \, d\mu(\gamma). \tag{42}$$

Then we consider the initial value problem

$$\frac{d}{dt} \langle F, \mu_t \rangle = \langle LF, \mu_t \rangle, \quad t > 0, \quad \mu_t\big|_{t=0} = \mu_0, \tag{43}$$

where F is an arbitrary function from a proper set, e.g. $F \in K\big(B_{\mathrm{bs}}(\Gamma_0)\big) \subset \mathcal{F}_{\mathrm{cyl}}(\Gamma)$. In fact, the solution to (43) describes the time evolution of distributions instead of the evolution of initial points in the Markov process. We rewrite (43) in the following heuristic form

$$\frac{\partial}{\partial t} \mu_t = L^* \mu_t, \tag{44}$$

where L^* is the (informally) adjoint operator of L with respect to the pairing (42).

In the physical literature, (44) is referred to the *Fokker–Planck equation.* The Markovian property of L yields that (44) might have a solution in the class of probability measures. However, the mere existence of the corresponding Markov process will not give us much information about properties of the solution to (44), in particular, about its moments or correlation functions. To do this, we suppose now that a solution $\mu_t \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ to (43) exists and remains locally absolutely continuous with respect to the Poisson measure π for all $t > 0$ provided μ_0 has such a property. Then one can consider the correlation functionals $k_t := k_{\mu_t}$, $t \geq 0$.

Recall that we suppose (41). Then, one can calculate $K^{-1}LF$ using (29), and, by (31), we may rewrite (43) in the following way

$$\frac{d}{dt}\langle\!\langle K^{-1}F, k_t\rangle\!\rangle = \langle\!\langle K^{-1}LF, k_t\rangle\!\rangle, \quad t > 0, \quad k_t\big|_{t=0} = k_0, \tag{45}$$

for all $F \in K\big(B_{\mathrm{bs}}(\Gamma_0)\big) \subset \mathcal{F}_{\mathrm{cyl}}(\Gamma)$. Here the pairing between functions on Γ_0 is given by

$$\langle\!\langle G,\, k\rangle\!\rangle := \int_{\Gamma_0} G(\eta)k(\eta)\, d\lambda(\eta). \tag{46}$$

Let us recall that then, by (25),

$$\langle\!\langle G, k\rangle\!\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G^{(n)}(x_1, \ldots, x_n)k^{(n)}(x_1, \ldots, x_n)\, dx_1 \ldots dx_n, \tag{47}$$

Next, if we substitute $F = KG$, $G \in B_{\mathrm{bs}}(\Gamma_0)$ in (45), we derive

$$\frac{d}{dt}\langle\!\langle G, k_t\rangle\!\rangle = \langle\!\langle \widehat{L}G, k_t\rangle\!\rangle, \quad t > 0, \quad k_t\big|_{t=0} = k_0, \tag{48}$$

for all $G \in B_{\mathrm{bs}}(\Gamma_0)$. Here the operator

$$(\widehat{L}G)(\eta) := (K^{-1}LKG)(\eta), \quad \eta \in \Gamma_0 \tag{49}$$

is defined point-wise for all $G \in B_{\mathrm{bs}}(\Gamma_0)$ under conditions (41). As a result, we are interested in a weak solution to the equation

$$\frac{\partial}{\partial t} k_t = \widehat{L}^* k_t, \quad t > 0, \quad k_t\big|_{t=0} = k_0, \tag{50}$$

where $\widehat{L}^*$ is dual operator to $\widehat{L}$ with respect to the duality (46), namely,

$$\int_{\Gamma_0} (\widehat{L}G)(\eta)k(\eta)\, d\lambda(\eta) = \int_{\Gamma_0} G(\eta)(\widehat{L}^* k)(\eta)\, d\lambda(\eta). \tag{51}$$

The procedure of deriving the operator $\widehat{L}$ for a given L is fully combinatorial meanwhile to obtain the expression for the operator $\widehat{L}^*$ we need an analog of integration by parts formula. For a difference operator L considered in (40) this discrete integration by parts rule is presented in the following well-known lemma (see e.g. [35]):

Lemma 3.1. *For any measurable function* $H : \Gamma_0 \times \Gamma_0 \times \Gamma_0 \to \mathbb{R}$

$$\int_{\Gamma_0} \sum_{\xi \subset \eta} H\left(\xi, \eta \setminus \xi, \eta\right) d\lambda\left(\eta\right) = \int_{\Gamma_0} \int_{\Gamma_0} H\left(\xi, \eta, \eta \cup \xi\right) d\lambda\left(\xi\right) d\lambda\left(\eta\right) \tag{52}$$

if both sides of the equality make sense.

We recall that any function function on Γ_0 may be identified with an infinite vector of symmetric functions of the growing number of variables. In this approach, the operator $\widehat{L}^*$ in (50) will be realized as an infinite matrix $\left(\widehat{L}^*_{n,m}\right)_{n,m\in\mathbb{N}_0}$, where $\widehat{L}^*_{n,m}$ is a mapping from the space of symmetric functions of n variables into the space of symmetric functions of m variables. As a result, instead of equation (43) for infinite-dimensional objects we obtain an infinite system of equations for functions $k_t^{(n)}$ each of them is a function of a finite number of variables, namely

$$\begin{aligned} &\frac{\partial}{\partial t} k_t^{(n)}(x_1, \ldots, x_n) = \left(\widehat{L}^*_{n,m} k_t^{(n)}\right)(x_1, \ldots, x_n), \quad t > 0, \quad n \in \mathbb{N}_0, \\ &k_t^{(n)}(x_1, \ldots, x_n)\big|_{t=0} = k_0^{(n)}(x_1, \ldots, x_n). \end{aligned} \tag{53}$$

Of course, in general, for a fixed n, any equation from (53) itself is not closed and includes functions $k_t^{(m)}$ of other orders $m \neq n$, nevertheless, the system (53) is a closed linear system. The chain evolution equations for $k_t^{(n)}$ consists the so-called *hierarchy* which is an analog of the BBGKY hierarchy for Hamiltonian systems, see e.g. [11].

One of the main aims of our considerations is to study the classical solution to (50) in a proper functional space. The choice of such a space might be based on estimates (22), or more generally, (36). However, even the correlation functions (21) of the Poisson measures shows that it is rather natural to study the solutions to the equation (50) in weighted L^∞-type space of functions with the Ruelle-type bounds. The integrable correlation functions are not natural for the dynamics on the spaces of locally finite configurations. For example, it is well-known that the Poisson measure π_ρ with integrable density $\rho(x)$ is concentrated on the space Γ_0 of finite configurations (since in this case on can consider $\mathbb{R}^d$ instead of Λ in (9)). Therefore, typically, the case of integrable correlation functions yields that effectively our stochastic dynamics evolves through finite configurations only. Note that the case of an integrable first order correlation function is referred to *zero density* case in statistical physics.

We restrict our attention to the so-called *sub-Poissonian* correlation functions. Namely, for a given $C > 0$ we consider the following Banach space

$$\mathcal{K}_C := \left\{ k : \Gamma_0 \to \mathbb{R} \;\middle|\; k \cdot C^{-|\cdot|} \in L^\infty(\Gamma_0, d\lambda) \right\} \tag{54}$$

with the norm

$$\|k\|_{\mathcal{K}_C} := \|C^{-|\cdot|} k(\cdot)\|_{L^\infty(\Gamma_0,\lambda)}. \tag{55}$$

It is clear that $k \in \mathcal{K}_C$ implies, cf. (22),

$$\left|k(\eta)\right| \leq \|k\|_{\mathcal{K}_C}\, C^{|\eta|} \qquad \text{for } \lambda\text{-a.a. } \eta \in \Gamma_0. \tag{56}$$

One can distinguish at least two possibilities for a study of the initial value problem (50). We may try to solve this equation in one space $\mathcal{K}_C$. The well-posedness of the initial value problem in this case is equivalent with an existence of the strongly continuous semigroup (C_0-semigroup in the sequel) in the space $\mathcal{K}_C$ with a generator $\widehat{L}^*$. However, the space $\mathcal{K}_C$ is isometrically isomorphic to the space $L^\infty(\Gamma_0, C^{|\cdot|}d\lambda)$ whereas, by the H. Lotz theorem [43], [3], in the L^∞ space any C_0-semigroup is uniformly continuous, that is it has a bounded generator. Typically, for the difference operator L given in (40), any operator $\widehat{L}^*_{n,m}$, cf. (53), might be bounded as an operator between two spaces of bounded symmetric functions of n and m variables whereas the whole operator $\widehat{L}^*$ is unbounded in $\mathcal{K}_C$.

To avoid this difficulties we use a trick which goes back to R. Phillips [50]. The main idea is to consider the semigroup in L^∞ space not itself but as a dual semigroup T_t^* to a C_0-semigroup T_t with a generator A in the pre-dual L^1 space. In this case T_t^* appears strongly continuous semigroup not on the whole L^∞ but on the closure of the domain of A^* only.

In our case this leads to the following scheme. We consider the pre-dual Banach space to $\mathcal{K}_C$, namely, for $C > 0$,

$$\mathcal{L}_C := L^1\big(\Gamma_0, C^{|\cdot|}d\lambda\big). \tag{57}$$

The norm in $\mathcal{L}_C$ is given by

$$\|G\|_C := \int_{\Gamma_0} \big|G(\eta)\big| C^{|\eta|}\, d\lambda(\eta) = \sum_{n=0}^\infty \frac{C^n}{n!} \int_{(\mathbb{R}^d)^n} \big|G^{(n)}(x_1,\ldots,x_n)\big|\, dx_1 \ldots dx_n. \tag{58}$$

Consider the initial value problem, cf. (48), (50),

$$\frac{\partial}{\partial t} G_t = \widehat{L} G_t, \quad t > 0, \quad G_t\big|_{t=0} = G_0 \in \mathcal{L}_C. \tag{59}$$

Whereas (59) is well-posed in $\mathcal{L}_C$ there exists a C_0-semigroup $\widehat{T}(t)$ in $\mathcal{L}_C$. Then using Philips' result we obtain that the restriction of the dual semigroup $\widehat{T}^*(t)$ onto $\overline{\mathrm{Dom}(\widehat{L}^*)}$ will be C_0-semigroup with generator which is a part of $\widehat{L}^*$. This provides a solution to (50) which continuously depends on an initial data from $\overline{\mathrm{Dom}(\widehat{L}^*)}$. And after we would like to find a more useful universal subspace of $\mathcal{K}_C$ which is not depend on the operator $\widehat{L}^*$. As a result, we obtain the classical solution to (50) for $t > 0$ in a class of sub-Poissonian functions which satisfy the Ruelle-type bound (56). Of course, after this we need to verify existence and uniqueness of measures whose correlation functions are solutions to (50), cf. Proposition 2.3 above. This usually can be done using proper approximation schemes.

Another possibility for a study of the initial value problem (50) is to consider this evolutional equation in a proper scale of spaces $\{\mathcal{K}_C\}_{C_* \leq C \leq C^*}$. In this case we will have typically that the solution is local in time only. Namely, there exists $T > 0$ such that for any $t \in [0, T)$ there exists a unique solution to (50) and $k_t \in \mathcal{K}_{C_t}$ for some $C_t \in [C_*, C^*]$. We realize this approach using the so-called Ovsyannikov method [21, 48]. This method provides less restrictions on systems parameters, however, the price for this is a finite time interval. And, of course, the question about possibility to recover measures via solutions to (50) should be also solved separately in this case.

4 Birth-and-death evolutions in the continuum

One of the most important classes of Markov evolution in the continuum is given by the birth-and-death Markov processes in the space Γ of all configurations from $\mathbb{R}^d$. These are processes in which an infinite number of individuals exist at each instant, and the rates at which new individuals appear and some old ones disappear depend on the instantaneous configuration of existing individuals [29]. The corresponding Markov generators have a natural heuristic representation in terms of birth and death intensities. The birth intensity $b(x,\gamma) \geq 0$ characterizes the appearance of a new point at $x \in \mathbb{R}^d$ in the presence of a given configuration $\gamma \in \Gamma$. The death intensity $d(x,\gamma) \geq 0$ characterizes the probability of the event that the point x of the configuration γ disappears, depending on the location of the remaining points of the configuration, $\gamma \setminus x$. Heuristically, the corresponding Markov generator is described by the following expression, cf. (40),

$$(LF)(\gamma) := \sum_{x\in\gamma} d(x,\gamma\setminus x)\,[F(\gamma\setminus x) - F(\gamma)] + \int_{\mathbb{R}^d} b(x,\gamma)\,[F(\gamma\cup x) - F(\gamma)]\,dx, \quad (60)$$

for proper functions $F : \Gamma \to \mathbb{R}$.

The study of spatial birth-and-death processes was initiated by C. Preston [51]. This paper dealt with a solution of the backward Kolmogorov equation (39) under the restriction that only a finite number of individuals are alive at each moment of time. Under certain conditions, corresponding processes exist and are temporally ergodic, that is, there exists a unique stationary distribution. Note that a more general setting for birth-and-death processes only requires that the number of points in any compact set remains finite at all times. A further progress in the study of these processes was achieved by R. Holley and D. Stroock in [29]. They described in detail an analytic framework for birth-and-death dynamics. In particular, they analyzed the case of a birth-and-death process in a bounded region.

Stochastic equations for spatial birth-and-death processes were formulated in [26], through a spatial version of the time-change approach. Further, in [27], these processes were represented as solutions to a system of stochastic equations, and conditions for the existence and uniqueness of solutions to these equations, as well as for the corresponding martingale problems, were given. Unfortunately, quite restrictive assumptions on the birth and death rates in [27] do not allow an application of these results to several particular models that are interesting for applications (see e.g. [6, 7, 10, 19, 45, 47]).

A growing interest to the study of spatial birth-and-death processes, which we have recently observed, is stimulated by (among others) an important role which these processes play in several applications. For example, in spatial plant ecology, a general approach to the so-called individual based models was developed in a series of works, see e.g. [6, 7, 10, 45] and the references therein. These models are described as birth-and-death Markov processes in the configuration space Γ with specific rates b and d which reflect biological notions such as competition, establishment, fecundity etc. Other examples of birth-and-death processes may be found in mathematical physics. In particular, the Glauber-type stochastic dynamics in Γ is properly associated with the grand canonical Gibbs measures for classical gases. This gives a possibility to study these Gibbs measures as equilibrium states for specific birth-and-death Markov evolutions [4]. Starting with a Dirichlet form for a given Gibbs measure, one can consider an equilibrium stochastic dynamics [34]. However, these dynamics give the time evolution of initial distributions from a quite narrow class. Namely, the class of admissible initial distributions is essentially reduced to the states which are absolutely continuous with respect to the invariant measure. In [18], we construct non-equilibrium stochastic dynamics which may have a much wider class of initial states.

This approach was successfully applied before to the construction and analysis of state evolutions for different versions of the Glauber dynamics [17, 20, 33] and for some spatial ecology models [16]. Each of the considered models required its own specific version of the construction of a semigroup, which takes into account particular properties of corresponding birth and death rates.

In [18], we realized a general approach considered in Section 2 to the construction of the state evolution corresponding to the birth-and-death Markov generators. We presented there conditions on the birth and death intensities which are sufficient for the existence of corresponding evolutions as strongly continuous semigroups in proper Banach spaces of correlation functions satisfying the Ruelle-type bounds. Also, in papers [15, 21, 22], we considered weaker assumptions on these intensities which provide the corresponding evolutions for finite time intervals in scales of Banach spaces as above.

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