Interdisciplinary Studies of Complex Systems
No. 25 (2024) 5–15

https://doi.org/10.31392/iscs.2024.25.005 UDC 517.921:517.17]:511.75

On preservation of singularity, absolute continuity and discreteness under transformation of probability spaces

Sergio Albeverio[1], *Maryna Lupain*[2,3], *Roman Nikiforov*[2,4], *Grygoriy Torbin*[2,5]

Abstract. The paper is devoted to the study of conditions for the preservation of mutual singularity resp. absolute continuity, and discreteness of probability measures under measurable mappings of probability spaces. Under very general assumptions we have found such conditions for the preservations. At the same time a series of important counterexamples are presented. The results obtained can simplified essentially the study the Lebesgue structure (i.e., finding necessary and sufficient conditions for the singular continuity, absolute continuity and discreteness of a wide spectra of probability measures with independent digits of symbolic expansions of real numbers and their multidimensional generalizations.

Keywords: probability measures, singularity, absolute continuity, discreteness, measurable mapping, bimeasurable mapping, probability spaces, image measure

1 Introduction

It is well known that there exist only three types of pure probability distributions: discrete, absolutely continuous, and singularly continuous.

Discrete and absolutely continuous distributions are the most studied; however, in many textbooks the class of continuous distributions is often incorrectly identified with the class of absolutely continuous ones. Singularly continuous distributions, which gained interest in the early 20th century after the development of Lebesgue measure theory, have experienced periods of growth and decline in scientific attention. The development of the theory of singular measures was stimulated, in particular, by its connections with the problems of harmonic analysis (especially with the theory of trigonometric series), the theory of dynamical systems, and spectral theory. At the end of the 20th century, with the advent of fractal theory, a new toolkit for the study of singular measures appeared, which gave a new impetus to the development of this direction. Despite a rather long history of research, the problem of

[1] Institute for Applied Mathematics and Hausdorff Center for Mathematics, University of Bonn, Bonn, Germany. albeverio@iam.uni-bonn.de
[2] Dragomanov Ukrainian State University, Kyiv, Ukraine
[3] m.l.lupayin@npu.edu.ua, https://orcid.org/0000-0001-9553-3157
[4] r.o.nikiforov@udu.edu.ua, https://orcid.org/0000-0001-9553-3157
[5] g.m.torbin@npu.edu.ua, https://orcid.org/0000-0003-3088-1614

finding necessary and sufficient conditions for singular continuity resp. absolute continuity remains open for many classes of probability distributions (see, e.g., [11, 12, 15, 16]). After the proving singularity, the issues of studying the fractal properties of the corresponding measures become important. The paper is devoted to the study of conditions for the preservation of mutual singularity resp. absolute continuity, and discreteness of probability measures under measurable mappings of probability spaces. Under very general assumptions we have found such conditions for the preservations. At the same time a series of important counterexamples are presented. The results obtained can simplified essentially the study the Lebesgue structure (i.e., finding necessary and sufficient conditions for the singular continuity, absolute continuity and discreteness of a wide spectra of probability measures with independent digits of symbolic expansions of real numbers (see, e.g., [1, 1–6, 6, 14, 17, 18, 22] and references therein) and their multidimensional generalizations.

2 On preservation of singularity, absolute continuity and discreteness under transformation of probability spaces

Let $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ be measurable spaces, and let μ_1, ν_1 be probability measures on $\mathcal{A}_1$.

Let f be a measurable mapping:

$$(\Omega_1, \mathcal{A}_1, \mu_1) \xrightarrow{f} (\Omega_2, \mathcal{A}_2, \mu_2),$$

$$(\Omega_1, \mathcal{A}_1, \nu_1) \xrightarrow{f} (\Omega_2, \mathcal{A}_2, \nu_2),$$

where the measures μ_2 and ν_2 are image measures of the measures μ_1 and ν_1 under the mapping f :

$$\mu_2(E_2) := \mu_1(f^{-1}(E_2)), \forall E_2 \in \mathcal{A}_2,$$

$$\nu_2(E_2) := \nu_1(f^{-1}(E_2)), \forall E_2 \in \mathcal{A}_2.$$

Theorem 1. *If $\mu_1 \ll \nu_1$, then $\mu_2 \ll \nu_2$.*

Proof. Let us assume that $\nu_2(E_2) = 0$ for some subset $E_2 \in \mathcal{A}_2$. Then

$$\nu_1(f^{-1}(E_2)) = \nu_2(E_2) = 0.$$

Since $\mu_1 \ll \nu_1$, we deduce that $\mu_1(f^{-1}(E_2)) = \mu_2(E_2) = 0$. So, the measure μ_2 is absolutely continuous w.r.t. the measure ν_2. □

Remark 1. The implication $\mu_2 \ll \nu_2 \Rightarrow \mu_1 \ll \nu_1$ is false.

Example 1. Let

$$\Omega_1 = \{0, 1, 2\}, \mathcal{A}_1 = 2^{\Omega_1},$$

$$\Omega_2 = \{a, b\}, \mathcal{A}_2 = 2^{\Omega_2},$$

$$f(0) = f(1) = a, f(2) = b.$$

Let us define measures μ_1 and ν_1 as follows:

$$\mu_1(\{0\}) = 0, \quad \mu_1(\{1\}) = \frac{1}{2}, \quad \mu_1(\{2\}) = \frac{1}{2};$$
$$\nu_1(\{0\}) = \frac{1}{2}, \quad \nu_1(\{1\}) = 0, \quad \nu_1(\{2\}) = \frac{1}{2}.$$

Since $\nu_1(\{1\}) = 0$, but $\mu_1(\{1\}) > 0$, we see that the measure μ_1 is not absolutely continuous w.r.t. the measure ν_1.

It is clear that

$$\mu_2(\{a\}) = \mu_1(f^{-1}\{a\}) = \mu_1(\{0,1\}) = \frac{1}{2},$$

$$\mu_2(\{b\}) = \mu_1(f^{-1}\{b\}) = \mu_1(\{2\}) = \frac{1}{2}.$$

$$\nu_2(\{a\}) = \nu_1(f^{-1}\{a\}) = \nu_1(\{0,1\}) = \frac{1}{2},$$

$$\nu_2(\{b\}) = \nu_1(f^{-1}\{b\}) = \nu_1(\{2\}) = \frac{1}{2}.$$

Therefore, $\mu_2 \ll \nu_2$ (moreover. $\mu_2 \equiv \nu_2$). At the same time the measure μ_1 is not absolutely continuous w.r.t. the measure ν_1.

Remark 2. There exists mutual singular probability measures μ_1 and ν_1, and a bimeasurable mapping f such that their image measures μ_2 and ν_2 coincide.

Example 2. Let

$$\Omega_1 = \{0,1,2,3\}, \quad \mathcal{A}_1 = 2^{\Omega_1},$$
$$\Omega_2 = \{0,1\}, \quad \mathcal{A}_2 = 2^{\Omega_2},$$
$$f(0) = f(1) = 0, \quad f(2) = f(3) = 1.$$

Let us define measures μ_1 and ν_1 as follows:

$$\mu_1(\{0\}) = \mu_1(\{2\}) = \frac{1}{2};$$
$$\nu_1(\{1\}) = \nu_1(\{3\}) = \frac{1}{2}.$$

Then $\nu_1(\{0,2\}) = 0$, $\mu_1(\{0,2\}) = 1$ and $\mu_1(\{1,3\}) = 0$, $\nu_1(\{1,3\}) = 1$. So, $\mu_1 \perp \nu_1$.

From the definition of measures μ_2 and ν_2 it follows that

$$\mu_2(\{0\}) = \mu_1(f^{-1}(\{0\})) = \mu_1(\{0,1\}) = \frac{1}{2},$$
$$\mu_2(\{1\}) = \mu_1(f^{-1}(\{1\})) = \mu_1(\{2,3\}) = \frac{1}{2},$$
$$\nu_2(\{0\}) = \nu_1(f^{-1}(\{0\})) = \nu_1(\{0,1\}) = \frac{1}{2},$$
$$\nu_2(\{1\}) = \nu_1(f^{-1}(\{1\})) = \nu_1(\{2,3\}) = \frac{1}{2}.$$

So, $\mu_1 \perp \nu_1$, but the corresponding measures μ_2 and ν_2 coincide.

Theorem 2. *If $\mu_2 \perp \nu_2$, then $\mu_1 \perp \nu_1$.*

Proof. If $\mu_2 \perp \nu_2$, then there exists a subset $E_2 \in \mathcal{A}_2$ such that $\nu_2(E_2) = 0$ and $\mu_2(\Omega_2 \setminus E_2) = 0$. From the definition of the measures μ_2 and ν_2 we have $\mu_1(f^{-1}(E_2)) = \mu_2(E_2) = 0$ and $\nu(f^{-1}(\Omega_2 \setminus E_2)) = \nu_2(\Omega_2 \setminus E_2) = 0$. It is clear that $f^{-1}(\Omega_2 \setminus E_2)) \cap f^{-1}(E_2) = \emptyset$. So, $\mu_1 \perp \nu_1$. □

Remark 3. The inverse implication is false even under additional assumption on bijectivity of f.

Example 3. Let $\Omega_1 = \Omega_2 = \{0, 1, 2, 3\}$, $\mathcal{A}_1 = 2^{\Omega_1}$, $\mathcal{A}_2 = \{\Omega_2, \emptyset, \{0, 1\}, \{2, 3\}\}$.

Let $\mu_1(\{0\}) = \mu_1(\{2\}) = \frac{1}{2}$, $\nu_1(\{1\}) = \nu_1(\{3\}) = \frac{1}{2}$ and $f(i) = i, i \in \{0, 1, 2, 3\}$.

Since $\nu_1(\{0, 2\}) = 0$ and $\mu_1(\{0, 2\}) = 1$, we get $\mu_1 \perp \nu_1$.

At the same time

$$\mu_2(\{0,1\}) = \mu_1(\{0,1\}) = \frac{1}{2},$$

$$\mu_2(\{2,3\}) = \mu_1(\{2,3\}) = \frac{1}{2},$$

$$\nu_2(\{0,1\}) = \nu_1(\{0,1\}) = \frac{1}{2},$$

$$\nu_2(\{2,3\}) = \nu_1(\{2,3\}) = \frac{1}{2}.$$

Therefore, $\mu_1 \perp \nu_1$, the mapping f is measurable and bijective, but $\mu_2 \equiv \nu_2$.

Theorem 3. *Let f be a measurable and bijective mapping. Then*

$$\mu_1 \ll \nu_1 \Leftrightarrow \mu_2 \ll \nu_2,$$

$$\mu_1 \perp \nu_1 \Leftrightarrow \mu_2 \perp \nu_2.$$

Proof. If f is a measurable and bijective mapping, then $A = f^{-1}(f(A))$ and $\mu_1(A) = \mu_2(f(A))$.

In a similar way we get $\nu_1(A) = \nu_2(f(A))$, $\forall A \in \mathcal{A}_1$.

Taking into account Theorem 1, to prove the first statement of the theorem it sufficient to show that from the condition $\mu_2 \ll \nu_2$ it follows that $\mu_1 \ll \nu_1$.

Let us assume that $\nu_1(A) = 0$ for some measurable subset $A \subset \Omega_1$. Then $\nu_2(f(A)) = \nu_1(A) = 0$. Since $\mu_2 \ll \nu_2$, we get $\mu_2(f(A)) = \mu_1(A) = 0$, Which proves the absolute continuity of the measure μ_1 w.r.t. the measure ν_1.

Taking into account Theorem 2, to prove the second statement of the theorem it sufficient to show that from $\mu_1 \perp \nu_1$, it follows that $\mu_2 \perp \nu_2$.

From $\mu_1 \perp \nu_1$, it follows that there exists a subset $A \in \mathcal{A}_1$ such that $\nu_1(A) = 0$, $\mu_1(A) = 1$. Then $\nu_1(A) = \nu_2(f(A)) = 0$. Let $f(A) =: A'$. From bimeasurability of f it follows that $A' \in \mathcal{A}_2$, $\mu_2(A') = \mu_2(f(A)) = \mu_1(A) = 1$. Therefore, $\mu_2 \perp \nu_2$. □

In many important cases the corresponding mapping f is not bijective. Such cases were studied partially in papers [5, 10]. The following theorem gives a rather general conditions for the preservation of relations "to be mutually singular" and "to be absolutely continuous" for probability measures under bimeasurable mappings.

Theorem 4. *Let μ_1 and ν_1 be probability measures on a measurable space $(\Omega_1, \mathcal{A}_1)$. Let $(\Omega_2, \mathcal{A}_2)$ be a measurable space and let f be a bimeasurable mapping from $(\Omega_1, \mathcal{A}_1)$ into $(\Omega_2, \mathcal{A}_2)$.*

Let μ_2 and ν_2 be image measures of μ_1 and ν_1 under f, i.e.,

$$\mu_2(E_2) := \mu_1(f^{-1}(E_2)), \forall E_2 \in \mathcal{A}_2,$$

$$\nu_2(E_2) := \nu_1(f^{-1}(E_2)), \forall E_2 \in \mathcal{A}_2.$$

Assume that there exists a subset $\Omega_0 \in \mathcal{A}_1$, such that $\nu_2(f(\Omega_0)) = 0$, and the mapping f from $(\Omega_1 \setminus \Omega_0)$ into Ω_2 is bijective.

Then

$$\mu_1 \ll \nu_1 \Leftrightarrow \mu_2 \ll \nu_2,$$

$$\mu_1 \perp \nu_2 \Leftrightarrow \mu_2 \perp \nu_2.$$

Proof. **1.** Firstly let us prove that

$$\mu_1 \ll \nu_1 \Leftrightarrow \mu_2 \ll \nu_2.$$

1a. Assume that $\mu_2 \ll \nu_2$, and show that $\mu_1 \ll \nu_1$.

Let E_1 be an arbitrary subset from $\mathcal{A}_1$ with $\nu_1(E_1) = 0$. Let us prove that $\mu_1(E_1) = 0$.

Define $E_{10} := E_1 \cap \Omega_0$, $E_{11} := E_1 \cap \overline{\Omega}_0$.

Let $E_2 := f(E_1)$, $E_{21} := f(E_{11})$, $E_{20} := f(E_{10})$.

Since $E_1 \in \mathcal{A}_1$ and $\Omega_0 \in \mathcal{A}_1$, we get $E_{10} \in \mathcal{A}_1$ and $E_{11} \in \mathcal{A}_1$.

From bimeasurability of $f-$ it follows that $E_2 \in \mathcal{A}_2$, $E_{20} \in \mathcal{A}_2$, $E_{21} \in \mathcal{A}_2$. It can happen that $E_{20} \cap E_{21} \neq \emptyset$.

$$\nu_2(E_2) = \nu_2(E_{20} \cup E_{21}) \leq \nu_2(E_{20}) + \nu_2(E_{21}).$$

Since the mapping f from $\Omega_1 \setminus \Omega_0$ into Ω_2 is bijective, we get

$$f^{-1}(E_{21}) \cap (\Omega_1 \setminus \Omega_0) = E_{11}.$$

Therefore,

$$f^{-1}(E_{21}) = E_{11} \cup E_{11}^*, E_{11}^* \subset \Omega_0, E_{11}^* \in \mathcal{A}_1.$$

Then

$$\nu_2(E_{21}) = \nu_1(f^{-1}(E_{21})) = \nu_1(E_{11} \cup E_{11}^*) = \nu_1(E_{11}) + \nu_1(E_{11}^*).$$

$\nu_1(E_{11}) = 0$, because $E_{11} \subset E_1$ and $\nu_1(E_1) = 0$.

From $E_{11}^* \subset \Omega_0$ it follows $f(E_{11}^*) \subset f(\Omega_0)$. So, $f^{-1}(f(E_{11}^*)) \subset f^{-1}(f(\Omega_0))$.

Therefore,

$$\nu_1(f^{-1}(f(E_{11}^*))) \leq \nu_1(f^{-1}(f(\Omega_0))).$$

Since $\nu_1(f^{-1}(f(\Omega_0))) = 0$, we have $\nu_1(E_{11}^*) = 0$. So, $\nu_2(E_{21}) = 0$.

From $E_{10} \subset \Omega_0$, it follows that $E_{20} \subset f(\Omega_0)$ and $f^{-1}(E_{20}) \subset f^{-1}(f(\Omega_0))$. Therefore, $\nu_2(E_{20}) = \nu_1(f^{-1}(E_{20})) \leq \nu_1(f^{-1}(f(\Omega_0))$. So, $\nu_2(E_{20}) = 0$.

Hence,

$$\nu_2(E_2) = \nu_2(E_{20} \cup E_{21}) \leq \nu_2(E_{20}) + \nu_2(E_{21}) = 0 + 0 = 0.$$

From $\mu_2 \ll \nu_2$ it follows that $\mu_2(E_2) = 0$.

Since $\mu_2(E_2) = \mu_1(f^{-1}(E_2)) = 0$ and $E_1 \subset f^{-1}(f(E_1)) = f^{-1}(E_2)$, we deduce that $\mu_1(E_1) \le \mu_2(E_2) = 0$. So, $\mu_1 \ll \nu_1$.

1b. The implication

$$\mu_1 \ll \nu_1 \Rightarrow \mu_2 \ll \nu_2$$

follows directly from Theorem 1.

2. Let us prove that $\mu_1 \perp \nu_1 \Leftrightarrow \mu_2 \perp \nu_2$.

2a. The implication $\mu_2 \perp \nu_2 \Rightarrow \mu_1 \perp \nu_1$ follows directly from Theorem 2.

2b. Let us prove the implication $\mu_1 \perp \nu_1 \Rightarrow \mu_2 \perp \nu_2$

From the definition of singularity $\mu_1 \perp \nu_1$ it follows that there exists a subset $E_1 \in \mathcal{A}_1$ such that $\nu_1(E_1) = 0$, $\mu_1(E_1) = 1$. Let us show that there exists a subset $E_2 \in \mathcal{A}_2$ such that $\nu_2(E_2) = 0$, and $\mu_2(E_2) = 1$.

Let $E_{10} := E_1 \cap \Omega_0$, $E_{11} := E_1 \cap \overline{\Omega}_0$, $f(E_1) =: E_2$, $f(E_{11}) =: E_{21}$, $f(E_{10}) =: E_{20}$.

$$\nu_2(E_2) = \nu_2(E_{20} \cup E_{21}) \le \nu_2(E_{20}) + \nu_2(E_{21}).$$

From the definition of the measure ν_2 it follows that

$$\nu_2(E_{21}) = \nu_1(f^{-1}(E_{21})),$$

where $f^{-1}(E_{21}) = E_{11} \cup E_{11}^*$, $E_{11}^* \subset \Omega_0$.

Then

$$\nu_2(E_{21}) = \nu_1(E_{11} \cup E_{11}^*) = \nu_1(E_{11}) + \nu_1(E_{11}^*).$$

Since $E_{11} \subset E_1$, $\nu_1(E_1) = 0$, we have $\nu_1(E_{11}) = 0$.

Since $E_{11}^* \subset \Omega_0$, we deduce that $f^{-1}(f(E_{11}^*)) \subset f^{-1}(f(\Omega_0))$. From assumptions of the theorem it follows that $\nu_1(f^{-1}(f(\Omega_0))) = 0$. Therefore, $\nu_1(E_{11}^*) = 0$. So,

$$\nu_2(E_{21}) = \nu_1(E_{11}) + \nu_1(E_{11}^*) = 0 + 0 = 0.$$

Let us determine $\nu_2(E_{20})$.

$\nu_2(E_{20}) = \nu_1(f^{-1}(E_{20}))$. Since $E_2 \subset f(\Omega_0)$, we have

$$f^{-1}(E_{20}) \subset f^{-1}(f(\Omega_0)).$$

Then

$$\nu_1(f^{-1}(E_{20})) \le \nu_1(f^{-1}(f(\Omega_0))) = 0.$$

Therefore,

$$\nu_2(E_{20}) = \nu_1(f^{-1}(E_{20})) = 0.$$

So,

$$\nu_2(E_2) \le \nu_2(E_{20}) + \nu_2(E_{21}) = 0 + 0 = 0.$$

Let us show that $\mu(E_2) = 1$. From the definition of the measure μ_2 it follows that

$$\mu_2(E_2) = \mu_1(f^{-1}(E_2)).$$

Since $f^{-1}(E_2) = f^{-1}(f(E_1))$, and $E_1 \subset f^{-1}(f(E_1))$, we get $\mu_1(f^{-1}(E_2)) \ge \mu_1(E_1) = 1$. So, $\mu_2(E_2) = 1$ and therefore $\mu_2 \perp \nu_2$. □

Remark 4. The assumption $\nu_1(f^{-1}f(\Omega_0)) = 0$ is important and can not be replaced by the assumption $\nu_1(\Omega_0) = 0$.

Example 4. Let

$$\Omega_1 = \{0, 1, 2\}, \mathcal{A}_1 = 2^{\Omega_1},$$

$$\Omega_2 = \{a, b\}, \mathcal{A}_2 = 2^{\Omega_1},$$

$$f(0) = f(1) = a, f(2) = b.$$

Define the measure μ_1 and ν_1 in the following way:

$$\mu_1(\{0\}) = \frac{1}{2}, \mu_1(\{1\}) = 0, \mu_1(\{2\}) = \frac{1}{2};$$

$$\nu_1(\{0\}) = 0, \nu_1(\{1\}) = \frac{1}{2}, \nu_1(\{2\}) = \frac{1}{2}.$$

Since $\nu_1(\{0\}) = 0$, and $\mu_1(\{0\}) > 0$, we deduce that the measure μ_1 is not absolutely continuous w.r.t. the measure ν_1.

Let $\Omega_0 = \{0\}$. Then $\Omega_1 \setminus \Omega_0 = \{1, 2\}$, and the mapping $f : \Omega_1 \setminus \Omega_0 \longrightarrow \Omega_2$ is bijective and bimeasurable.

At the same time

$$\mu_2(\{a\}) = \mu_1(f^{-1}\{a\}) = \mu_1(\{0, 1\}) = \frac{1}{2},$$

$$\mu_2(\{b\}) = \mu_1(f^{-1}\{b\}) = \mu_1(\{2\}) = \frac{1}{2}.$$

Similarly

$$\nu_2(\{a\}) = \nu_1(f^{-1}\{a\}) = \nu_1(\{0, 1\}) = \frac{1}{2},$$

$$\nu_2(\{b\}) = \nu_1(f^{-1}\{b\}) = \nu_1(\{2\}) = \frac{1}{2}.$$

So, $\mu_2 \ll \nu_2$ ($\mu_2 \equiv \nu_2$), but the measure μ_1 is not absolutely continuous w.r.t. the measure ν_1.

Theorem 5. *Let μ_1 and ν_1 be probability measures on a measurable space $(\Omega_1, \mathcal{A}_1)$. Let $(\Omega_2, \mathcal{A}_2)$ be a measurable space and let f be a bimeasurable mapping from $(\Omega_1, \mathcal{A}_1)$ into $(\Omega_2, \mathcal{A}_2)$.*

Let μ_2 and ν_2 be image measures of μ_1 and ν_1 under f, i.e.,

$$\mu_2(E_2) := \mu_1(f^{-1}(E_2)), \forall E_2 \in \mathcal{A}_2,$$

$$\nu_2(E_2) := \nu_1(f^{-1}(E_2)), \forall E_2 \in \mathcal{A}_2.$$

Assume that there exists a subset $\Omega_0 \in \mathcal{A}_1$, such that

$$\mu_1(\Omega_0) = 0, \nu_1(\Omega_0) = 0,$$

and the mapping f from $(\Omega_1 \setminus \Omega_0)$ into Ω_2 is bijective.

Then

$$\mu_1 \ll \nu_1 \Leftrightarrow \mu_2 \ll \nu_2,$$

$$\mu_1 \perp \nu_2 \Leftrightarrow \mu_2 \perp \nu_2.$$

Proof. To prove the first statement of the theorem we must prove the implication

$$\mu_2 \ll \nu_2 \Rightarrow \mu_1 \ll \nu_1.$$

Assume that the measure μ_1 is not absolutely continuous w.r.t. ν_1. Then there exists a subset $E_1 \in \mathcal{A}_1$ such that $\nu_1(E_1) = 0$ and $\mu_1(E_1) > 0$.

Let us split the set E_1 into two parts $E_1 = E_1^* \cup E_0$, where $E_1^* = E_1 \cap \Omega'$, $E_0 \subset \Omega_0$, $\Omega' = \Omega_1 \setminus \Omega_0$.

From our assumptions it follows that $\nu_1(E_1) = 0$ and $E_1^* \subset E_1$. Hence, $\nu_1(E_1^*) = 0$.

Similarly, taking into account that $\mu_1(E_1) > 0$ and $\mu_1(E_0) = 0$, we get $\mu_1(E_1^*) > 0$.

Let $E_2^* := f(E_1^*)$. From the definition of the measure ν_2:

$$\nu_2(E_2^*) = \nu_1(f^{-1}f(E_1^*)).$$

Since the mapping f from Ω' into Ω_2 is bijective, we have

$$f^{-1}(E_2^*) = E_1^* \cup E_1',$$

where $E_1 \subset \Omega_0$.

Then $\nu_1(E_1') \leq \nu_1(\Omega_0) = 0$. Therefore, $\nu_1(E_1') = 0$. So,

$$\nu_2(E_2^*) = \nu_1(E_1^*) + \nu_1(E_1') = 0 + 0 = 0.$$

Let us determine $\mu_2(E_2^*)$.

$$\mu_2(E_2^*) = \mu_1(E_1^*) + \mu_1(E_1').$$

Since $\mu_1(E_1') \leq \mu_1(\Omega_0) = 0$ and $\mu_1(E_1^*) > 0$, we get $\mu_2(E_2^*) > 0$ which contradicts the absolute continuity of the measure μ_2 w.r.t. ν_2.

To prove the second statement of the theorem we must prove the implication

$$\mu_1 \perp \nu_1 \Rightarrow \mu_2 \perp \nu_2.$$

Let $\mu_1 \perp \nu_1$. Then there exists a subset $A_1 \in \mathcal{A}_1$ such that $\nu_1(A_1) = 0$ and $\mu_1(A_1) = 1$. Let us define $B_1 := \overline{A}_1 = \Omega_1 \setminus A_1$. Then $A_1 \cap B_1 = \emptyset$, $B_1 \in \mathcal{A}_1$ and $\mu_1(A_1) = 1$, $\nu_1(B_1) = 1$.

Let $A_1^* := A_1 \cap \Omega'$, $B_1^* := B_1 \cap \Omega'$, where $\Omega' = \Omega_1 \setminus \Omega_0$. Since $A_1 \setminus A_1^* \subset \Omega_0$ and $\mu_1(\Omega_0) = 0$, we get $\mu_1(A_1^*) = 1$. Similarly, $\nu_1(B_1^*) = 1$.

Let $A_2^* := f(A_1^*)$. Since f is bimeasurable, we have $A_2^* \in \mathcal{A}_2$. From the definition of the measure μ_2 :

$$\mu_2(A_2^*) = \mu_1(f^{-1}(A_2^*)).$$

Since $f^{-1}(f(A_1^*)) \supset A_1^*$ and $\mu_1(A_1^*) = 1$, we deduce that $\mu_2(A_2^*) = 1$. Similarly, we have $\nu_2(B_2^*) = 1$, where $B_2^* := f(B_1^*)$.

Since $A_1^* \cap B_1^* = \emptyset$ and the mapping $f : \Omega' \to \Omega_2$ is bijective, and $A_1^* \subset \Omega'$, $B_1^* \subset \Omega'$, we get $A_2^* \cap B_2^* = \emptyset$.

From $\nu_2(B_2^*) = 1$ and $A_2^* \cap B_2^* = \emptyset$, it follows that $\nu_2(A_2^*) = 0$.

Since $\mu_2(A_2^*) = 1$, we get a desirable relation $\mu_2 \perp \nu_2$, which proves the second part of the theorem. □

Theorem 6. *Let f be a mapping from $(\Omega_1, \mathcal{A}_1, \mu_1)$ into $(\Omega_2, \mathcal{A}_2, \mu_2)$, and the measure μ_2 is the image of the measure μ_1 under f.*

If the mapping f is measurable, then the discreteness of the measure μ_1 implies the discreteness of the measure μ_2.

Proof. If the measure μ_1 is discrete (atomic), then there exists an at most countable subset $E_d \in \mathcal{A}_1$ such that $\mu_1(E_d) = 1$.

Let $E'_d := f(E_d)$. Since f is a bimeasurable mapping, we deduce that $E'_d \in \mathcal{A}_2$ and E'_d is an at most countable set.

To determine $\mu_2(E'_d)$ let us consider the set $E^*_d := f^{-1}(E'_d) = f^{-1}(f(E_d))$. From the definition of the measure μ_2 :

$$\mu_2(E'_d) = \mu_1(f^{-1}(E'_d)) = \mu_1(E^*_d).$$

It is clear that $E^*_d \supset E_d$.

Hence, $\mu_1(E_d*) \geq \mu_1(E_d) = 1$. So, $\mu_2(E'_d) = 1$, which proves the discreteness of the measure μ_2. □

Remark 5. The implication "μ_2 is discrete $\Rightarrow$ μ_1 is discrete" is false.

Example 5. Let $\Omega_1 = [0, 1]$, and let $\mathcal{A}_1 = \mathcal{B}[0, 1]$, let μ_1 be Lebesgue measure on $[0, 1]$.

Let $\Omega_2 = [0, 1]$, $\mathcal{A}_2 = \mathcal{B}[0, 1]$.

Let $f(x) = \frac{1}{2}, \forall x \in [0, 1]$.

It is clear that f is bimeasurable and μ_2 is pure discrete with a unique atom at the point $\frac{1}{2}$.

Theorem 7. *If the mapping f is bimeasurable and for any point $y \in \Omega_2$ the set $f^{-1}(y)$ is an at most countable, then μ_2 is discrete if and only if μ_1 is discrete.*

Proof. Let μ_2 be discrete. Then there exists an at most countable subset $E'_d \in \mathcal{A}_2$ such that $\mu_2(E'_d) = 1$. Let $E_d := f^{-1}(E'_d)$.

From the definition of the measure μ_2:

$$\mu_2(E'_d) = \mu_1(f^{-1}(E'_d)) = \mu_1(E_d) = 1, \quad E_d \in \mathcal{A}_1.$$

From the assumption of the theorem for any point $y \in \Omega_2$ the set $f^{-1}(y)$ is an at most countable. Therefore, the set E_d is an at most countable. So, from the discreteness of the measure μ_2 we get the discreteness of the measure μ_1, which proves the theorem. □

Acknowledgment

This work was partly supported by a grant from the Simons Foundation (1290607, Torbin G.) and by the Ministry of Science and Education of Ukraine (projects "Multifractal analysis of probability distributions and its application for modeling of complex dynamical systems" and "Dynamics of complex systems on different time scales").

References

[1] S. Albeverio, V. Koshmanenko, M. Pratsiovytyi, G. Torbin. 2011. On fine structure of singularly continuous probability measures and random variables with independent $\tilde{Q}$-symbols. *Methods of Functional Analysis and Topology* 17(1). P. 97–111. http://mfat.imath.kiev.ua/article/?id=570

[2] S. Albeverio, G. Torbin. 2005. Fractal properties of singularly continuous probability distributions with independent Q^*-digits. *Bull. Sci. Math.* 129(4). P. 356–367. https://doi.org/10.1016/j.bulsci.2004.12.001

[3] S. Albeverio, V. Koshmanenko, G. Torbin. 2003. Fine structure of the singular continuous spectrum. *Methods of Functional Analysis and Topology* 9(2). P. 101–119. http://mfat.imath.kiev.ua/article/?id=236

[4] S. Albeverio, Ya. Gontcharenko, M. Pratsiovytyi, G. Torbin. 2006. Jessen—Wintner type random variables and fractal properties of their distributions. *Mathematische Nachrichten* 279(15). P. 1619–1633. https://doi.org/10.1002/mana.200310441

[5] S. Albeverio, G. Torbin. 2004. Image measures of infinite product measures and generalized Bernoulli convolutions. *Transactions of the National Pedagogical University (Phys.-Math. Sci.)* 5. P. 248–264.

[6] S. Albeverio, G. Torbin. 2008. On fine fractal properties of generalized infinite Bernoulli convolutions. *Bull. Sci. Math.* 132(8). P. 711–727. https://doi.org/10.1016/j.bulsci.2008.03.002

[7] S. Albeverio, O. Baranovskyi, M. Pratsiovytyi, G. Torbin. 2007. The Ostrogradsky series and related Cantor-like sets. *Acta Arithm.* 130(3). P. 215–230. http://eudml.org/doc/278451

[8] S. Albeverio, Yu. Kondratiev, R. Nikiforov, G. Torbin. 2017. On new fractal phenomena connected with infinite linear IFS. *Mathematische Nachrichten* 290(8–9). P. 1163–1176. https://doi.org/10.1002/mana.201500471

[9] S. Albeverio, M. Pratsiovytyi , G. Torbin. 2008. Transformations preserving the Hausdorff–Besicovitch dimension. *Central European Journal of Mathematics* 6(1). P. 119–128. https://doi.org/10.2478/s11533-008-0007-y

[10] G. Ivanenko, R. Nikiforov, G. Torbin. 2006. Ergodic approach to investigations of singular probability measures. *Transactions of the National Pedagogical University (Phys.-Math. Sci.)* 7. P. 126–142.

[11] B. Jessen, A. Wintner. 1935. Distribution function and Rieman Zeta-function. *Trans.Amer.Math.Soc.* 38. P. 48–88. https://doi.org/10.1090/S0002-9947-1935-1501802-5

[12] S. Kakutani. 1948.Equivalence of infinite product measures. *Ann. of Math.* 49(1). P. 214–224. https://doi.org/10.2307/1969123

[13] M. Lupain, G. Torbin. 2014. On new fractal phenomena related to distributions of random variables with independent GLS-symbols. *Transactions of the National Pedagogical University (Phys.-Math. Sci.)* 16(2). P. 25–39.

[14] R. Nikiforov, G. Torbin. 2013. Fractal properties of random variables with independent Q_∞-digits. *Theory Probab. Math. Stat.* 86. P. 169–182. https:

//doi.org/10.1090/S0094-9000-2013-00896-5

[15] Y. Peres, B. Solomyak. 1996. Absolute continuity of Bernoulli convolutions, a simple proof. *Math. Res. Lett.* 3(2). P. 231–239. https://dx.doi.org/10.4310/MRL.1996.v3.n2.a8

[16] B. Solomyak. 1995. On the random series $\sum \pm\lambda^n$ (an Erdös problem). *Ann. of Math.* 142(3). P. 611–625. https://doi.org/10.2307/2118556

[17] G. Torbin. 2002. Fractal properties of the distributions of random variables with independent Q-symbols. *Transactions of the National Pedagogical University (Phys.-Math. Sci.)* 3. P. 241–252.

[18] G. Torbin, M. Pratsiovytyi. 1992. Random variables with independent Q^*-symbols, Institute for Mathematics of NASU: Random evolutions: theoretical and applied problems. P. 95–104.