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Time dependent ecological model

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Abstract. We study the asymptotic behaviour of an ecological model in the presence of a time dependent birth rate.

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1 Introduction

Dynamics of many models in spatial ecology may be described by means of Markov processes on the configuration spaces in the continuum. The most elaborated here is the case of birth-and-death processes. For this class of Markov processes we have several technical approaches to construct such stochastic processes or the dynamics of the states of considered systems in the course of stochastic evolutions. Powerful methods in the study of such dynamics are related, in particular, with the possibility of the reformulation of dynamical problems in terms of hierarchical equations for correlation functions and applications of the semigroup theory. We refer to [1], [2], [3] for the review on these subjects and extended list of related literature.

The study of birth-and-death processes in the spatial ecology are restricted, essentially, to the time homogeneous models. But in the applications the time dependent birth and death rates are very essential. In particular, in the study of plant populations is necessary to take into account the season periodic fluctuations for these rates.

In this note we study an easy population model which is the contact model in the continuum with time dependent rates. For this model we show the existence of two regimes. We have an easy criteria for the unbounded growth of the density of the system or for the asymptotic degeneration of such model.

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2 Time dependent contact model in the continuum

The contact model in the continuum was introduced in [6] as a Markov process in the space of locally finite configuration $\Gamma(\mathbb{R}^d)$ over $\mathbb{R}^d$. This process X_t describe branching of particles with a dispersion kernel $a(x-y)$ and independent deaths with a mortality intensity m. We assume that a is non-negative and integrable on $\mathbb{R}^d$ and $m \geq 0$, For simplicity of notations we will assume that $a(x)$ is even and spherically symmetric. If μ_0 is the distribution in time 0 for the process, then the marginal distribution of X_t is a probability measure μ_t on $\Gamma(\mathbb{R}^d)$. The latest is an evolution of states in the contact process. As usual in the theory of point processes, the measure μ_t may be characterized by the collection of correlation functions $k_t^{(n)}(x_1, ..., x_n), n \geq 0$. These functions are symmetric w.r.t. permutations and satisfy certain positive definiteness condition [4].

The evolution of states in this process has very easy recurrent description of correlation functions evolution, see [7, 9]. In particular, the evolution equation for the density $k_t^{(1)}(x) = u(t,x)$ of the system has form

$$\frac{\partial}{\partial t}u(t,x) = \int_{\mathbb{R}^d} a(x-y)u(t,y)dy - mu(t,x).$$

It is useful to rewrite this equation in terms of the jump generator:

$$\frac{\partial}{\partial t}u(t,x) = \int_{\mathbb{R}^d} a(x-y)[u(t,y) - u(t,x)]dy + (\langle a\rangle - m)u(t,x)$$

$$= (L_a + (\langle a\rangle - m))u(t.x).$$

Here

$$\langle a\rangle = \int_{\mathbb{R}^d} a(x)dx.$$

It is clear that the case $m = \langle a\rangle$ gives stationary solution. For the initial density $u(0,x) = 1$ in this case $u(t,x) = 1$ for all $x \in \mathbb{R}^d$ [5].

Our first problem is related to the study of the case of time dependent birth rate

$$a_t(x,y) = \lambda(t)a(x-y)$$

with continuous non-negative $\lambda(t)$. Definitely, $a_t(x,y)$ may have much more general structure but in this paper we consider the simplest case to be able to show possible effects in the most easy technical situation.

The equation for the density of the system will have form

$$\frac{\partial}{\partial t}u(t,x) = \lambda(t)\int_{\mathbb{R}^d} a(x-y)[u(t,y) - u(t,x)]dy + (\lambda(t) - m)u(t,x), \qquad (1)$$

where without the lost of generality we assume $\langle a\rangle = 1$, i.e., $a(x)$ is a probability density.

Of course, the situation with time dependent mortality $m(t)$ is also important in the applications, especially, to ecological models. Our considerations below may be easily modified to this case.

Lemma 1. *For any initial data $u(0,\cdot) \in L^\infty(\mathbb{R}^d)$ the solution to equation (1) has the representation*

$$u(t,x) = \exp(A(t)L_a)u(0,x)\exp(A(t) - mt), \tag{2}$$

where

$$A(t) = \int_0^t \lambda(s)ds.$$

For each $t > 0$ this solution belongs to $L^\infty(\mathbb{R}^d)$.

Proof. Because L_a is a Markov generator the operator $\exp(A(t)L_a)$ is the positive contraction in all spaces $L^p(\mathbb{R}^d), p \in (1,\infty]$. Using directly differentiation in t in (2) we show immediately that $u(t,x)$ satisfies equation (1). □

Our next aim is to consider the case of periodic birth intensity. To this end we suppose the following assumptions.

$$\lambda \in C(\mathbb{R}), \quad \lambda \geq 0 \tag{3}$$

$\lambda(t)$ is periodic with the period 1.

Theorem 2. *Consider equation (1) with the initial data $u(0,\cdot) \in L^\infty(\mathbb{R}^d)$ and assume periodicity condition (3).*

1) If

$$\int_0^1 \lambda(s)ds > m,$$

then

$$u(t,x) \to \infty, \ t \to \infty$$

for a.a. $x \in \mathbb{R}^d$.

2) If

$$\int_0^1 \lambda(s)ds < m,$$

then

$$u(t,x) \to 0, \ t \to \infty$$

uniformly w.r.t. a.a. $x \in \mathbb{R}^d$.

Proof. From Lemma 1 follows that the asymptotic of $u(t,x)$ is determined by the factor

$$\exp(A(t) - mt).$$

The periodicity of $\lambda(t)$ gives

$$A(t) = \int_0^t \lambda(s)ds \sim tA(1), t \to \infty.$$

Therefore,

$$\exp(A(t) - mt) \sim \exp t(\int_0^1 \lambda(s)ds - m)$$

and the statement of the theorem follows immediately.

□

Remark 3. For each particular type of the population there is possible maximal intensity of the birth rate, say $\lambda_+ > 0$. The condition of the growing density for the population without season oscillation is $\lambda_+ > m$. If $\epsilon \in (0.1)$ is the part of the period in which $\lambda(t) > 0$ and $\lambda(t) = 0, t \in (\epsilon, 1))$ then unbounded growth of the density will be still true if

$$\int_0^\epsilon \lambda(s)ds > m.$$

Additionally, if $\epsilon\lambda_+ < m$, then the population will be surely degenerated. In particular, if the vegetation part of the year is quite short, then we can not expect a growth of plant populations. This observation is applicable for several particular regions.

There is a natural question about the case of exact equality

$$\int_0^1 \lambda(s)ds = m.$$

Assume we have the initial density $u(0, x) = 1$. For moments of time $t = n \in \mathbb{N}$ representation (2) gives

$$u(n, x) = 1, x \in \mathbb{R}^d.$$

It means that the density is stabile for these times. Again, from the same representation follows that

$$\forall t > 0 \ \ u(t, x) = v(t), x \in \mathbb{R}^d,$$

where an oscillation

$$v(t) = e^{\int_{[t]}^t \lambda(s)ds - m(t-[t])} \le e^m.$$

Therefore, in the regime

$$\int_0^1 \lambda(s)ds = m$$

the density $u(t, x)$ may be different form the stationary solution only by the uniformly bounded in time oscillation.

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