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Microscopic dynamics and kinetic description of spatial ecology models

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Abstract. We consider a method for the construction of Markov statistical dynamics for a class of birth-and-death ecological models in the continuum. Mesoscopic scaling limits for these dynamics lead to the kinetic equations for the density of a population. The resulting evolution equations are non-local and non-linear ones. We discuss properties of solutions to kinetic equations which strongly depend on characteristics of the models considered. The survey paper is devoted to giving an overview of our recent progress on the subject and it is not intended to be a complete review of the field.

1 Introduction

Dynamics of interacting particle systems appear in several areas of the complex systems theory. In particular, we observe a growing activity in the study of Markov dynamics for continuous systems. The latter fact is motivated, in particular, by modern problems of mathematical physics, ecology, mathematical biology, and genetics, see e.g. [16, 17, 20, 35] and literature cited therein. Moreover, Markov dynamics are used for the construction of social, economic and demographic models. Note that Markov processes for continuous systems are considering in the stochastic analysis as dynamical point processes [28, 29, 31] and they appear even in the representation theory of big groups [7, 8].

A mathematical formalization of the problem may be described as the following. As a phase space of the system we use the space $\Gamma(\mathbb{R}^d)$ of locally finite configurations in the Euclidean space $\mathbb{R}^d$. An heuristic Markov generator which describes considered model is given by its expression on a proper set of functions (observables) over $\Gamma(\mathbb{R}^d)$. With this operator we can relate two evolution equations. Namely, backward Kolmogorov equation for observables and Kolmogorov forward equation on probability measures on the phase space $\Gamma(\mathbb{R}^d)$ (macroscopic states of the system). The latter equation is a.k.a. Fokker–Planck equation in the mathematical physics terminology. Comparing with the usual situation in the stochastic analysis, there is an essential technical difficulty: corresponding Markov process in the configuration space may be

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constructed only in very special particular cases. As a result, a description of Markov dynamics in terms of random trajectories is absent for most of models under considerations.

As an alternative approach we use a concept of the statistical dynamics that substitutes the notion of a Markov stochastic process. A central object now is an evolution of states of the system that shall be defined by mean of the Fokker–Planck equation. This evolution equation with respect to probability measures on $\Gamma(\mathbb{R}^d)$ may be reformulated as a hierarchical chain of equations for correlation functions of considered measures. Such kind of evolution equations are well known in the study of Hamiltonian dynamics for classical gases as BBGKY chains but now they appear as a tool for construction and analysis of Markov dynamics. As an essential technical step, we consider related pre-dual evolution chains of equations on the so-called quasi-observables. As it will be shown in the paper, such hierarchical equations may be analyzed in the framework of semigroup theory with the use of powerful techniques of perturbation theory for the semigroup generators etc. Considering the dual evolution for the constructed semigroup on quasi-observables we introduce then the dynamics on correlation functions. Such a scheme of constructing the dynamics comes as a surprise because one cannot expect any perturbation techniques for the initial Kolmogorov evolution equations. The point is that states of infinite interacting particle systems are given by measures which are, in general, mutually orthogonal. As a result, we can not compare their evolutions or apply a perturbative approach. But under quite general assumptions they have correlation functions and corresponding dynamics may be considered in a common Banach space of correlation functions. Proper choice of this Banach space means, in fact, that we find an admissible class of initial states for which the statistical dynamics may be constructed. There we see again a crucial difference comparing with the framework of Markov stochastic processes, where the evolution is defined for any initial distribution.

Another interesting topic is related to the study of different scalings of the microscopic systems. Among others, the crucial role from the point of view of applications is played mesoscopic (Vlasov) description of the mentioned above microscopic systems. Originally, the notion of the Vlasov scaling was related to the Hamiltonian dynamics of interacting particle systems. This is a mean field scaling limit when the influence of weak long-range forces is taken into account. Rigorously, this limit was studied by W. Braun and K. Hepp in [5] for the Hamiltonian dynamics, and by R. L. Dobrushin [11] for more general deterministic dynamical systems. In [15], we proposed a general scheme for a Vlasov-type scaling of stochastic Markovian dynamics. Our approach is based on a proper scaling of the evolutions of correlation functions proposed by H. Spohn in [46] for the Hamiltonian dynamics. The present paper is meant to provide a comprehensive overview of our recent approaches to the birth and death stochastic dynamics. In particular, the approach proposed in [15] gives us a rigorous framework for the study of convergence of the scaled hierarchical equations to a solution of the limiting Vlasov hierarchy, and for the derivation of a resulting non-linear evolutional equation for the density of the limiting system. We consider some special birth-and-death models to show how the general conditions proposed in the paper may be verified in applications.

In the last section we study the kinetic (Vlasov) equation which corresponds to the birth-death Bolker–Pacala–Dieckman–Law (BDLP) model [9]. Namely, we consider a non-linear non-local evolution equation with non-local terms, which are convolutions with probability densities. We demonstrate that the long-time behavior of the solution depends on the asymptotic of the birth kernel and the initial condition, where either constant speed of the propagation or acceleration may be observed. Under additional assumptions, we also prove existence and uniqueness of traveling waves.

The results introduced in this article do not pretend to be novel. The present survey work provides a thorough summary of our papers [14–16, 18, 19, 21, 22, 24–26] as well as our understanding of fundamental ideas and results on the subject.

The structure of the paper is following. Section 2 contains a brief summary of the mathematical description of complex systems. In Section 3 we discuss general concept of statistical dynamics for Markov evolutions in the continuum and introduce necessary mathematical structures. Then, in Section 4, this concept is applied to an important class of Markov dynamics of continuous systems, namely, to birth-and-death models. Here general conditions for the existence of a semigroup evolution in a space of quasi-observables are obtained. Then we construct evolutions of correlation functions as dual objects. It is shown how to apply general results to the study of particular models of statistical dynamics coming from mathematical physics and ecology. In Section 5 we discuss the Vlasov-type scaling for birth-and-death stochastic dynamics. Finally, in Section 6 we study the kinetic (Vlasov) equation for the birth-death BDLP model.

2 Mathematical description of complex systems

We proceed to the mathematical realization of complex systems.

Let $\mathcal{B}(\mathbb{R}^d)$ be the family of all Borel sets in $\mathbb{R}^d$, $d \geq 1$; $\mathcal{B}_\mathrm{b}(\mathbb{R}^d)$ denotes the system of all bounded sets from $\mathcal{B}(\mathbb{R}^d)$.

The configuration space over space $\mathbb{R}^d$ consists of all locally finite subsets (configurations) of $\mathbb{R}^d$. Namely,

$$\Gamma = \Gamma\big(\mathbb{R}^d\big) := \Big\{\gamma \subset \mathbb{R}^d \,\Big|\, |\gamma_\Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d)\Big\}.$$

Here $|\cdot|$ means the cardinality of a set, and $\gamma_\Lambda := \gamma \cap \Lambda$. We may identify each $\gamma \in \Gamma$ with the non-negative Radon measure $\sum_{x\in\gamma} \delta_x \in \mathcal{M}(\mathbb{R}^d)$, where δ_x is the Dirac measure with unit mass at x, $\sum_{x\in\emptyset} \delta_x$ is, by definition, the zero measure, and $\mathcal{M}(\mathbb{R}^d)$ denotes the space of all non-negative Radon measures on $\mathcal{B}(\mathbb{R}^d)$. This identification allows us to endow Γ with the topology induced by the vague topology on $\mathcal{M}(\mathbb{R}^d)$, i.e. the weakest topology on Γ with respect to which all mappings

$$\Gamma \ni \gamma \mapsto \sum_{x\in\gamma} f(x) \in \mathbb{R}$$

are continuous for any $f \in C_0(\mathbb{R}^d)$, the set of all continuous functions on

$\mathbb{R}^d$ with compact supports. It is worth noting the vague topology may be metrizable in such a way that Γ becomes a Polish space (see e.g. [33] and references therein).

Corresponding to the vague topology the Borel σ-algebra $\mathcal{B}(\Gamma)$ appears the smallest σ-algebra for which all mappings

$$\Gamma \ni \gamma \mapsto N_\Lambda(\gamma) := |\gamma_\Lambda| \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \tag{2.1}$$

are measurable for any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$, see e.g. [3].

Among all measurable functions $F : \Gamma \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ we mark out the set $\mathcal{F}_0(\Gamma)$ consisting of such of them for which $|F(\gamma)| < \infty$ at least for all $|\gamma| < \infty$. The important subset of $\mathcal{F}_0(\Gamma)$ formed by cylindric functions on Γ. Any such a function is characterized by a set $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ such that $F(\gamma) = F(\gamma_\Lambda)$ for all $\gamma \in \Gamma$. The class of cylindric functions we denote by $\mathcal{F}_{\mathrm{cyl}}(\Gamma) \subset \mathcal{F}_0(\Gamma)$.

Functions on Γ are usually called *observables* . This notion is borrowed from statistical physics and means that typically in course of empirical investigation we may estimate, check, see only some quantities derived from the system as a whole rather then look into the system itself.

We denote the class of all probability measures on $\big(\Gamma, \mathcal{B}(\Gamma)\big)$ by $\mathcal{M}^1(\Gamma)$. Given a distribution $\mu \in \mathcal{M}^1(\Gamma)$ one can consider a collection of random variables $N_\Lambda(\cdot)$, $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ defined in (2.1). They describe random numbers of elements inside bounded regions. The natural assumption is that these random variables should have finite moments. Thus, we consider the class $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ of all measures from $\mathcal{M}^1(\Gamma)$ such that

$$\int_\Gamma |\gamma_\Lambda|^n \, \mathrm{d}\mu(\gamma) < \infty, \qquad \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d), n \in \mathbb{N}.$$

Example 2.1. Let σ be a non-atomic Radon measure on $\big(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\big)$. Then the *Poisson measure* π_σ with intensity measure σ is defined on $\mathcal{B}(\Gamma)$ by

$$\pi_\sigma\big(\{\gamma \in \Gamma \big| N_\Lambda(\gamma) = |\gamma_\Lambda| = n\}\big) = \frac{\big(\sigma(\Lambda)\big)^n}{n!} \exp\big\{-\sigma(\Lambda)\big\}, \quad \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d), n \in \mathbb{N}_0.$$

In the case of the Lebesgue measure $\sigma(dx) = dx$ one can say about the homogeneous Poisson distribution (measure) $\pi := \pi_{dx}$ with constant intensity 1.

The space of (finite) configuration which belong to a bounded domain $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ will be denoted by $\Gamma(\Lambda)$. The σ-algebra $\mathcal{B}(\Gamma(\Lambda))$ may be generated by family of mappings $\Gamma(\Lambda) \ni \gamma \mapsto N_{\Lambda'}(\gamma) \in \mathbb{N}_0$, $\Lambda' \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$, $\Lambda' \subset \Lambda$. A measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is called locally absolutely continuous with respect to the Poisson measure π if for any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ the projection of μ onto $\Gamma(\Lambda)$ is absolutely continuous with respect to (w.r.t.) the projection of π onto $\Gamma(\Lambda)$. More precisely, if we consider the projection mapping $p_\Lambda : \Gamma \to \Gamma(\Lambda)$, $p_\Lambda(\gamma) := \gamma_\Lambda$ then $\mu^\Lambda := \mu \circ p_\Lambda^{-1}$ is absolutely continuous w.r.t. $\pi_\Lambda := \pi \circ p_\Lambda^{-1}$.

By e.g. [32], for any $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ which is locally absolutely continuous w.r.t the Poisson measure, there exists the family of (symmetric) *correlation functions* $k_\mu^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}_+ := [0, \infty)$ which defined as follows. For any symmetric function $f^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}$ with finite support the following equality holds

$$\int_\Gamma \sum_{\{x_1,\ldots,x_n\}\subset\gamma} f^{(n)}(x_1,\ldots,x_n)\,\mathrm{d}\mu(\gamma)$$
$$= \frac{1}{n!}\int_{(\mathbb{R}^d)^n} f^{(n)}(x_1,\ldots,x_n)k_\mu^{(n)}(x_1,\ldots,x_n)\,\mathrm{d}x_1\ldots\mathrm{d}x_n \quad (2.2)$$

for $n\in\mathbb{N}$, and $k_\mu^{(0)} := 1$.

The meaning of this notion is the following: the correlation function $k_\mu^{(n)}(x_1,\ldots,x_n)$ describes the non-normalized density of probability to have points of our systems in the positions $x_1,\ldots,x_n$.

The symmetric function of n variables from $\mathbb{R}^d$ can be considered as functions on n-point subsets from $\mathbb{R}^d$. We proceed now to the exact constructions.

The space of n-point configurations in $Y\in\mathcal{B}(\mathbb{R}^d)$ is defined by

$$\Gamma^{(n)}(Y) := \bigl\{\eta\subset Y \bigm| |\eta| = n\bigr\}, \qquad n\in\mathbb{N}.$$

We put $\Gamma^{(0)}(Y) := \{\emptyset\}$. As a set, $\Gamma^{(n)}(Y)$ may be identified with the symmetrization of

$$\widetilde{Y^n} = \bigl\{(x_1,\ldots,x_n)\in Y^n \bigm| x_k\neq x_l \text{ if } k\neq l\bigr\}.$$

Hence, one can introduce the corresponding Borel σ-algebra, which we denote by $\mathcal{B}\bigl(\Gamma^{(n)}(Y)\bigr)$. The space of finite configurations in $Y\in\mathcal{B}(\mathbb{R}^d)$ is defined as

$$\Gamma_0(Y) := \bigsqcup_{n\in\mathbb{N}_0}\Gamma^{(n)}(Y).$$

This space is equipped with the topology of the disjoint union. Let $\mathcal{B}\bigl(\Gamma_0(Y)\bigr)$ denote the corresponding Borel σ-algebra. In the case of $Y=\mathbb{R}^d$ we will omit the index Y in the previously defined notations. Namely,

$$\Gamma_0 := \Gamma_0(\mathbb{R}^d), \qquad \Gamma^{(n)} := \Gamma^{(n)}(\mathbb{R}^d), \quad n\in\mathbb{N}_0.$$

The restriction of the Lebesgue product measure $(dx)^n$ to $\bigl(\Gamma^{(n)},\mathcal{B}(\Gamma^{(n)})\bigr)$ we denote by $m^{(n)}$. We set $m^{(0)} := \delta_{\{\emptyset\}}$. The Lebesgue–Poisson measure λ on Γ_0 is defined by

$$\lambda := \sum_{n=0}^{\infty}\frac{1}{n!}m^{(n)}. \quad (2.3)$$

For any $\Lambda\in\mathcal{B}_\mathrm{b}(\mathbb{R}^d)$ the restriction of λ to $\Gamma_0(\Lambda)=\Gamma(\Lambda)$ will be also denoted by λ.

Remark 2.1. The space $\bigl(\Gamma,\mathcal{B}(\Gamma)\bigr)$ is the projective limit of the family of measurable spaces $\bigl\{\bigl(\Gamma(\Lambda),\mathcal{B}(\Gamma(\Lambda))\bigr)\bigr\}_{\Lambda\in\mathcal{B}_\mathrm{b}(\mathbb{R}^d)}$. The Poisson measure π on $\bigl(\Gamma,\mathcal{B}(\Gamma)\bigr)$ from Example 2.1 may be defined as the projective limit of the family of measures $\{\pi^\Lambda\}_{\Lambda\in\mathcal{B}_\mathrm{b}(\mathbb{R}^d)}$, where $\pi^\Lambda := e^{-m(\Lambda)}\lambda$ is a probability measure on $\bigl(\Gamma(\Lambda),\mathcal{B}(\Gamma(\Lambda))\bigr)$ and $m(\Lambda)$ is the Lebesgue measure of $\Lambda\in\mathcal{B}_\mathrm{b}(\mathbb{R}^d)$ (see e.g. [3] for details).

Functions on Γ_0 will be called *quasi-observables* . Any $\mathcal{B}(\Gamma_0)$-measurable function G on Γ_0 is, in fact, defined by a sequence of functions $\{G^{(n)}\}_{n\in\mathbb{N}_0}$ where $G^{(n)}$ is a $\mathcal{B}(\Gamma^{(n)})$-measurable function on $\Gamma^{(n)}$. We preserve the same notation for the function $G^{(n)}$ considered as a symmetric function on $(\mathbb{R}^d)^n$. Note that $G^{(0)} \in \mathbb{R}$.

A set $M \in \mathcal{B}(\Gamma_0)$ is called bounded if there exists $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that

$$M \subset \bigsqcup_{n=0}^{N} \Gamma^{(n)}(\Lambda).$$

The set of bounded measurable functions on Γ_0 with bounded support we denote by $B_{\mathrm{bs}}(\Gamma_0)$, i.e., $G \in B_{\mathrm{bs}}(\Gamma_0)$ iff $G \upharpoonright_{\Gamma_0 \setminus M} = 0$ for some bounded $M \in \mathcal{B}(\Gamma_0)$. For any $G \in B_{\mathrm{bs}}(\Gamma_0)$ the functions $G^{(n)}$ have finite supports in $(\mathbb{R}^d)^n$ and may be substituted into (2.2). But, additionally, the sequence of $G^{(n)}$ vanishes for big n. Therefore, one can sum up equalities (2.2) over $n \in \mathbb{N}_0$. This requires the following definition.

Let $G \in B_{\mathrm{bs}}(\Gamma_0)$, then we define the function $KG : \Gamma \to \mathbb{R}$ by

$$\begin{aligned} (KG)(\gamma) &:= \sum_{\eta \Subset \gamma} G(\eta) \\ &= G^{(0)} + \sum_{n=1}^{\infty} \sum_{\{x_1,\dots,x_n\} \subset \gamma} G^{(n)}(x_1,\dots,x_n), \quad \gamma \in \Gamma, \end{aligned} \tag{2.4}$$

see e.g. [32, 37, 38]. The summation in (2.4) is taken over all finite subconfigurations $\eta \in \Gamma_0$ of the (infinite) configuration $\gamma \in \Gamma$; we denote this by the symbol, $\eta \Subset \gamma$. The mapping K is linear, positivity preserving, and invertible, with

$$(K^{-1}F)(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad \eta \in \Gamma_0. \tag{2.5}$$

By [32], for any $G \in B_{\mathrm{bs}}(\Gamma_0)$, we have $KG \in \mathcal{F}_{\mathrm{cyl}}(\Gamma)$, moreover, there exists $C = C(G) > 0$, $\Lambda = \Lambda(G) \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$, and $N = N(G) \in \mathbb{N}$ such that

$$|KG(\gamma)| \le C\big(1 + |\gamma_\Lambda|\big)^N, \quad \gamma \in \Gamma.$$

The expression (2.4) can be extended to the class of all nonnegative measurable $G : \Gamma_0 \to \mathbb{R}_+$, in this case, evidently, $KG \in \mathcal{F}_0(\Gamma)$. Stress that the left hand side (l.h.s.) of (2.5) has a meaning for any $F \in \mathcal{F}_0(\Gamma)$, moreover, in this case $(KK^{-1}F)(\gamma) = F(\gamma)$ for any $\gamma \in \Gamma_0$.

For G as above we may sum up (2.2) over n and rewrite the result in a compact form:

$$\int_\Gamma (KG)(\gamma)\, \mathrm{d}\mu(\gamma) = \int_{\Gamma_0} G(\eta) k_\mu(\eta)\, \mathrm{d}\lambda(\eta). \tag{2.6}$$

As was shown in [32], the equality (2.4) may be extended on all functions G such that the l.h.s. of (2.6) is finite. In this case (2.4) holds for μ-a.a. $\gamma \in \Gamma$ and (2.6) holds, too.

3 Statistical descriptions of Markov evolutions

Spatial Markov processes in $\mathbb{R}^d$ may be described as stochastic evolutions of configurations $\gamma \subset \mathbb{R}^d$. In course of such evolutions points of configurations may disappear (die), move (continuously or with jumps from one position to another), or new particles may appear in a configuration (that is birth). The rates of these random events may depend on whole configuration that reflect an interaction between elements of the system.

The construction of a spatial Markov process in the continuum is a highly difficult question which is not solved in a full generality at present, see e.g. the review [44] and more detail references about birth-and-death processes in Section 3. Meanwhile, for discrete systems the corresponding processes have been constructed under quite general assumptions, see e.g. [39]. One of the main difficulties for continuous systems includes the necessity to control number of elements in a bounded region. Note that the construction of spatial processes on bounded sets from $\mathbb{R}^d$ is typically well understood, see e.g. [23].

The existing Markov process $\Gamma \ni \gamma \mapsto X_t^\gamma \in \Gamma$, $t > 0$ provides solution to the backward Kolmogorov equation for bounded continuous functions:

$$\frac{\mathrm{d}}{\mathrm{d}t} F_t = L F_t,$$

where L is the Markov generator of the process X_t. The question about existence for a Markov process with a generator L is still open. On the other hand, the evolution of a state in the course of a stochastic dynamics is an important question in its own right. A mathematical formulation of this question may be realized through the forward Kolmogorov equation for probability measures (states) on the configuration space Γ. Namely, we consider the pairing between functions and measures on Γ given by

$$\langle F, \mu \rangle := \int_\Gamma F(\gamma)\ \mathrm{d}\mu(\gamma). \tag{3.1}$$

Then we consider the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle F, \mu_t \rangle = \langle LF, \mu_t \rangle, \quad t > 0, \quad \mu_t \big|_{t=0} = \mu_0, \tag{3.2}$$

where F is an arbitrary function from a proper set, e.g. $F \in K\big(B_{\mathrm{bs}}(\Gamma_0)\big) \subset \mathcal{F}_{\mathrm{cyl}}(\Gamma)$. In fact, the solution to (3.2) describes the time evolution of distributions instead of the evolution of initial points in the Markov process. We rewrite (3.2) in the heuristic form

$$\frac{\mathrm{d}}{\mathrm{d}t} \mu_t = L^* \mu_t, \tag{3.3}$$

where L^* is the (informally) adjoint operator of L with respect to the pairing (3.1).

In the physical literature, (3.3) is referred to the *Fokker–Planck equation.* The Markovian property of L yields that (3.3) might have a solution in the class of probability measures. However, the mere existence of the corresponding Markov process will not give us much information about properties of the solution to (3.3), in particular, about its moments or correlation functions. To get it, we suppose now that a solution $\mu_t \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ to (3.2) exists and remains locally absolutely continuous with respect to the Poisson measure π for all $t > 0$ provided μ_0 has such a property. Then one can consider the correlation function $k_t := k_{\mu_t}$, $t \geq 0$. If we suppose that

$$LF \in \mathcal{F}_0(\Gamma) \qquad \text{for all } F \in \mathcal{F}_{\mathrm{cyl}}(\Gamma), \tag{3.4}$$

then, one can calculate $K^{-1}LF$ using (2.5), and, by (2.6), we may rewrite (3.2) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\!\langle K^{-1}F, k_t\rangle\!\rangle = \langle\!\langle K^{-1}LF, k_t\rangle\!\rangle, \quad t > 0, \quad k_t\big|_{t=0} = k_0, \tag{3.5}$$

for all $F \in K\big(B_{\mathrm{bs}}(\Gamma_0)\big) \subset \mathcal{F}_{\mathrm{cyl}}(\Gamma)$. Here the pairing between functions on Γ_0 is given by

$$\langle\!\langle G,\, k\rangle\!\rangle := \int_{\Gamma_0} G(\eta)k(\eta)\ \mathrm{d}\lambda(\eta). \tag{3.6}$$

Let us recall that then, by (2.3),

$$\langle\!\langle G, k\rangle\!\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G^{(n)}(x_1, \ldots, x_n) k^{(n)}(x_1, \ldots, x_n)\ \mathrm{d}x_1 \ldots \mathrm{d}x_n,$$

Next, if we substitute $F = KG$, $G \in B_{\mathrm{bs}}(\Gamma_0)$ in (3.5), we derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\!\langle G, k_t\rangle\!\rangle = \langle\!\langle \widehat{L}G, k_t\rangle\!\rangle, \quad t > 0, \quad k_t\big|_{t=0} = k_0, \tag{3.7}$$

for all $G \in B_{\mathrm{bs}}(\Gamma_0)$. Here the operator

$$(\widehat{L}G)(\eta) := (K^{-1}LKG)(\eta), \quad \eta \in \Gamma_0$$

is defined point-wise for all $G \in B_{\mathrm{bs}}(\Gamma_0)$ under conditions (3.4). Consequently, we are interested in a weak solution to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} k_t = \widehat{L}^* k_t, \quad t > 0, \quad k_t\big|_{t=0} = k_0, \tag{3.8}$$

where $\widehat{L}^*$ is dual operator to $\widehat{L}$ with respect to the duality (3.6), namely,

$$\int_{\Gamma_0} (\widehat{L}G)(\eta)k(\eta)\ \mathrm{d}\lambda(\eta) = \int_{\Gamma_0} G(\eta)(\widehat{L}^*k)(\eta)\ \mathrm{d}\lambda(\eta). \tag{3.9}$$

The procedure of deriving the operator $\widehat{L}$ for a given L is fully combinatorial meanwhile to obtain the expression for the operator $\widehat{L}^*$ we need an analog of integration by parts formula.

We recall that any function on Γ_0 may be identified with an infinite vector of symmetric functions of the growing number of variables. In this approach, the operator $\widehat{L}^*$ in (3.8) will be realized as an infinite matrix $\big(\widehat{L}^*_{n,m}\big)_{n,m\in\mathbb{N}_0}$, where $\widehat{L}^*_{n,m}$ is a mapping from the space of symmetric functions of n variables into the space of symmetric functions of m variables. As a result, instead of equation (3.2) for infinite-dimensional objects we obtain an infinite system of equations for functions $k_t^{(n)}$ each of them is a function of a finite number of variables, namely

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}k_t^{(m)}(x_1,\ldots,x_m) &= \sum_{n\in\mathbb{N}_0}\big(\widehat{L}^*_{n,m}k_t^{(n)}\big)(x_1,\ldots,x_m), \quad t>0,\ m\in\mathbb{N}_0,\\ k_t^{(m)}(x_1,\ldots,x_m)\big|_{t=0} &= k_0^{(m)}(x_1,\ldots,x_m).\end{aligned}\tag{3.10}$$

Of course, in general, for a fixed n, any equation from (3.10) itself is not closed and includes functions $k_t^{(m)}$ of other orders $m\neq n$, nevertheless, the system (3.10) is a closed linear system. The chain evolution equations for $k_t^{(n)}$ consists the so-called *hierarchy* which is an analog of the BBGKY hierarchy for Hamiltonian systems, see e.g. [12].

In the present paper the restrict our attention to the so-called *sub-Poissonian* correlation functions. Namely, for a given $C>0$ we consider the following Banach space

$$\mathcal{K}_C := \big\{k:\Gamma_0\to\mathbb{R}\ \big|\ k\cdot C^{-|\cdot|}\in L^\infty(\Gamma_0,d\lambda)\big\}\tag{3.11}$$

with the norm

$$\|k\|_{\mathcal{K}_C} := \|C^{-|\cdot|}k(\cdot)\|_{L^\infty(\Gamma_0,d\lambda)}.$$

It is clear that $k\in\mathcal{K}_C$ implies,

$$\big|k(\eta)\big|\leq\|k\|_{\mathcal{K}_C}\,C^{|\eta|}\qquad\text{for }\lambda\text{-a.a. }\eta\in\Gamma_0.\tag{3.12}$$

In the following we study the initial value problem (3.8) using the following scheme. We solve this equation in space $\mathcal{K}_C$. The well-posedness of the initial value problem in this case is equivalent with an existence of the strongly continuous semigroup (C_0-semigroup in the sequel) in the space $\mathcal{K}_C$ with a generator $\widehat{L}^*$. However, the space $\mathcal{K}_C$ is isometrically isomorphic to the space $L^\infty(\Gamma_0,C^{|\cdot|}d\lambda)$ whereas, by the H. Lotz theorem [40], [1], in a L^∞ space any C_0-semigroup is uniformly continuous, that is it has a bounded generator. Typically, for the operator L, any operator $\widehat{L}^*_{n,m}$, cf. (3.10), might be bounded as an operator between two spaces of bounded symmetric functions of n and m variables whereas the whole operator $\widehat{L}^*$ is unbounded in $\mathcal{K}_C$.

To avoid this difficulties we use a trick which goes back to R. Phillips [45]. The main idea is to consider the semigroup in L^∞ space not itself but as a dual semigroup $T^*(t)$ to a C_0-semigroup $T(t)$ with a generator A in the pre-dual L^1 space. In this case $T^*(t)$ appears strongly continuous semigroup not on the whole L^∞ but on the closure of the domain of A^* only.

In our case this leads to the following scheme. We consider the pre-dual Banach space to $\mathcal{K}_C$, namely, for $C>0$,

$$\mathfrak{L}_C := L^1\big(\Gamma_0,C^{|\cdot|}d\lambda\big).\tag{3.13}$$

The norm in $\mathfrak{L}_C$ is given by

$$\begin{aligned}\|G\|_C : &= \int_{\Gamma_0}\big|G(\eta)\big|C^{|\eta|}\ \mathrm{d}\lambda(\eta)\\ &= \sum_{n=0}^\infty\frac{C^n}{n!}\int_{(\mathbb{R}^d)^n}\big|G^{(n)}(x_1,\ldots,x_n)\big|\ \mathrm{d}x_1\ldots\,\mathrm{d}x_n.\end{aligned}$$

Consider the initial value problem, cf. (3.7), (3.8),

$$\frac{\mathrm{d}}{\mathrm{d}t} G_t = \widehat{L} G_t, \quad t > 0, \quad G_t\big|_{t=0} = G_0 \in \mathfrak{L}_C. \tag{3.14}$$

As long as (3.14) is well-posed in $\mathfrak{L}_C$ there exists a C_0-semigroup $\widehat{T}(t)$ in $\mathfrak{L}_C$. Then using Philips' result we see that the restriction of the dual semigroup $\widehat{T}^*(t)$ onto $\overline{\mathrm{Dom}(\widehat{L}^*)}$ will be C_0-semigroup with generator which is a part of $\widehat{L}^*$ (see Section 4 below for details). This provides a solution to (3.8) which continuously depends on an initial data from $\overline{\mathrm{Dom}(\widehat{L}^*)}$. And after we would like to find a more useful universal subspace of $\mathcal{K}_C$ which is not depend on the operator $\widehat{L}^*$. The realization of this scheme for a birth-and-death operator L is presented in Section 4 below. As a result, we obtain the classical solution to (3.8) for $t > 0$ in a class of sub-Poissonian functions which satisfy the Ruelle-type bound (3.12). Of course, after this we need to verify existence and uniqueness of measures whose correlation functions are solutions to (3.8). This usually can be done using proper approximation schemes, see e.g. Section 5.

4 Birth-and-death evolutions in the continuum

4.1 Microscopic description

One of the most important classes of Markov evolution in the continuum is given by the birth-and-death Markov processes in the space Γ of all configurations in $\mathbb{R}^d$. These are processes in which an infinite number of individuals exist at each instant, and the rates at which new individuals appear and some old ones disappear depend on the current configuration of existing individuals [31]. The corresponding Markov generators have a natural heuristic representation in terms of birth and death intensities. The birth intensity $b(x,\gamma) \geq 0$ characterizes the appearance of a new point at $x \in \mathbb{R}^d$ in the presence of a given configuration $\gamma \in \Gamma$. The death intensity $d(x,\gamma) \geq 0$ characterizes the probability of the event that the point x of the configuration γ disappears, depending on the location of the remaining points of the configuration $\gamma \setminus \{x\}$ (in the sequel $\gamma \setminus x$). Heuristically, the corresponding Markov generator is described by the following expression,

$$(LF)(\gamma) := \sum_{x\in\gamma} d(x,\gamma\setminus x)\,[F(\gamma\setminus x) - F(\gamma)] \\ + \int_{\mathbb{R}^d} b(x,\gamma)\,[F(\gamma\cup x) - F(\gamma)]\,\mathrm{d}x, \tag{4.1}$$

for proper functions $F : \Gamma \to \mathbb{R}$.

4.2 Expressions for $\widehat{L}$ and $\widehat{L}^*$. Examples of rates b and d

We always suppose that rates $d, b : \mathbb{R}^d \times \Gamma \to [0; +\infty]$ from (4.1) satisfy the following assumptions

$$d(x,\eta), b(x,\eta) > 0, \qquad \eta \in \Gamma_0 \setminus \{\emptyset\},\ x \in \mathbb{R}^d \setminus \eta,$$

$$
\begin{aligned}
&d(x,\eta), b(x,\eta) < \infty, && \eta \in \Gamma_0,\ x \in \mathbb{R}^d \setminus \eta,\\
&\int_M \big(d(x,\eta) + b(x,\eta)\big)\, \mathrm{d}\lambda(\eta) < \infty, && M \in \mathcal{B}(\Gamma_0) \text{ bounded, a.a. } x \in \mathbb{R}^d,\\
&\int_\Lambda \big(d(x,\eta) + b(x,\eta)\big)\, \mathrm{d}x < \infty, && \eta \in \Gamma_0,\ \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d).
\end{aligned}
$$

We start with the expression for $\widehat{L} = K^{-1}LK$,

Proposition 4.1 ([20, Proposition 5]). *For any $G \in B_{\mathrm{bs}}(\Gamma_0)$ the following formula holds*

$$
\begin{aligned}
(\widehat{L}G)(\eta) = &-\sum_{\xi\subset\eta} G(\xi) \sum_{x\in\xi} \big(K^{-1}d(x, \cdot\cup\xi\setminus x)\big)(\eta\setminus\xi)\\
&+\sum_{\xi\subset\eta}\int_{\mathbb{R}^d} G(\xi\cup x)\big(K^{-1}b(x,\cdot\cup\xi)\big)(\eta\setminus\xi)\,\mathrm{d}x, \qquad \eta\in\Gamma_0.
\end{aligned}
\tag{4.2}
$$

Using this, one can derive the explicit form of $\widehat{L}^*$.

Proposition 4.2 ([20, Corollary 9]). *For any $k \in B_{\mathrm{bs}}(\Gamma_0)$ the following formula holds*

$$
\begin{aligned}
(\widehat{L}^*k)(\eta) = &-\sum_{x\in\eta}\int_{\Gamma_0} k(\zeta\cup\eta)\big(K^{-1}d(x,\cdot\cup\eta\setminus x)\big)(\zeta)\,\mathrm{d}\lambda(\zeta)\\
&+\sum_{x\in\eta}\int_{\Gamma_0} k(\zeta\cup(\eta\setminus x))\big(K^{-1}b(x,\cdot\cup\eta\setminus x)\big)(\zeta)\,\mathrm{d}\lambda(\zeta),
\end{aligned}
$$

where $\widehat{L}^*k$ is defined by (3.9).

4.3 Semigroup evolutions in the space of quasi-observables

We proceed now to the construction of a semigroup in the space $\mathfrak{L}_C$, $C>0$, see (3.13), which has a generator, given by $\widehat{L}$, with a proper domain. To define such domain, let us set

$$
D(\eta) := \sum_{x\in\eta} d(x,\eta\setminus x) \geq 0, \quad \eta\in\Gamma_0; \tag{4.3}
$$

$$
D := \{G\in\mathfrak{L}_C \mid D(\cdot)\,G\in\mathfrak{L}_C\}. \tag{4.4}
$$

Note that $B_{\mathrm{bs}}(\Gamma_0)\subset D$ and $B_{\mathrm{bs}}(\Gamma_0)$ is a dense set in $\mathfrak{L}_C$. Therefore, D is also a dense set in $\mathfrak{L}_C$. We will show now that $(\widehat{L}, D)$ given by (4.2), (4.4) generates C_0-semigroup on $\mathfrak{L}_C$ if only 'the *full energy of death*', given by (4.3), is big enough.

Theorem 4.3 ([18, Theorem 3.2]). *Suppose that there exists $a_1\geq 1$, $a_2>0$ such that for all $\xi\in\Gamma_0$ and a.a. $x\in\mathbb{R}^d$*

$$
\sum_{x\in\xi}\int_{\Gamma_0}\left|K^{-1}d(x,\cdot\cup\xi\setminus x)\right|(\eta)\,C^{|\eta|}\,\mathrm{d}\lambda(\eta) \leq a_1 D(\xi), \tag{4.5}
$$

$$\sum_{x\in\xi}\int_{\Gamma_0}\left|K^{-1}b\left(x,\cdot\cup\xi\setminus x\right)\right|(\eta)\,C^{|\eta|}\,\mathrm{d}\lambda\left(\eta\right)\le a_2 D(\xi) \tag{4.6}$$

and, moreover,

$$a_1+\frac{a_2}{C}<\frac{3}{2}. \tag{4.7}$$

Then $(\widehat{L},D)$ *is the generator of a holomorphic semigroup* $\widehat{T}\left(t\right)$ *on* $\mathfrak{L}_C$*.*

4.4 Evolutions in the space of correlation functions

In this Subsection we will use the semigroup $\widehat{T}(t)$ acting oh the space of quasi-observables for a construction of a solution to the evolution equation (3.8) on the space of correlation functions.

We denote $d\lambda_C := C^{|\cdot|}d\lambda$; and the dual space $(\mathfrak{L}_C)' = \bigl(L^1(\Gamma_0, d\lambda_C)\bigr)' = L^\infty(\Gamma_0, d\lambda_C)$. As was mentioned before the space $(\mathfrak{L}_C)'$ is isometrically isomorphic to the Banach space $\mathcal{K}_C$ considered in (3.11)–(3.12). The isomorphism is given by the isometry R_C

$$(\mathfrak{L}_C)'\ni k\longmapsto R_C k := k\cdot C^{|\cdot|}\in\mathcal{K}_C. \tag{4.8}$$

Recall, one may consider the duality between the Banach spaces $\mathfrak{L}_C$ and $\mathcal{K}_C$ given by (3.6) with

$$|\langle\!\langle G,k\rangle\!\rangle|\le\|G\|_C\cdot\|k\|_{\mathcal{K}_C}.$$

Let $\bigl(\widehat{L}', \mathrm{Dom}(\widehat{L}')\bigr)$ be an operator in $(\mathfrak{L}_C)'$ which is dual to the closed operator $\bigl(\widehat{L}, D\bigr)$. We consider also its image on $\mathcal{K}_C$ under the isometry R_C. Namely, let $\widehat{L}^* = R_C\widehat{L}'R_{C^{-1}}$ with the domain $\mathrm{Dom}(\widehat{L}^*) = R_C\mathrm{Dom}(\widehat{L}')$. Similarly, one can consider the adjoint semigroup $\widehat{T}'(t)$ in $(\mathfrak{L}_C)'$ and its image $\widehat{T}^*(t)$ in $\mathcal{K}_C$.

The space $\mathfrak{L}_C$ is not reflexive, hence, $\widehat{T}^*(t)$ is not C_0-semigroup in whole $\mathcal{K}_C$. The last semigroup will be weak*-continuous, weak*-differentiable at 0 and $\widehat{L}^*$ will be weak*-generator of $\widehat{T}^*(t)$. Therefore, one has an evolution in the space of correlation functions. In fact, we have a solution to the evolution equation (3.8), in a weak*-sense. This subsection is devoted to the study of a classical solution to this equation. The restriction $\widehat{T}^{\odot}(t)$ of the semigroup $\widehat{T}^*(t)$ onto its invariant Banach subspace $\overline{\mathrm{Dom}(\widehat{L}^*)}$ (here and below all closures are in the norm of the space $\mathcal{K}_C$) is a strongly continuous semigroup. Moreover, its generator $\widehat{L}^{\odot}$ will be a part of $\widehat{L}^*$, namely,

$$\mathrm{Dom}(\widehat{L}^{\odot}) = \Bigl\{k\in\mathrm{Dom}(\widehat{L}^*)\ \Bigm|\ \widehat{L}^*k\in\overline{\mathrm{Dom}(\widehat{L}^*)}\Bigr\} \tag{4.9}$$

and $\widehat{L}^{\odot}k = \widehat{L}^*k$ for any $k\in\mathrm{Dom}(\widehat{L}^{\odot})$.

One can consider the restriction $\widehat{T}^{\odot\alpha}(t)$ of the semigroup $\widehat{T}^{\odot}(t)$ onto $\overline{\mathcal{K}_{\alpha C}}$. It will be strongly continuous semigroup with the generator $\widehat{L}^{\odot\alpha}$ which is a restriction of $\widehat{L}^{\odot}$ onto $\overline{\mathcal{K}_{\alpha C}}$. Namely, cf. 4.9,

$$\mathrm{Dom}(\widehat{L}^{\odot\alpha}) = \Bigl\{k\in\overline{\mathcal{K}_{\alpha C}}\ \Bigm|\ \widehat{L}^*k\in\overline{\mathcal{K}_{\alpha C}}\Bigr\},$$

and $\widehat{L}^{\odot\alpha}k = \widehat{L}^{\odot}k = \widehat{L}^{*}k$ for any $k \in \overline{\mathcal{K}_{\alpha C}}$. In the other words, $\widehat{L}^{\odot\alpha}$ is a part of $\widehat{L}^{*}$.

And now we may proceed to the main statement of this Subsection.

Theorem 4.4 ([19, Theorem 3.16])**.** *Let* (4.5), (4.6) *hold together with the following assumptions*

$$d\,(x,\xi) \le A(1+|\xi|)^N \nu^{|\xi|}, \tag{4.10}$$

$$1 \le \nu < \frac{C}{a_2}\Big(\frac{3}{2} - a_1\Big). \tag{4.11}$$

and let α *be chosen in the following way*

$$\frac{a_2}{C\big(\frac{3}{2} - a_1\big)} < \alpha < \frac{1}{\nu}.$$

Then for any $k_0 \in \overline{\mathcal{K}_{\alpha C}}$ *there exists a unique classical solution to* (3.8) *in the space* $\overline{\mathcal{K}_{\alpha C}}$*, and this solution is given by* $k_t = \widehat{T}^{\odot\alpha}(t)k_0$*. Moreover,* $k_0 \in \mathcal{K}_{\alpha C}$ *implies* $k_t \in \mathcal{K}_{\alpha C}$ *for* $t > 0$*.*

Example 4.1. (BDLP model) This example describes a generalization of the model of plant ecology (see [14] and references therein). Let L be given by (4.1) with

$$d(x,\gamma\setminus x) = m(x) + \varkappa^-(x)\sum_{y\in\gamma\setminus x} a^-(x-y), \qquad x\in\gamma,\ \gamma\in\Gamma,$$

$$b(x,\gamma) = \varkappa^+(x)\sum_{y\in\gamma} a^+(x-y), \qquad x\in\mathbb{R}^d\setminus\gamma,\ \gamma\in\Gamma,$$

where $0 < m \in L^\infty(\mathbb{R}^d)$, $0 \le \varkappa^\pm \in L^\infty(\mathbb{R}^d)$, $0 \le a^\pm \in L^1(\mathbb{R}^d,dx)\cap L^\infty(\mathbb{R}^d,dx)$, $\int_{\mathbb{R}^d} a^\pm(x)\,\mathrm{d}x = 1$. Let us suppose, cf. [14], that there exists $\delta > 0$ such that

$$(4+\delta)C\varkappa^-(x) \le m(x), \quad x\in\mathbb{R}^d, \tag{4.12}$$

$$(4+\delta)\varkappa^+(x) \le m(x), \quad x\in\mathbb{R}^d, \tag{4.13}$$

$$4\varkappa^+(x)a^+(x) \le C\varkappa^-(x)a^-(x). \quad x\in\mathbb{R}^d, \tag{4.14}$$

Then

$$d(x,\xi) + C\varkappa^-(x) \le d(x,\xi) + \frac{m(x)}{4+\delta} \le \Big(1 + \frac{1}{4+\delta}\Big)d(x,\xi),$$

$$b(x,\xi) + C\varkappa^+(x) \le \frac{C}{4}\varkappa^-(x)\sum_{y\in\xi} a^-(x-y) + \frac{Cm(x)}{4+\delta} < \frac{C}{4}d(x,\xi),$$

Hence, (4.5), (4.6) hold with $a_1 = 1 + \dfrac{1}{4+\delta}, a_2 = \dfrac{C}{4}$, that fulfills (4.7). Next, under conditions (4.12), (4.14), we have

$$d(x,\xi) \le \|m\|_{L^\infty(\mathbb{R}^d)} + \|\varkappa^-\|_{L^\infty(\mathbb{R}^d)}\|a^-\|_{L^\infty(\mathbb{R}^d)}|\xi|, \quad \xi\in\Gamma_0,$$

and hence (4.10) holds with $\nu = 1$, which makes (4.11) obvious.

Remark 4.5. It was shown in [14] that, for the case of constant $m, \varkappa^\pm$, the condition like (4.12) is essential. Namely, if $m > 0$ is arbitrary small the operator $\widehat{L}$ will not even be accretive in $\mathfrak{L}_C$.

5 Vlasov-type scalings

For the reader convenience, we start from the idea of the Vlasov-type scaling. The general scheme for the birth-and-death dynamics as well as for the conservative ones may be found in [15]. The realizations of this approach for the Glauber dynamics (Example 1 with $s = 0$) and for the BDLP dynamics (Example 2) were considered in [16, 17], correspondingly. The idea of the Vlasov-type scaling consists in the following.

We would like to construct some scaling L_ε, $\varepsilon > 0$, of the generator L, such that the following scheme holds. Suppose that we have a semigroup $\hat{U}_\varepsilon(t)$ with the generator $\hat{L}_\varepsilon$ in some $\mathcal{L}_{C_\varepsilon}$, $\varepsilon > 0$. Consider the dual semigroup $\hat{U}^*_\varepsilon(t)$. Let us choose an initial function of the corresponding Cauchy problem with a singularity in ε. Namely, $\varepsilon^{|\eta|} k_0^{(\varepsilon)}(\eta) \sim r_0(\eta)$, $\varepsilon \to 0$, $\eta \in \Gamma_0$ for some function r_0, which is independent of ε. The scaling $L \mapsto L_\varepsilon$ should be chosen in such a way that first of all the corresponding semigroup $\hat{U}^*_\varepsilon(t)$ preserves the order of the singularity:

$$\varepsilon^{|\eta|} (\hat{U}^*_\varepsilon(t) k_0^{(\varepsilon)})(\eta) \sim r_t(\eta), \quad \varepsilon \to 0, \quad \eta \in \Gamma_0,$$

and, secondly, the dynamics $r_0 \mapsto r_t$ preserves the Lebesgue–Poisson exponents. There exists explicit (in general, nonlinear) differential equation for ρ_t:

$$\frac{\mathrm{d}}{\mathrm{d}t} \rho_t(x) = \upsilon(\rho_t)(x) \tag{5.1}$$

which will be called the Vlasov-type equation.

Now we explain an informal way to realize such a scheme. Let us consider for any $\varepsilon > 0$ the following mapping (cf. (4.8)) defined for functions on Γ_0

$$(R_\varepsilon r)(\eta) := \varepsilon^{|\eta|} r(\eta).$$

This mapping is “self-dual” with respect to the duality (3.6), moreover, $R_\varepsilon^{-1} = R_{\varepsilon^{-1}}$. Having $R_\varepsilon k_0^{(\varepsilon)} \sim r_0$, $\varepsilon \to 0$, we need $r_t \sim R_\varepsilon \hat{U}^*_\varepsilon(t) k_0^{(\varepsilon)} \sim R_\varepsilon \hat{U}^*_\varepsilon(t) R_{\varepsilon^{-1}} r_0$, $\varepsilon \to 0$. Therefore, we have to show that for any $t \geq 0$ the operator family $R_\varepsilon \hat{U}^*_\varepsilon(t) R_{\varepsilon^{-1}}$, $\varepsilon > 0$ has limiting (in a proper sense) operator $U(t)$ and

$$U(t) e_\lambda(\rho_0) = e_\lambda(\rho_t). \tag{5.2}$$

But, heuristically, $\hat{U}^*_\varepsilon(t) = \exp\{t \hat{L}^*_\varepsilon\}$ and $R_\varepsilon \hat{U}^*_\varepsilon(t) R_{\varepsilon^{-1}} = \exp\{t R_\varepsilon \hat{L}^*_\varepsilon R_{\varepsilon^{-1}}\}$. Let us consider the “renormalized” operator

$$\hat{L}^*_{\varepsilon,\,\mathrm{ren}} := R_\varepsilon \hat{L}^*_\varepsilon R_{\varepsilon^{-1}}. \tag{5.3}$$

In fact, we need that there exists an operator $\hat{L}^*_V$ such that $\exp\{t R_\varepsilon \hat{L}^*_\varepsilon R_{\varepsilon^{-1}}\} \to \exp\{t \hat{L}^*_V\} =: U(t)$ satisfying (5.2). Therefore, an heuristic way to produce scaling $L \mapsto L_\varepsilon$ is to demand that

$$\lim_{\varepsilon \to 0} \left(\frac{\mathrm{d}}{\mathrm{d}t} e_\lambda(\rho_t, \eta) - \hat{L}^*_{\varepsilon,\,\mathrm{ren}} e_\lambda(\rho_t, \eta) \right) = 0, \qquad \eta \in \Gamma_0$$

provided ρ_t satisfies (5.1). The point-wise limit of $\hat{L}^*_{\varepsilon,\,\mathrm{ren}}$ will be natural candidate for $\hat{L}^*_V$.

Note that (5.3) implies informally that $\hat{L}_{\varepsilon,\mathrm{ren}} = R_{\varepsilon^{-1}}\hat{L}_\varepsilon R_\varepsilon$. We propose below the scheme to give rigorous meaning to the idea introduced above. We consider, for a proper scaling L_ε, the "renormalized" operator $\hat{L}_{\varepsilon,\mathrm{ren}}$ and prove that it is a generator of a strongly continuous contraction semigroup $\hat{U}_{\varepsilon,\mathrm{ren}}(t)$ in $\mathcal{L}_C$. Next, we show that the formal limit $\hat{L}_V$ of $\hat{L}_{\varepsilon,\mathrm{ren}}$ is a generator of a strongly continuous contraction semigroup $\hat{U}_V(t)$ in $\mathcal{L}_C$. Finally, we prove that $\hat{U}_{\varepsilon,\mathrm{ren}}(t) \to \hat{U}_V(t)$ strongly in $\mathcal{L}_C$. This implies weak*-convergence of the dual semigroups $\hat{U}^*_{\varepsilon,\mathrm{ren}}(t)$ to $\hat{U}^*_V(t)$. We explain also in which sense $\hat{U}^*_V(t)$ satisfies the properties above.

Let us consider for any $\varepsilon \in (0;1]$ the following scaling of (4.1)

$$\begin{aligned}(L_\varepsilon F)(\gamma) :=& \sum_{x\in\gamma} d_\varepsilon(x,\gamma\setminus x)\left[F(\gamma\setminus x) - F(\gamma)\right]\\ &+ \varepsilon^{-1}\int_{\mathbb{R}^d} b_\varepsilon(x,\gamma)\left[F(\gamma\cup x) - F(\gamma)\right]\mathrm{d}x,\end{aligned}$$

and define the renormalized operator $\hat{L}_{\varepsilon,\mathrm{ren}} := R_{\varepsilon^{-1}}K^{-1}L_\varepsilon K R_\varepsilon$. Using the same arguments as in the proof of Proposition 4.1, we get

$$\begin{aligned}(\hat{L}_{\varepsilon,\mathrm{ren}}G)(\eta) =& -\sum_{\xi\subset\eta} G(\xi)\varepsilon^{-|\eta\setminus\xi|}\sum_{x\in\xi}\big(K_0^{-1}d_\varepsilon(x,\cdot\cup\xi\setminus x)\big)(\eta\setminus\xi)\\ &+\sum_{\xi\subset\eta}\int_{\mathbb{R}^d} G(\xi\cup x)\varepsilon^{-|\eta\setminus\xi|}\big(K_0^{-1}b_\varepsilon(x,\cdot\cup\xi)\big)(\eta\setminus\xi)\,\mathrm{d}x.\end{aligned}$$

For $\varepsilon \in (0;1]$, $$D_\varepsilon(\eta) := \sum_{x\in\eta} d_\varepsilon(x,\eta\setminus x);$$

Suppose that there exists $a_1 \geq 1$, $a_2 > 0$, such that for all $\xi \in \Gamma_0$, for a.a. $x \in \mathbb{R}^d$, and for any $\varepsilon \in (0;1]$

$$\sum_{x\in\xi}\int_{\Gamma_0}\left|K_0^{-1}d_\varepsilon\left(x,\cdot\cup\xi\setminus x\right)\right|(\eta)\,\varepsilon^{-|\eta|}C^{|\eta|}\,\mathrm{d}\lambda\left(\eta\right) \leq a_1 D_\varepsilon\left(\xi\right), \tag{5.4}$$

$$\sum_{x\in\xi}\int_{\Gamma_0}\left|K_0^{-1}b_\varepsilon\left(x,\cdot\cup\xi\setminus x\right)\right|(\eta)\,\varepsilon^{-|\eta|}C^{|\eta|}\,\mathrm{d}\lambda\left(\eta\right) \leq a_2 D_\varepsilon\left(\xi\right), \tag{5.5}$$

$$a_1 + \frac{a_2}{C} < \frac{3}{2}. \tag{5.6}$$

For all $\eta,\xi \in \Gamma_0$ and a.a. $x \in \mathbb{R}^d$ the following limits exist and coincide:

$$\lim_{\varepsilon\to 0}\varepsilon^{-|\eta|}\big(K_0^{-1}d_\varepsilon\left(x,\cdot\cup\xi\right)\big)(\eta) = \lim_{\varepsilon\to 0}\varepsilon^{-|\eta|}\big(K_0^{-1}d_\varepsilon\left(x,\cdot\right)\big)(\eta) =: D_x^V(\eta); \tag{5.7}$$

$$\lim_{\varepsilon\to 0}\varepsilon^{-|\eta|}\big(K_0^{-1}b_\varepsilon\left(x,\cdot\cup\xi\right)\big)(\eta) = \lim_{\varepsilon\to 0}\varepsilon^{-|\eta|}\big(K_0^{-1}b_\varepsilon\left(x,\cdot\right)\big)(\eta) =: B_x^V(\eta). \tag{5.8}$$

We would like to emphasize, that above limits should not depend on ξ. The collection of examples for such d_ε, b_ε can be found in [15].

Now we are able to state result about convergence in $\mathcal{L}_C$.

Theorem 5.1 ([18, Theorem 4.4]). *Let conditions* (5.4), (5.5), *and* (5.6) *are satisfied. Suppose that convergences* (5.7), (5.8) *take place for all* $\eta \in \Gamma_0$ *as well as in the sense of* $\mathcal{L}_C$. *Assume also that there exists* $\sigma > 0$ *such that either*

$$d_\varepsilon(x,\xi) \le \sigma D_x^V(\emptyset) \qquad \text{or} \qquad d_\varepsilon(x,\xi) \ge \sigma D_x^V(\emptyset)$$

is satisfied for all $\xi \in \Gamma_0$ *and for a.a.* $x \in \mathbb{R}^d$. *Then* $\hat{U}_\varepsilon(t) \xrightarrow{s} \hat{U}_V(t)$ *in* $\mathcal{L}_C$ *uniformly on finite time intervals.*

Example 5.1 (BDLP model, revisited). Let

$$d_\varepsilon(x,\gamma\setminus x) = m + \varepsilon\varkappa^- \sum_{y\in\gamma\setminus x} a^-(x-y),$$
$$b_\varepsilon(x,\gamma) = \varepsilon\varkappa^+ \sum_{y\in\gamma} a^+(x-y).$$

Comparing with the previous notations we have changed $\varkappa^\pm$ onto $\varepsilon\varkappa^\pm$. Clearly, conditions (4.12), (4.14) implies the same inequalities for $\varepsilon\varkappa^\pm$. Note also that d_ε is decreasing in $\varepsilon \to 0$. Therefore, to apply all results of this section to BDLP-model we should prove the convergence (5.7), (5.8) in $\mathcal{L}_C$. Note, that

$$\begin{aligned}\varepsilon^{-|\eta|}K_0^{-1} d_\varepsilon\left(x,\cdot\cup\xi\right)(\eta) &= d_\varepsilon(x,\xi)0^{|\eta|} + 1\!\!1_{\Gamma^{(1)}}(\eta)\sum_{y\in\eta} a^-(x-y)\\ &\to m0^{|\eta|} + 1\!\!1_{\Gamma^{(1)}}(\eta)\sum_{y\in\eta} a^-(x-y) =: D_x^V(\eta)\end{aligned}$$

and, analogously,

$$\begin{aligned}\varepsilon^{-|\eta|}K_0^{-1} b_\varepsilon\left(x,\cdot\cup\xi\right)(\eta) &= b_\varepsilon(x,\xi)0^{|\eta|} + 1\!\!1_{\Gamma^{(1)}}(\eta)\sum_{y\in\eta} a^+(x-y)\\ &\to 1\!\!1_{\Gamma^{(1)}}(\eta)\sum_{y\in\eta} a^+(x-y) =: B_x^V(\eta).\end{aligned}$$

The convergence in $\mathcal{L}_C$ is obvious now. The kinetic (Vlasov) equation has the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t(x) = \varkappa^+(a^+ * \rho_t)(x) - \varkappa^-\rho_t(x)(a^- * \rho_t)(x) - m\rho_t(x). \tag{5.9}$$

We study the obtained equation in the following section.

Remark 5.2. By duality (3.6), Theorem 5.1 yields weak*-convergence of the semigroups $\hat{U}_\varepsilon^{\odot\alpha}(t)$ to $\hat{U}_V^{\odot\alpha}(t)$ in $\overline{\mathcal{K}_{\alpha C}}$. To prove such convergence in the strong sense we need additional analysis of their generators. The problem concerns the fact that we have explicit expression for the generator $\hat{L}_V^{\odot\alpha} = \hat{L}_V^*$ only on the core $\left\{k \in \mathcal{K}_{\alpha C} \;\middle|\; \hat{L}_V^* k \in \overline{\mathcal{K}_{\alpha C}}\right\}$. However, we are able to show such convergence for the Glauber dynamics described in Example 1 for $s=0$ using modified technique (see [16]).

6 Kinetic equation of a spatial ecology model

6.1 Introduction

In this section we study the mesoscopic equation of the BDLP model (5.9) from different perspectives. Namely we will deal with the following nonlinear nonlocal evolution equation, for $x \in \mathbb{R}^d$,

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t}(x,t) = \varkappa^+(a^+ * u)(x,t) - mu(x,t) - \varkappa^- u(x,t)(a^- * u)(x,t), & t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{6.1}$$

which we will study in a class of bounded in x nonnegative functions.

The solution $u = u(x,t)$ to (6.1) describes approximately a density (at the moment of time t and at the position x of the space $\mathbb{R}^d$) for a particle system evolving in the continuum. In course of the evolution, particles might reproduce themselves, die, and compete (say, for resources). Namely, a particle located at a point $y \in \mathbb{R}^d$ may produce a 'child' at a point $x \in \mathbb{R}^d$ with the intensity $\varkappa^+$ and according to the dispersion kernel $a^+(x-y)$. Next, any particle may die with the constant intensity m. And additionally, a particle located at x may die according to the competition with the rest of particles; the intensity of the death because of a competitive particle located at y is equal to $\varkappa^-$ and the distribution of the competition is described by $a^-(x-y)$.

This model was originally proposed in mathematical ecology, see [9]. Rigorous mathematical constructions were done in [14, 23]. In [14], the mathematical approach was realized using the theory of Markov statistical dynamics on the so-called configuration spaces expressed in terms of evolution of time-dependent correlation functions of the system, see e.g. [20, 32, 34].

Here $m > 0$, $\varkappa^\pm > 0$ are constants, and functions $0 \leq a^\pm \in L^1(\mathbb{R}^d)$ are probability densities:

$$\int_{\mathbb{R}^d} a^+(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} a^-(y)\,\mathrm{d}y = 1.$$

Here and below, for a function $u = u(y,t)$, which is (essentially) bounded in $y \in \mathbb{R}^d$, and a function (a kernel) $a \in L^1(\mathbb{R}^d)$, we denote

$$(a * u)(x,t) := \int_{\mathbb{R}^d} a(x-y)u(y,t)\,\mathrm{d}y.$$

We assume that u_0 is a bounded function on $\mathbb{R}^d$. For technical reasons, we will consider two Banach spaces of bounded real-valued functions on $\mathbb{R}^d$: the space $C_{ub}(\mathbb{R}^d)$ of bounded uniformly continuous functions on $\mathbb{R}^d$ with sup-norm and the space $L^\infty(\mathbb{R}^d)$ of essentially bounded (with respect to the Lebesgue measure) functions on $\mathbb{R}^d$ with esssup-norm. Let also $C_b(\mathbb{R}^d)$ and $C_0(\mathbb{R}^d)$ denote the spaces of continuous functions on $\mathbb{R}^d$ which are bounded and have compact supports, correspondingly.

Let E be either $C_{ub}(\mathbb{R}^d)$ or $L^\infty(\mathbb{R}^d)$. Consider the equation (6.1) in E; in particular, u must be continuously differentiable in t, for $t > 0$, in the sense of the norm in E. Moreover, we consider u as an element from the space

$C_b(I \to E)$ of continuous bounded functions on I (including 0) with values in E and with the following norm

$$\|u\|_{C_b(I\to E)} = \sup_{t\in I} \|u(\cdot,t)\|_E.$$

Such a solution is said to be a classical solution to (6.1); in particular, u will continuously (in the sense of the norm in E) depend on the initial condition u_0.

We will also use the space $C_b(I \to E)$ with $I = [T_1, T_2]$, $T_1 > 0$. For simplicity of notations, we denote

$$\mathcal{X}_{T_1,T_2} := C_b\big([T_1,T_2] \to E\big), \qquad T_2 > T_1 \geq 0,$$

and the corresponding norm will be denoted by $\|\cdot\|_{T_1,T_2}$. We set also $\mathcal{X}_T := \mathcal{X}_{0,T}$, $\|\cdot\|_T := \|\cdot\|_{0,T}$, and

$$\mathcal{X}_\infty := C_b\big(\mathbb{R}_+ \to E\big)$$

with the corresponding norm $\|\cdot\|_\infty$. The upper index '+' will denote the cone of nonnegative functions in the corresponding space, namely,

$$\mathcal{X}_\sharp^+ := \{u \in \mathcal{X}_\sharp \mid u \geq 0\},$$

where $\sharp$ is one of the subscripts above.

We will also omit the subscript for the norm $\|\cdot\|_E$ in E, if it is clear whether we are working with sup- or esssup-norm.

6.2 Basic properties

The following theorem yields existence and uniqueness of a solution

Theorem 6.1. *Let $u_0 \in E$ and $u_0(x) \geq 0$, $x \in \mathbb{R}^d$. Then, for any $T > 0$, there exists a unique nonnegative solution u to the equation* (6.1) *in E, such that $u \in \mathcal{X}_T$.*

Proof. The proof is based on the fixed point argument applied to the map $u = \Phi_\tau v$, where, for a fixed $0 \leq v \in \mathcal{X}_T$, the function u solves the following equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = -mu(x,t) - \varkappa^- u(x,t)(a^- * v)(x,t) + \varkappa^+(a^+ * v)(x,t), \quad t \in (\tau, T], \\ u(x,\tau) = u_\tau(x). \end{cases}$$

One can show that, for $\tau = 0$, Φ_τ will be a contraction mapping on a time interval $[0, T_0]$. Hence a fixed point $u = \Phi u$ exists on $[0, T_0]$. There exists $T_1 > T_0$ such that, for $\tau = T_0$, Φ_τ is a contraction mapping on $[T_0, T_1]$ and the fixed point u may be extended to $[0, T_1]$. Iterating this scheme, we obtain a sequence $\{T_n\}_{n\in\mathbb{N}}$, such that $T_n \to \infty$ and $u = \Phi u$ on each $[0, T_n]$. Hence $u = \Phi u$ on $[0, \infty)$. It is left to note that u is a fixed point of Φ if and only if it satisfies (6.1).

For the details see [21, Theorem 2.2]. □

Below, $|\cdot| = |\cdot|_{\mathbb{R}^d}$ denotes the Euclidean norm in $\mathbb{R}^d$, $B_r(x)$ is a closed ball in $\mathbb{R}^d$ with the center at $x \in \mathbb{R}^d$ and the radius $r > 0$; and b_r is a volume of this ball.

The following theorem is an extension of Theorem 6.1 for $E = C_{ub}(\mathbb{R}^d)$, in which case the global boundedness of the solutions holds in both space and time under additional weak assumptions.

Theorem 6.2. *Suppose that there exists $r_0 > 0$ such that*

$$\alpha := \inf_{|x|\le r_0} a^-(x) > 0,$$

and, for some $\varepsilon, A > 0$, one have $a^+(x) \le \dfrac{A}{1+|x|^{d+\varepsilon}}$, for all $x \in \mathbb{R}^d$. Then, the solution $u \ge 0$ to (6.1), with $0 \le u_0 \in C_{ub}(\mathbb{R}^d)$, belongs to $C_{ub}(\mathbb{R}^d \times \mathbb{R}_+)$.

Proof. The idea of the proof goes back to [30, Theorem 1.3]. We consider

$$v(x,t) := (1\!\!1_{B_r(0)} * u)(x,t) = \int_{B_r(x)} u(y,t)\,dy.$$

It is possible to prove by contradiction that under conditions of the theorem v is globally bounded, which implies that u is bounded on $\mathbb{R}^d \times \mathbb{R}_+$. For the details see [21, Theorem 2.8]. □

The main difficulty in studying non-local monostable evolution equations is the lack of techniques for the class of equations. In particular, variational methods may be hardly applied here because of the type of the non-linear ('reaction') term, which is not a potential operator. Nevertheless, under restrictions on the kernels a^+, a^-, a version of the comparison principle may be proven. This result will be needed in the rest of the article. Let $T > 0$ be fixed. Define the sets $\mathcal{X}_T^1$ of functions from $\mathcal{X}_T$, which are continuously differentiable on $(0,T]$ in the sense of the norm in E. Here and below we consider the left derivative at $t = T$ only. For any u from $\mathcal{X}_T^1$ one can define the following function

$$\mathcal{F}u := \frac{\mathrm{d}u}{\mathrm{d}t} - \varkappa^+ a^+ * u + mu + \varkappa^- u(a^- * u), \qquad t \in (0,T],\ x \in \mathbb{R}^d. \tag{6.2}$$

Theorem 6.3. *Let there exist $c > 0$, such that*

$$\varkappa^+ a^+(x) \ge c\varkappa^- a^-(x), \quad a.a.\ x \in \mathbb{R}^d.$$

Let $T \in (0,\infty)$ be fixed and functions $u_1, u_2 \in \mathcal{X}_T^1$ be such that, for any $(x,t) \in \mathbb{R}^d \times (0,T]$,

$$(\mathcal{F}u_1)(x,t) \le (\mathcal{F}u_2)(x,t), \tag{6.3}$$
$$u_1(x,t) \ge 0, \qquad 0 \le u_2(x,t) \le c, \qquad u_1(x,0) \le u_2(x,0).$$

Then $u_1(x,t) \le u_2(x,t)$, for all $(x,t) \in \mathbb{R}^d \times [0,T]$. In particular, $u_1 \le c$.

Proof. We consider

$$v(x,t) := e^{Kt}(u_2(x,t) - u_1(x,t)), \quad x \in \mathbb{R}^d,\ t \in [0,T].$$

By the fixed point method applied to the integral equation that is satisfied by v we can show that $v \geq 0$, provided $K > 0$ is sufficiently large. Hence $u_2 \geq u_1$. Also see [21, Theorem 3.1]. □

For $E = C_{ub}(\mathbb{R}^d)$ one can prove a refined version of Theorem 6.3 for non-differentiable in time functions. For any $T \in (0, \infty]$, define the set D_T of all functions $u : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$, such that, for all $t \in [0, T)$, $u(\cdot, t) \in C_{ub}(\mathbb{R}^d)$, and, for all $x \in \mathbb{R}^d$, the function $f(x, t)$ is absolutely continuous in t on $[0, T)$. Then, for any $u \in D_T$, one can define the function (6.2), for all $x \in \mathbb{R}^d$ and a.a. $t \in [0, T)$.

Proposition 6.4 ([21, Proposition 3.3])**.** *The statement of Theorem 6.3 remains true, if we assume that $u_1, u_2 \in D_T$ and, for any $x \in \mathbb{R}^d$, the inequality* (6.3) *holds for a.a. $t \in (0, T)$ only.*

We introduce a notation for the non-zero constant solution

$$\theta := \frac{\varkappa^+ - m}{\varkappa^-} \in \mathbb{R}. \tag{6.4}$$

It is easy to show using Duhamel's principle, that if $\varkappa^+ < m$, then the solutions to (6.1) converges to 0 exponentially fast and uniformly in space, as time tends to infinity. The case $\varkappa^+ = m$ was partially considered by Terra and Wolanski (see [48, 49]) and we omit it in the present article. Hence we make the following assumption in the rest of the article,

$$\varkappa^+ > m. \tag{A1}$$

It yields in particular that the constant solution θ is greater than zero. We will study solutions with initial conditions, that are non-negative and bounded by θ.

Definition 6.5. For $\theta > 0$, given by (6.4), consider the following sets

$$\begin{aligned} U_\theta &:= \{f \in C_{ub}(\mathbb{R}^d) \mid 0 \leq f(x) \leq \theta,\ x \in \mathbb{R}^d\}, \\ L_\theta &:= \{f \in L^\infty(\mathbb{R}^d) \mid 0 \leq f(x) \leq \theta,\ \text{for a.a. } x \in \mathbb{R}^d\}, \\ E_\theta &:= \{f \in E \mid 0 \leq f(x) \leq \theta,\ x \in \mathbb{R}^d\}. \end{aligned}$$

Hence E_θ is either U_θ or L_θ.

By virtue of Theorem 6.3, we assume,

$$\varkappa^+ a^+(x) \geq (\varkappa^+ - m) a^-(x), \quad \text{a.a. } x \in \mathbb{R}^d. \tag{A2}$$

Let us mention an important consequence of Theorem 6.3.

Proposition 6.6 ([21, Proposition 3.4])**.** *Suppose that* (A1) *and* (A2) *hold. Let $0 \leq u_0 \in E_\theta$ be an initial condition to* (6.1) *and $u \in \mathcal{X}_T$ be the corresponding solutions on any $[0, T]$, $T > 0$. Then $u \in \mathcal{X}_\infty$, with $\|u\|_\infty \leq \theta$.*

Let $v_0 \in E_\theta$ be another initial condition to (6.1) *such that $u_0(x) \leq v_0(x)$, $x \in \mathbb{R}^d$; and $v \in \mathcal{X}_\infty$ be the corresponding solution. Then*

$$u(x, t) \leq v(x, t), \quad x \in \mathbb{R}^d, t \geq 0.$$

Let us show that if $u_0 \not\equiv 0$, then the solutions to (6.1) are strictly positive; this is quite common feature of linear parabolic equations, however, in general, it may fail for nonlinear ones. It is required that

$$\text{there exists } \rho, \delta > 0 \text{ such that } a^+(x) \geq \rho, \text{ for a.a. } x \in B_\delta(0). \tag{A3}$$

Proposition 6.7 ([21, Proposition 3.8]). *Let* (A1), (A2), (A3) *hold. Let* $u_0 \in U_\theta$, $u_0 \not\equiv 0$, $u_0 \not\equiv \theta$, *be the initial condition to* (6.1), *and* $u \in \mathcal{X}_\infty$ *be the corresponding solution. Then*

$$u(x,t) > \inf_{\substack{y\in\mathbb{R}^d\\ s>0}} u(y,s) \geq 0, \qquad x \in \mathbb{R}^d, t > 0.$$

As a matter of fact, under (A4), a much stronger statement than unattainability of θ does hold. To show this we assume that

$$\begin{gathered}\text{there exists } \rho, \delta > 0, \text{ such that}\\ J_\theta(x) = \varkappa^+ a^+(x) - (\varkappa^+ - m)a^-(x) \geq \rho, \text{ for a.a. } x \in B_\delta(0).\end{gathered} \tag{A4}$$

Theorem 6.8 ([21, Theorem 3.9]). *Let* (A1), (A2), (A4) *hold. Let* $u_1, u_2 \in \mathcal{X}_\infty$ *be two solutions to* (6.1), *such that* $u_1(x,0) \leq u_2(x,0)$, $x \in \mathbb{R}^d$, *are from* U_θ. *Then either* $u_1(x,t) = u_2(x,t)$, $x \in \mathbb{R}^d$, $t \geq 0$ *or* $u_1(x,t) < u_2(x,t)$, $x \in \mathbb{R}^d$, $t > 0$.

By choosing $u_2 \equiv \theta$ in Theorem 6.8, we immediately get the following

Corollary 6.9 ([21, Corollary 3.10]). *Let* (A1), (A2), (A4) *hold. Let* $u_0 \in U_\theta$, $u_0 \not\equiv \theta$, *be the initial condition to* (6.1), *and* $u \in \mathcal{X}_\infty$ *be the corresponding solution. Then* $u(x,t) < \theta$, $x \in \mathbb{R}^d$, $t > 0$.

6.3 Traveling waves

For simplicity, we consider one-dimensional space ($d = 1$) in the following. For many-dimensional analogues of the statements, see [21, 22, 24–26].

Traveling waves were studied intensively for the original Fisher–KPP equation, see e.g. [4, 13, 36]; for locally nonlinear equation with nonlocal diffusion, see e.g. [10, 47, 51]; and for nonlocal nonlinear equation with local diffusion, see e.g. [2, 6, 30, 43].

Through this section we will mainly work in L^∞-setting. Recall that we will always assume that (A1) and (A2) hold, and $\theta > 0$ is given by (6.4).

Let us give a brief overview for the results of this Section. The existence and properties of the traveling wave solutions will be considered under the so-called Mollison condition (A5), cf. e.g. [41, 42]. Namely, in Theorem 6.12 we will prove that, for any $\xi \in S^{d-1}$, there exists $c_*(\xi) \in \mathbb{R}$, such that, for any $c \geq c_*(\xi)$, there exists a traveling wave with the speed c in the direction ξ, and, for any $c < c_*(\xi)$, such a traveling wave does not exist. Moreover, we will find an expression for $c_*(\xi)$, see (6.7). We will prove that the profile of a traveling wave with a non-zero speed is smooth, whereas the zero-speed traveling wave (provided it exists, i.e. if $c_*(\xi) \leq 0$) has a continuous profile (Proposition 6.13, Corollary 6.14). Next, we will demonstrate the uniqueness (up to shifts) of a traveling wave wave profile with a given speed $c \geq c_*(\xi)$ (Theorem 6.18).

Definition 6.10. Let $\mathcal{M}_\theta(\mathbb{R})$ denote the set of all decreasing and right-continuous functions $f : \mathbb{R} \to [0, \theta]$.

Definition 6.11. Let $\tilde{\mathcal{X}}^1_\infty := \tilde{\mathcal{X}}_\infty \cap C^1((0,\infty) \to L^\infty(\mathbb{R}^d))$. A function $u \in \tilde{\mathcal{X}}^1_\infty$ is said to be a traveling wave solution to the equation (6.1) with a speed $c \in \mathbb{R}$ and in a direction $\xi \in S^{d-1}$ if and only if (iff, in the sequel) there exists a function $\psi \in \mathcal{M}_\theta(\mathbb{R})$, such that for all $t \geq 0$, a.a. $x \in \mathbb{R}^d$,

$$u(x,t) = \psi(x \cdot \xi - ct), \quad \psi(-\infty) = \theta, \quad \psi(+\infty) = 0. \tag{6.5}$$

Here and below the function ψ is said to be the profile for the traveling wave, whereas c is its speed.

For a given $\xi \in S^{d-1}$, consider the following assumption on a^+:

$$\text{There exists } \mu = \mu(\xi) > 0 \text{ such that } \mathfrak{a}_\xi(\mu) := \int_{\mathbb{R}^d} a^+(x) e^{\mu \xi \cdot x}\, \mathrm{d}x < \infty. \tag{A5}$$

Theorem 6.12. *Let* (A1) *and* (A2) *hold and* $\xi \in S^{d-1}$ *be fixed. Suppose also that* (A5) *holds. Then there exists* $c_*(\xi) \in \mathbb{R}$ *such that*

1. *for any* $c \geq c_*(\xi)$, *there exists a traveling wave solution (in direction* ξ*), in the sense of Definition 6.11, with a profile* $\psi \in \mathcal{M}_\theta(\mathbb{R})$ *and the speed* c,
2. *for any* $c < c_*(\xi)$, *such a traveling wave does not exist.*

Proof. Since the semi-flow generated by (6.1) is commutative with the translation in $\mathbb{R}^d$, there is no loss of generality in considering the one-dimensional space ($d = 1$). Then one can show there exists $\mu > 0$ such that

$$\varphi(s) := \theta \min\{e^{-\mu s}, 1\}, \quad s \in \mathbb{R},$$

is a super-solution to (6.1). Now one can apply [51, Theorem 5]. Also see [21, Theorem 4.9]. □

Next statements describe the properties of a traveling wave solution.

Proposition 6.13 ([21, Proposition 4.11])**.** *Let* $\psi \in \mathcal{M}_\theta(\mathbb{R})$ *and* $c \in \mathbb{R}$ *be such that there exists a solution* $u \in \tilde{\mathcal{X}}^1_\infty$ *to the equation* (6.1) *such that* (6.5) *holds, for some* $\xi \in S^{d-1}$. *Then* $\psi \in C^1(\mathbb{R} \to [0,\theta])$, *for* $c \neq 0$, *and* $\psi \in C(\mathbb{R} \to [0,\theta])$, *otherwise.*

Corollary 6.14 ([21, Corollary 4.12, Proposition 4.13])**.** *In conditions and notations of Proposition 6.13,* ψ *is a strictly decaying function, for any speed* c, *and for any speed* $c \neq 0$, *the profile* $\psi \in C^\infty_b(\mathbb{R})$.

We assume that the first moment of a^+ in direction $\xi \in S^{d-1}$ exists, namely,

$$\int_{\mathbb{R}^d} |x \cdot \xi|\, a^+(x)\, \mathrm{d}x < \infty. \tag{A6}$$

The following assumption is a weaker form of (A3).

$$\begin{gathered} \text{There exist } r = r(\xi) \geq 0,\ \rho = \rho(\xi) > 0,\ \delta = \delta(\xi) > 0, \text{ such that} \\ a^+(x) \geq \rho, \text{ for a.a. } x \in B_\delta(r\xi). \end{gathered} \tag{A7}$$

We set,

$$\check{a}^{\pm}(s) := \begin{cases} \displaystyle\int_{\mathbb{R}^{d-1}} a^{\pm}(\tau_1\eta_1 + \ldots + \tau_{d-1}\eta_{d-1} + s\xi)\, d\tau_1 \ldots d\tau_{d-1}, & d \geq 2, \\ a^{\pm}(s\xi), & d = 1. \end{cases}$$

There exists a critical situation: when the abscissa of the traveling wave coincides with the abscissa of the kernel a^+. In this case, properties of the traveling waves may be different from the 'generic' case. To distinguish this cases and simplify the further statements, we introduce the following two classes of functions.

Definition 6.15. Let $m > 0$, $\varkappa^{\pm} > 0$, $0 \leq a^- \in L^1(\mathbb{R}^d)$ be fixed, and (A1) holds. For an arbitrary $\xi \in S^{d-1}$, we denote by $\mathcal{V}_\xi$ the the class of all kernels $0 \leq a^+ \in L^1(\mathbb{R}^d)$ such that (A2), (A5)–(A7) and one of the following assumptions does hold:

1. $\lambda_0 := \sup\{\lambda \in \mathbb{R} : \mathfrak{a}_\xi(\lambda) < \infty\} = \infty$;
2. $\lambda_0 < \infty$ and $\mathfrak{a}_\xi(\lambda_0) = \infty$;
3. $\lambda_0 < \infty$, $\mathfrak{a}_\xi(\lambda_0) < \infty$ and $\mathfrak{t}_\xi(\lambda_0) \in [-\infty, m)$, where $\mathfrak{t}_\xi(\lambda)$ is given by

$$\mathfrak{t}_\xi(\lambda) := \varkappa^+ \int_{\mathbb{R}} (1 - \lambda s)\check{a}^+(s) e^{\lambda s}\, \mathrm{d}s \in [-\infty, \varkappa^+), \qquad \lambda \in [0, \lambda).$$

Correspondingly, we denote by $\mathcal{W}_\xi$ the class of all kernels such that $\lambda_0 < \infty$, $\mathfrak{a}_\xi(\lambda_0) < \infty$, and $\mathfrak{t}_\xi(\lambda_0) \in [m, \varkappa^+)$ instead of (1) – (3).

For $a^+ \in \mathcal{V}_\xi \cup \mathcal{W}_\xi$, denote the interval $I_\xi \subset (0, \infty)$ by

$$I_\xi := \begin{cases} (0, \infty), & \text{if } \ \lambda_0 = \infty, \\ \bigl(0, \lambda_0\bigr), & \text{if } \lambda_0 < \infty \ \text{ and } \ (\mathfrak{L}\check{a}^+)\bigl(\lambda_0\bigr) = \infty \\ \bigl(0, \lambda_0\bigr], & \text{if } \lambda_0 < \infty \ \text{ and } \ (\mathfrak{L}\check{a}^+)\bigl(\lambda_0\bigr) < \infty. \end{cases}$$

Consider the following complex-valued function

$$G_\xi(z) := \frac{\varkappa^+(\mathfrak{L}\check{a}^+)(z) - m}{z}, \quad \operatorname{Re} z > 0, \tag{6.6}$$

Proposition 6.16. *Let $\xi \in S^{d-1}$ be fixed and $a^+ \in \mathcal{V}_\xi \cup \mathcal{W}_\xi$. Then there exists a unique $\lambda_* = \lambda_*(\xi) \in I_\xi$ such that*

$$\inf_{\lambda > 0} G_\xi(\lambda) = \min_{\lambda \in I_\xi} G_\xi(\lambda) = G_\xi(\lambda_*) > \varkappa^+ \mathfrak{m}_\xi.$$

Moreover, G_ξ is strictly decreasing on $(0, \lambda_*]$ and G_ξ is strictly increasing on $I_\xi \setminus (0, \lambda_*]$ (the latter interval may be empty).

The following theorem provides expressions of for the minimal speed of traveling waves.

Theorem 6.17. *Let $\xi \in S^{d-1}$ be fixed and $a^+ \in \mathcal{V}_\xi \cup \mathcal{W}_\xi$. Let $c_*(\xi)$ be the minimal traveling wave speed according to Theorem 6.12, and let, for any $c \geq c_*(\xi)$, the function $\psi = \psi_c \in \mathcal{M}_\theta(\mathbb{R})$ be a traveling wave profile corresponding to the speed c. Let $\lambda_* \in I_\xi$ be the same as in Proposition 6.16.*

1. *The following relations hold*

$$c_*(\xi) = \min_{\lambda>0} \frac{\varkappa^+ \mathfrak{a}_\xi(\lambda) - m}{\lambda} = \frac{\varkappa^+ \mathfrak{a}_\xi(\lambda_*) - m}{\lambda_*} > \varkappa^+ \mathfrak{m}_\xi. \tag{6.7}$$

2. *For $a^+ \in \mathcal{V}_\xi$, there exists another representation for the minimal speed,*

$$c_*(\xi) = \varkappa^+ \int_{\mathbb{R}} x \cdot \xi \, a^+(x) e^{\lambda_* x\cdot\xi} \, \mathrm{d}x = \varkappa^+ \int_{\mathbb{R}} s \check{a}^+(s) e^{\lambda_* s} \, \mathrm{d}s > \varkappa^+ \mathfrak{m}_\xi.$$

Proof. First, we apply the Laplace transform to (6.1) with the traveling wave solution $u(x,t) = \psi(x\cdot\xi - ct)$. Then, analysis of the minimal speed $c_*(\xi)$ will be reduced to the analysis of the function G_ξ defined by (6.6). In particular, (6.7) follows from Proposition 6.16. For the details see [21, Theorem 4.23]. □

Now we will formulate the uniqueness (up to shifts) of a profile ψ for a traveling wave with the given speed $c \geq c_*(\xi)$, $c \neq 0$.

Theorem 6.18. *Let $\xi \in S^{d-1}$ be fixed and $a^+ \in \mathcal{V}_\xi \cup \mathcal{W}_\xi$. Suppose, additionally, that* (A4) *holds. Let $c_*(\xi)$ be the minimal traveling wave speed according to Theorem 6.12. For the case $a^+ \in \mathcal{W}_\xi$ with $m = \mathfrak{t}_\xi(\lambda_0)$, we will assume, additionally, that $\int_{\mathbb{R}} s^2 \check{a}^+(s) e^{\lambda_0 s} \, \mathrm{d}s < \infty$. Then, for any $c \geq c_*$, such that $c \neq 0$, there exists a unique, up to a shift, traveling wave profile ψ for* (6.1).

Proof. We will follow the sliding technique from [10]. Let $\psi_1, \psi_2 \in C^1(\mathbb{R}) \cap \mathcal{M}$ are traveling wave profiles with a speed $c \geq c_*$, $c \neq 0$. On can prove that, for any $\tau > 0$, there exists $T = T(\tau) > 0$, such that

$$\psi_1(s-\tau) > \psi_2(s), \quad s \geq T.$$

Then there exists $\nu > 0$, such that,

$$\psi_1(s-\nu) \geq \psi_2(s), \quad s \in \mathbb{R}.$$

Similarly, there exists $\tilde{\nu} > 0$, such that,

$$\psi_2(s-\tilde{\nu}) \geq \psi_1(s), \quad s \in \mathbb{R}.$$

We can shift ψ_1 and ψ_2 such that $\nu = \tilde{\nu} = 0$. As a result $\psi_1 = \psi_2$. See [21, Theorem 4.33] for the detailed proof. □

6.4 Propagation with a constant speed

We will study here the behavior of $u(tx,t)$, where u solves (6.1), for big $t \geq 0$. The results of Section 6.3 together with the comparison principle imply that if an initial condition $u_0(x)$ to (6.1) has a minorant/majorant which has a form $\psi(x\cdot\xi)$, $\xi \in S^{d-1}$, where $\psi \in \mathcal{M}_\theta(\mathbb{R})$ is a traveling wave profile in the direction ξ with a speed $c \geq c_*(\xi)$, then for the corresponding solution u to (6.1), the function $u(tx,t)$ will have the minorant/majorant $\psi(t(x\cdot\xi - c))$, correspondingly. In particular, if the initial condition is "below" of any traveling wave in a given direction, then one can estimate the corresponding value of $u(tx,t)$ (Theorem 6.19). Considering such a behavior in different directions, one can obtain a (bounded) set, out of which the solution exponentially decays to 0 (Theorem 6.20). Inside of this set the solution will uniformly converge to θ (Theorem 6.21).

Here and below, for any measurable $A \subset \mathbb{R}$, we define $tA := \{tx \mid x \in A\} \subset \mathbb{R}$.

$$E_{\lambda,\xi}(\mathbb{R}^d) := \{f \in L^\infty(\mathbb{R}^d) | \|f\|_{\lambda,\xi} := \sup_{x\in\mathbb{R}^d} |f(x)| e^{\lambda x\cdot\xi} < \infty\}.$$

We are going to explain now how a solution $u(x,t)$ to (6.1) behaves outside of the sets

$$\Upsilon_{t,\xi} = \bigl\{x \in \mathbb{R}^d \mid x\cdot\xi \leq tc_*(\xi)\bigr\}, \quad \xi \in S^{d-1}.$$

Theorem 6.19. *Let $\xi \in S^{d-1}$ and $a^+ \in \mathcal{V}_\xi \cup \mathcal{W}_\xi$; i.e. all conditions of Definition 6.15 hold. Let $\lambda_* = \lambda_*(\xi) \in I_\xi$ be the same as in Proposition 6.16. Suppose that $u_0 \in E_{\lambda_*,\xi}(\mathbb{R}^d) \cap E_\theta$ and let $u \in \mathcal{X}_\infty$ be the corresponding solution to (6.1). Let $O_\xi \subset \mathbb{R}$ be an open set, such that $\Upsilon_{1,\xi} \subset O_\xi$ and $\delta := \operatorname{dist}(\Upsilon_{1,\xi}, \mathbb{R}^d \setminus O_\xi) > 0$. Then the following estimate holds*

$$\sup_{x\notin tO_\xi} u(x,t) \leq \|u_0\|_{\lambda_*,\xi} e^{-\lambda_*\delta t}, \quad t > 0.$$

Proof. is based on the proof of Theorem 6.1. We consider the map $\Phi(v)$ in the weighted L^∞-space $E_{\lambda,\xi}(\mathbb{R}^d)$. We can show there exists λ_* such that

$$0 \leq u(x,t) \leq \|u_0\|_{\lambda_*,\xi} \exp\bigl\{p_* t - \lambda_* x\cdot\xi\bigr\}, \quad \text{a.a. } x \in \mathbb{R}^d,$$

where $p_* = \varkappa^+ \int_{\mathbb{R}^d} a^+(x) e^{\lambda_* x\cdot\xi}\, dx - m$.

Also see [21, Theorem 5.4]. □

We are going to consider now the global long-time behavior along both directions $\xi \in S^{d-1}$ simultaneously. Define,

$$\Upsilon_T = \bigcap_{\xi\in S^{d-1}} \Upsilon_{T,\xi} = \bigcap_{\xi\in S^{d-1}} T\Upsilon_{1,\xi} = T\Upsilon_1, \quad T > 0.$$

We are ready now to state a result about the long-time behavior at infinity in space.

$$a^+ \in L^\infty(\mathbb{R}^d). \tag{A8}$$

$$\text{There exists } \mu_d > 0, \text{ such that } \int_{\mathbb{R}^d} a^+(x) e^{\mu_d|x|}\, \mathrm{d}x < \infty. \tag{A9}$$

Clearly, (A9) implies

$$\int_{\mathbb{R}^d} |x| a^+(x)\, \mathrm{d}x < \infty. \tag{6.8}$$

Theorem 6.20. *Let the conditions* (A1), (A2), (A3), (A8), (A9) *hold. Let* $u_0 \in E_\theta$ *be such that*

$$|||u_0||| := \max_{\xi \in S^{d-1}} \|u_0\|_{\lambda_*(\xi),\xi} < \infty,$$

and let $u \in \mathcal{X}_\infty$ *be the corresponding solution to* (6.1). *Then, for any open set* $O \supset \Upsilon_1$, *there exists* $\nu = \nu(O) > 0$, *such that*

$$\sup_{x \notin tO} u(x,t) \le |||u_0|||e^{-\nu t}, \quad t > 0.$$

Proof. The proof follows from Theorem 6.19. See [21, Theorem 5.9] for details. □

Our second main result about the long-time behavior states that the solution $u \in \mathcal{X}_\infty$ uniformly converges to θ inside the set $t\Upsilon_1 = \Upsilon_t$.

For a closed set $A \subset \mathbb{R}^d$, we denote by $\mathrm{int}(A)$ the interior of A.

Theorem 6.21. *Let the conditions* (A1), (A2), (A4), (A8), (A9) *hold. Let* $u_0 \in U_\theta$, $u_0 \not\equiv 0$, *and* $u \in \mathcal{X}_\infty$ *be the corresponding solution to* (6.1). *Then, for any compact set* $C \subset \mathrm{int}(\Upsilon_1)$,

$$\lim_{t\to\infty} \min_{x \in tC} u(x,t) = \theta. \tag{6.9}$$

Proof. The result of the theorem is a special case of the general result for dynamical systems on the space of bounded continuous functions by H. Weinberger [50]. See [21, Theorem 5.10] for the detailed proof. □

All result above about traveling waves and long-time behavior of the solutions were obtained under exponential integrability assumptions, cf. (A5) or (A9). In [27], it was proved, in the case of the local competition (e.g. $a^- = \delta_0$), on $\mathbb{R}$ with local nonlinear term, that the case with a^+ which does not satisfy such conditions leads to 'accelerating' solutions, i.e. in this case the equality like (6.9) holds for arbitrary big compact $C \subset \mathbb{R}$. The detailed analysis of the propagation for the slow decaying a^+ is done in the following section.

We will formulate an analog of the first statement in [27, Theorem 1].

Theorem 6.22 ([21, Theorem 5.21]). *Let the conditions* (A1), (A2), (A4), (A8), *and* (6.8) *hold. Suppose also (cf.* (A9)*), that for any* $\lambda > 0$ *and for any* $\xi \in S^{d-1}$, $\mathfrak{a}_\xi(\lambda) = \infty$. *Let* $u_0 \in E_\theta$ *be such that there exist* $x_0 \in \mathbb{R}$, $\eta > 0$, $r > 0$, *with* $u_0 \ge \eta$, *for a.a.* $x \in B_r(x_0)$. *Let* $u \in \mathcal{X}_\infty$ *be the corresponding solution to* (6.1). *Then, for any compact set* $\mathcal{K} \subset \mathbb{R}^d$,

$$\lim_{t\to\infty} \inf_{x \in t\mathcal{K}} u(x,t) = \theta.$$

6.5 Accelerating propagation

The main result of this subsection is Theorem 6.25, where we demonstrate the accelerated propagation of solutions to (6.1) in the case when either of the dispersion kernel or the initial condition has regularly heavy tails at ∞, perhaps different. We show that, in such case, the propagation is fully determined by either the kernel or the initial condition. Our approach in this subsection is based, in particular, on the extension of the theory of sub-exponential distributions, which we introduced early in [25].

To formulate our main result, we start with the following definition.

Definition 6.23. A function $b : \mathbb{R} \to \mathbb{R}_+$ is said to be

- *(right-side) long-tailed*, if there exists $\rho = \rho_b \geq 0$, such that $b(s) > 0$ for all $s \geq \rho$; and, for any $\tau \geq 0$,
$$\lim_{s\to\infty} \frac{b(s+\tau)}{b(s)} = 1;$$
- *(right-side) tail-decreasing*, if there exists $\rho = \rho_b \geq 0$, such that $b = b(s)$ is strictly decreasing on $[\rho, \infty)$ to 0. In particular, $b(s) > 0$, $s \geq \rho$;
- *(right-side) tail-log-convex*, if there exists $\rho = \rho_b > 0$, such that $b(s) > 0$, $s \geq \rho$, and the function $\log b$ is convex (and hence continuous) on (ρ, ∞).

Definition 6.24. Let $\widetilde{\mathcal{S}}_{\mathrm{reg},d}$ be the set of all bounded functions $b : \mathbb{R} \to \mathbb{R}_+$ such that

1. b is tail-decreasing and tail-log-convex with the same $\rho = \rho_b > 1$, such that $b(\rho) \leq 1$ (without loss of generality); and
$$\int_{-\infty}^{\rho} b(s)\, ds + \int_{\rho}^{\infty} b(s) s^{d-1}\, ds < \infty$$
2. there exist $\delta = \delta_b \in (0,1)$ and an increasing function $h = h_b : (0,\infty) \to (0,\infty)$, with $h(s) < \frac{s}{2}$ and $\lim_{s\to\infty} h(s) = \infty$, such that
$$\lim_{s\to\infty} \frac{b(s \pm h(s))}{b(s)} = 1,$$
$$\lim_{s\to\infty} b\big(h(s)\big) s^{1+\delta} = 0.$$
3. if $d > 1$, then we assume additionally that
   - either, for some $\mu, M > 0$,
$$b(s) = \frac{M}{(1+s)^{d+\mu}}, \quad s \in \mathbb{R}_+,$$
   - or, for all $\nu \geq 1$,
$$\lim_{s\to\infty} b(s) s^{\nu} = 0.$$

Any function which is asymptotically proportional at ∞ to either of

$$(\log s)^{\nu} s^{-(d+\delta)}, \quad s^{\nu} \exp\big(-D(\log s)^q\big), \quad s^{\nu} \exp\big(-s^{\alpha}\big), \quad s^{\nu} \exp\Big(-\frac{s}{(\log s)^{\gamma}}\Big),$$

belongs to the class $\widetilde{\mathcal{S}}_{\mathrm{reg},d}$, provided that $D, \delta > 0$, $q, \gamma > 1$, $\alpha \in (0,1)$, $\nu \in \mathbb{R}$.

We will choose an appropriate function $c:\mathbb{R}^d\to(0,\infty)$ and set

$$\Lambda(t,c) := \bigl\{x\in\mathbb{R}^d \bigm| c(x)\geq e^{-\beta t}\bigr\},$$

where $\beta := \varkappa^+ - m > 0$. Two model examples for us will be

$$c(x) = b(|x|) \quad \text{and} \quad c(x) = \int_{\Delta(x)} b(|y|)\,dy, \qquad x\in\mathbb{R}^d,$$

where $\Delta(x) := \bigl\{y\in\mathbb{R}^d : y_j\geq x_j,\ 1\leq j\leq d\bigr\}$.

We are aimed to show that, for a small enough $\varepsilon>0$,

$$\lim_{t\to\infty} \operatorname*{ess\,inf}_{x\in\Lambda_\varepsilon^-(t,c)} u(x,t) = \theta, \tag{6.10a}$$

$$\lim_{t\to\infty} \operatorname*{ess\,sup}_{x\notin\Lambda_\varepsilon^+(t,c)} u(x,t) = 0. \tag{6.10b}$$

We formulate now our main result.

Theorem 6.25. *Let $b,q:\mathbb{R}_+\to\mathbb{R}_+$ be bounded functions such that, for some $M,\mu,r,\delta>0$,*

$$b(s)+q(s)\leq \frac{M}{(1+s)^{d+\mu}} \quad \text{for a.a. } s\geq r,$$

and $q(s)\geq\delta$ for a.a. $s\in[0,\rho]$. Let (A1)–(A4), (6.8) *hold. Suppose that $a^+(x)=b(|x|)$, $x\in\mathbb{R}^d$. Let either of the following conditions holds*

$$\sup_{s\in\mathbb{R}_+}\frac{q(s)}{b(s)}<\infty, \tag{6.11}$$

$$\sup_{s\in\mathbb{R}_+}\frac{b(s)}{q(s)}<\infty. \tag{6.12}$$

1. *Let $q:\mathbb{R}\to[0,\theta]$ and*

$$u_0(x) = q(|x|), \quad x\in\mathbb{R}^d.$$

Then

   (a) if $b\in\widetilde{\mathcal{S}}_{\mathrm{reg},d}$ and (6.11) *holds, then* (6.10) *holds with $c=a^+$;*

   (b) if $q\in\widetilde{\mathcal{S}}_{\mathrm{reg},d}$ and (6.12) *holds, then* (6.10) *holds with $c=u_0$.*

2. *Let $\int_0^\infty q(s)s^{d-1}ds\in(0,\theta]$ and*

$$u_0(x) = \int_{\Delta(x)} q(|y|)dy, \quad x\in\mathbb{R}^d.$$

Then

   (a) if $b\in\widetilde{\mathcal{S}}_{\mathrm{reg},d}$ and (6.11) *holds, then* (6.10) *holds with*

$$c(x) := \int_{\Delta(x)} a^+(y)dy, \quad x\in\mathbb{R}^d;$$

   (b) if $q\in\widetilde{\mathcal{S}}_{\mathrm{reg},d}$ and (6.12) *holds, then* (6.10) *holds with $c=u_0$.*

Proof. See [22, Theorem 1.5]. □

Note that in [22] the case when (6.8) does not hold was also covered.

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