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Time Between Real and Imaginary: what Geometries Describe Universe near Big Bang?

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Abstract. For about a century, a great challenge for theoretical physics consisted in understanding the role of quantum mode of description of our Universe ("quantum gravity"). Einstein space–times on the scale of observable Universe do not easily submit to any naive quantization scheme. There are better chances to concoct a satisfying quantum picture of the very early space–time, near the Big Bang, where natural scales of events like inflation extrapolated from current observations resist any purely classical description and rather require quantum input.

Many physicists and mathematicians tried to understand the quantum early Universe, sometimes unaware of input of the other community. One of the goals of this article is to contribute to the communication of the two communities. In the main text, I present some ideas and results contained in the recent survey/research papers [Le13] (physicists) and [MaMar14], [MaMar15] (mathematicians).

Introduction and survey

0.1. Relativistic models of space–time: Minkowski signature. Most modern mathematical models in cosmology start with description of space–time as a 4–dimensional *pseudo–Riemannian manifold* M endowed with metric

$$ds^2 = \sum g_{ik} dx^i dx^k$$

of signature $(+,-,-,-)$ where $+$ refers to time–like tangent vectors, whereas the infinitesimal light–cone consists of null–directions. Each such manifold is a point in the infinite–dimensional configuration space of cosmological models.

Basic cosmological models are constrained by *Einstein equations*

$$R_{ik} - \frac{1}{2} R g_{ik} + \Lambda g_{ik} = 8\pi G T_{ik}$$

and/or additional *symmetry postulates,* of which the most essential for us here are the so called *Bianchi IX space–times*, here with symmetry group $SO(3)$, cf. [To13] and [Ne13] for a recent context.

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In this model, the space–time is fibered over the semi–axis of a global ("cosmological") time t. Fibres are homogeneous spaces over $SO(3)$, and the negative Einstein metric $-ds^2$ induces on them a metric of constant curvature. In order to write ds^2 in convenient coordinates, we choose a fixed time–like geodesic ("observer's history") along which ds^2 is dt^2, and coordinatize each space section at the time t by the invariant distance r from the observer and two natural angle coordinates θ, ϕ on the sphere of radius r. By rescaling the radial coordinate, we may assume that the curvature constant k takes one of three values: $k = \pm 1$ or 0.

This rescaling produces the natural unit of length, when $k \neq 0$, and the respective unit of time is always chosen so that the speed of light is $c = 1$.

The Friedman–Robertson–Walker (FRW) metric is then given by the formula

$$ds^2 := dt^2 - R(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right] \tag{0.1}$$

0.2. Input of observations. One of the most counter–intuitive discoveries of the XX–th century cosmology was the "observability" of cosmological time t and possibility to estimate its natural scale ("age of our Universe"). We now know that it is about $14 \cdot 10^9$ years, or five million times longer than the age of human civilisation. Together with considerable homogeneity of the observable space section (local metric disturbances caused by galaxies are counted as negligible) this gives considerable weight to the results of mathematical studies of Bianchi IX $SO(3)$–models.

A robust version of observable global time is the inverse temperature $1/kT$ of the cosmic microwave background (CMB) radiation. It is accepted that the current value of it measures the global age of our Universe starting from the time when it stopped to be opaque for light, about $38 \cdot 10^4$ years after the Big Bang. Near the Big Bang our Universe was extremely hot, and its evolution is measured by its cooling.

Another version of time is furnished by measurements of the redshift of "standard candles" in observable galaxies, thus putting their current appearance on various cosmological time sections of our Universe (Hubble's Law).

Remarkably, generally accepted physical pictures of the Universe involve also unimaginably small periods of cosmological time: between 10^{-40} and 10^{-30} seconds after the Big Bang the radius of space sections has grown 10^{30} times ("inflation era"), with speed many orders of magnitude exceeding the speed of light. The inflation period is postulated in order to explain the homogeneity of space–time sections of observable Universe (on the scale where galaxies are negligible perturbations).

Last but not least: dynamical equations which must be satisfied by metrics of space–time are defined by the choice of Lagrangian (or Hamiltonian as soon as cosmological time variable is introduced). Besides the metric curvature, this Lagrangian may contain contributions from (models of) massive matter, electro–magnetic field etc. Observations led to the picture of the so called "dark matter" and "dark energy" participating only in gravitational interaction. Their cosmological influence far exceeds that of usual matter, say, content of galaxies. In particular, non–vanishing Einstein's cosmological constant Λ responsible for the "dark energy" effect must explain the observable *accelerating expansion* of the Universe.

For more details, see [AU], [Bal].

0.3. Primeval chaos: going backwards in time. As we have already stressed, in mathematical models of general relativity, the notion of time is local: along each oriented geodesic whose tangent vectors lie inside respective light cones, the differential of its time function dt is ds restricted to this geodesic. Applying this prescription formally, we see that even in a flat space–time, along space–like geodesics time becomes purely imaginary, whereas light–like geodesics along which time "stays still", form a wall. The respective wall–crossing in the space of geodesics produces the Wick rotation of time, from real axis to the pure imaginary axis. Along any light–like geodesic, "real" time stops, however "pure imaginary time flow" makes perfect sense appearing e. g., as a variable in wave–functions of photons.

In the main text, we will describe models (suggested in [MaMar14–15]) in which *cosmological time becomes imaginary also at the past boundary of the universe* $t = 0$. However, in these models the reverse Wick rotation does not happen instantly. Instead, it includes the movement of time along a random geodesic curve in the complex half plane endowed with its standard hyperbolic metric.

Moreover, the set of all such geodesics (modulo a subgroup of $PSL(2, \mathbf{Z})$) is endowed with much studied invariant measure, and we regard the resulting classical statistical system as an approximation to an (unknown) quantum description of the early Universe.

Our primary motivation (cf. [MaMar14]) was the desire to explain the pure formal coincidence of the dynamics of two very different systems:

A. *Mixmaster Universe.* In this model, one studies Bianchi IX $SO(3)$ with metric that in appropriate coordinates takes form $ds^2 = dt^2 - a(t)dx^2 - b(t)dy^2 - c(t)dz^2$, $t > 0$. It turns out that the respective Einstein equations have a family of Kasner's exact solutions $a(t) = t^{p_a}, b(t) = t^{p_b}, c(t) = t^{p_c}$. Moreover, mathematical methods of qualitative studies of dynamical systems suggest that *a generic solution* of the relevant Einstein equations, traced backwards in time towards the Big Bang moment $t = 0$, can be approximated by an infinite sequence of Kasner's solutions.

B. *Hyperbolic billiard.* The relevant dynamical system is the hyperbolic billiard on a standard fundamental domain for $PSL(2, \mathbf{Z})$ (or a finite index subgroup), encoded in the Poincaré return map with respect to the boards of this billiard: see [Ar24], [Se85], [Le13], [MaMar15].

However, accommodating Mixmaster Universe in the hyperbolic billiard picture seems to require an analytic continuation of Kasner's solutions. It is not known, and according to some computer assisted studies, time in Kasner's models does not admit the necessary analytic continuation involving space–like coordinates as well, cf. [LuCh13].

In [MaMar15], we avoided this obstacle by looking at the geometry of space–times from the perspective of imaginary time axis. This means that we start with space–times with metrics of the Euclidean signature $(+,+,+,+)$. In the framework of cosmology, they correspond to Bianchi IX $SU(2)$–symmetric space–times, where all coordinates generally can take complex values, so that it makes sense to trace time flow along the relevant geodesics.

0.4. Relativistic models of space–time: Euclidean signature. In these models, space–times satisfying a complexified version of Einstein equations are Bianchi IX four–dimensional manifolds, fibered over domains of complex plane of time, whose fibres are $SU(2)$–homogeneous spaces (rather than $SO(3)$–homogeneous spaces in the cases of Minkowski signature). By analogy with Yang–Mills instantons, they are sometimes called *gravitational instantons.*

More precisely, consider the $SU(2)$ Bianchi IX model with metric of the form

$$g = F\left(d\mu^2 + \frac{\sigma_1^2}{W_1^2} + \frac{\sigma_2^2}{W_2^2} + \frac{\sigma_3^2}{W_3^2}\right). \tag{0.2}$$

Here μ is the relevant version of the cosmological time, (σ_j) are $SU(2)$–invariant forms along space–sections with $d\sigma_i = \sigma_j \wedge \sigma_k$ for all cyclic permutations of $(1,2,3)$, and F is a conformal factor.

By analogy with the $SO(3)$ case and metric $dt^2 - a(t)^2dx^2 - b(t)^2dy^2 - c(t)^2dz^2$, in the main text we will treat W_i (as well as some natural monomials in W_i and F) as $SU(2)$*–scaling factors.*

It is important that, contrary to the $SO(3)$–case, generic anti–self–dual Einstein metrics (solutions of Einstein equations) in the $SU(2)$–case can be written explicitly in terms of elliptic modular functions whereas their chaotic behaviour along geodesics in the complex half–plane of time becomes only a reflection of the chaotic behaviour of the respective billiard ball trajectories.

A natural quantisation scheme of gravitational instantons involves non–commutative deformations of their toric space sections. Focussing on this quantisation scheme, in [MaMar15] we gave additional arguments about relationship between Mixmaster chaos and quantum mechanics of the Big Bang, but this time not involving Kasner's solutions at all: see section 2 of the main text.

0.5. Boundaries of space–times. The statement invoked above that the generic $SO(3)$ space–times traced back to $t \to 0$ can be approximated by an infinite sequence of Kasner's solutions is mathematically formulated and proved by considering a partial compactification of the respective phase–spaces and studying the geometry of separatrices on the boundary of a partial compactification of these phase spaces: see [BoNo73], [Bo85].

Another type of boundaries was considered in [MaMar14], where we tried to produce algebraic–geometric models of Roger Penrose's "aeons": see [Pe10] and [Pe64]–[Pe02]. According to his scheme, the moment $t = 0$ of our cosmological time might have been preceded by evolution of another Universe, the cold death of which was a prequel of our Big Bang. According to Penrose, conformal classes of the respective metrics furnish a continuous transition from the previous aeon to the next one.

Since a conformal change of the metric does not change the relevant light cone in the tangent space at any point of space–time, we suggested in [MaMar14] matching pairs of boundaries between aeons, in which the projective compactification of cold Minkowski space–time of previous aeon matches the blown up divisor over the Big Bang point of the next aeon.

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Cosmology has its own singular place in the body of scientific knowledge: the same quest for the meaning of Universe influences philosophy, poetry, faith (cf. two remarkable books [Lam07], [Lam15] about life, faith and research of Canon Georges Lemaitre, the first discoverer of Hubble's Law and Big Bang picture).

I will therefore close this introduction quoting the wonderful lines by Steven Weinberg ([We77]):

As I write this I happen to be in an airplane at 30,000 feet, flying over Wyoming en route home from San Francisco to Boston. Below, the earth looks very soft and comfortable — fluffy clouds here and there, snow turning pink as the sun sets, roads stretching straight across the country from one town to another. It is very hard to realize that this is just a tiny part of an overwhelmingly hostile universe. It is even harder to realise that this present universe has evolved from an unspeakably unfamiliar early condition, and faces a future extinction of endless cold or intolerable heat. The more the universe seems comprehensible, the more it also seems pointless.

But if there is no solace in the fruits of our research, there is at least some consolation in the research itself. Men and women are not content to comfort themselves with tales of gods and giants, or to confine their thoughts to the daily affairs of life; they also build telescopes and satellites and accelerators, and sit at their desks for endless hours working out the meaning of the data they gather. The effort to understand the universe is one of the very few things that lifts human life a little above the level of farce, and gives it some of the grace of tragedy.

Steven Weinberg. “The first three minutes.”

1 Cosmological time, elliptic integrals, and upper complex half–plane

1.1. Minkowski signature: late Universe. Following [To13] and [Ne13], we consider the cosmological time at the late stage of the FRW model (0.1).

It is convenient to replace r in (0.1) by the third dimensionless “angle” coordinate $\chi := r/R(t)$. Then (0.1) becomes

$$ds^2 := dt^2 - R(t)^2 \left[d\chi^2 + S_k^2(\chi)(d\theta^2 + \sin^2\theta\, d\phi^2) \right], \qquad (1.1)$$

where $S_k(\chi) = \sin\chi$ for $k = 1$; χ for $k = 0$; and $\sinh\chi$ for $k = -1$.

This rescaling produces the natural unit of length, when $k \neq 0$, and the respective unit of time is always chosen so that the speed of light is $c = 1$.

Dynamic in this model is described by one real function $R(t)$: it increases from zero at the Big Bang of one aeon to infinity.

We scale $R(t)$ by putting $R = 1$ “now”, as in [To13]. Notations in [To13] slightly differ from ours. In his formula for metric (2), r is our χ, and $f_k(r)$ is our $S_k(\chi)$.

This function is constrained by the Einstein–Friedman equations (here with cosmological constant $\Lambda = 3$), which leads to the introduction of the elliptic curve given by the equation in the (Y, R)–plane

$$Y^2 = R^4 + aR + b \tag{1.2}$$

(see [To13], equation (3), and [Ne13], eq. (9), where their S is the same as our R).

Besides the proper time t, and the scale factor $R(t)$, global time may be measured by its conformal version τ, which according to [To], formula (3), may be given as the Abelian integral along a real curve on the complex torus, Riemann surface of the elliptic curve (1.2):

$$\tau \cong \int_0^{R(t)} \frac{dR}{Y}. \tag{1.3}$$

Physical interpretation of the coefficients a, b as characterising matter and radiation sources in Einstein equations for this model for which we refer the reader to [OlPe05] and [To13], shows that in principle a, b also depend on time. Then (1.2) describes a family of elliptic curves parametrized in a way that is classic and well known to algebraic geometers. In particular, cosmological time variable moves along one of the versions of base families of elliptic curves.

Universal families of elliptic curves are parametrized by upper complex half–plane and its quotients (modular curves), and we see now that a family of elliptic curves (1.2) naturally emerges in the description of a late stage of evolution of the FRW model. In a pure mathematical context, the reader is invited to compare our suggestion with the treatment of the Painlevé VI equation in [Ma96] and the whole hierarchy of Painlevé equations in [Ta01].

Now we will discuss a totally different way in which the chaotic evolution in Mixmaster early Universe leads to the appearance of modular curves as well.

1.2. Minkowski signature: early Universe and Mixmaster chaos. As a model of the early universe emerging after the Big Bang we take here the Bianchi IX space–time, admitting $SO(3)$–symmetry of its space–like sections. We will choose coordinates in which its metric takes the following form:

$$ds^2 = dt^2 - a(t)^2 dx^2 - b(t)^2 dy^2 - c(t)^2 dz^2, \tag{1.4}$$

where the coefficients $a(t)$, $b(t)$, $c(t)$ are called scaling factors.

A family of such metrics satisfying Einstein equations is given by *Kasner solutions*,

$$a(t) = t^{p_1},\ b(t) = t^{p_2},\ c(t) = t^{p_3} \tag{1.5}$$

in which p_i are points on the real algebraic curve

$$\sum p_i = \sum p_i^2 = 1. \tag{1.6}$$

These metrics become singular at $t = 0$ which is the Big Bang moment.

Around 1970, V. Belinskii, I. M. Khalatnikov, E. M. Lifshitz and I. M. Lifshitz argued that almost every solution of the Einstein equations for (1.4) *traced* *backwards in time* $t \to +0$ can be approximately described by a sequence of solutions (1.5) or equivalently, of points (1.6): see [KLiKhShSi85] for a later and more comprehensive study. The n–th point of this sequence begins the respective n–th *Kasner era*, at the end of which a jump to the next point occurs, see below.

A mathematically careful treatment of this discovery in [BoNo73] has shown that this encoding is certainly applicable to *another dynamical system* which is defined on the boundary of a certain compactification of the phase space of this Bianchi IX model and in a sense is its limit.

Construction of this boundary involves a nontrivial real blow up at the $t = 0$, see details in [Bo85]. The resulting boundary is an attractor, it supports an array of fixed points and separatrices, and the jumps between separatrices which result from subtle instabilities account for jumps between successive Kasner's regimes, corresponding to different points of (1.6).

In what sense this picture approximates the actual trajectories, is a not quite trivial question: cf. the last three paragraphs of the section 2 of [KLiKhShSi85], where it is explained that among these trajectories there can exist "anomalous" cases when the description in terms of Kasner eras does not make sense, but that they are, in a sense, infinitely rare. See also the recent critical discussion in [LuCh13].

Here are some details of the classical description.

(a) Continued fractions. We denote by $\mathbf{Z}$, resp. $\mathbf{Z}_+$, the set of integers, resp. positive integers; $\mathbf{Q}$, resp. $\mathbf{R}$ is the field of rational, resp. real numbers. For $x \in \mathbf{R}$, we put $[x] := \max\{m \in \mathbf{Z} \,|\, m \le x\}$.

Irrational numbers $x > 1$ admit the canonical infinite continued fraction representation

$$x = k_0 + \cfrac{1}{k_1 + \cfrac{1}{k_2 + \dots}} =: [k_0, k_1, k_2, \dots],\ k_s \in \mathbf{Z}_+ \tag{1.7}$$

in which $k_0 := [x]$, $k_1 = [1/(x - k_0)]$ etc. Notice that our convention differs from that of [KLiKhShSi85]: their $[k_1, k_2, \dots]$ means our $[0, k_1, k_2, \dots]$.

(b) Transformation T. The (partial) map $\widetilde{T} :\ [0,1]^2 \to [0,1]^2$ is defined by

$$\widetilde{T} : (x, y) \mapsto \left(\frac{1}{x} - \left[\frac{1}{x}\right], \frac{1}{y + [1/x]} \right), \tag{1.8}$$

If both coordinates $(x, y) \in [0,1]^2$ are irrational (the complement is a subset of measure zero), we have for uniquely defined $k_s \in \mathbf{Z}_+$:

$$x = [0, k_0, k_1, k_2, \dots],\ y = [0, k_{-1}, k_{-2}, \dots].$$

Then

$$\frac{1}{x} - \left[\frac{1}{x}\right] = [0, k_1, k_2, \dots],$$

$$\frac{1}{y + [1/x]} = \frac{1}{k_0 + y} = [0, k_0, k_{-1}, k_{-2}, \dots].$$

On this subset, $\widetilde{T}$ is bijective and has invariant density

$$\frac{dx\,dy}{\ln 2\cdot(1+xy)^2}$$

(cf. [May87]).

Thus we may and will bijectively encode irrational pairs $(x,y)\in[0,1]^2$ by doubly infinite sequences

$$(k):=[\dots k_{-2},k_{-1},k_0,k_1,k_2,\dots],k_i\in\mathbf{Z}_+$$

in such a way that the map $\widetilde{T}$ above becomes the shift of such a sequence denoted T:

$$T(k)_s=k_{s+1}. \tag{1.9}$$

(c) Continued fractions and Mixmaster chaos. Any point (p_a,p_b,p_c) in (1.6) can be obtained by choosing a unique $u\in[1,\infty]$, putting

$$p_1^{(u)}:=-\frac{u}{1+u+u^2}\in[-1/3,0],\ p_2^{(u)}:=\frac{1+u}{1+u+u^2}\in[0,2/3],$$

$$p_3^{(u)}:=\frac{u(1+u)}{1+u+u^2}\in[2/3,1] \tag{1.10}$$

and then rearranging the exponents $p_1^{(u)}\le p_2^{(u)}\le p_3^{(u)}$ by a bijection $(1,2,3)\to(a,b,c)$.

As we have already explained, a "typical" solution γ of Einstein equations (vacuum, or with various energy momentum tensors) with $SO(3)$–symmetry of the Bianchi IX type, followed from an arbitrary (small) value $t_0>0$ in the reverse time direction $t\to+0$, oscillates close to a sequence of Kasner type solutions.

Somewhat more precisely, introduce the local logarithmic time Ω along this trajectory with inverted orientation. Its differential is $d\Omega:=-\dfrac{dt}{abc}$, and the time itself is counted from an arbitrary but fixed moment. Then $\Omega\to+\infty$ approximately as $-\log t$ as $t\to+0$, and we have the following picture.

As $\Omega\cong-\log t\to+\infty$, a "typical" solution γ of the Einstein equations determines a sequence of infinitely increasing moments $\Omega_0<\Omega_1<\dots<\Omega_n<\dots$ and a sequence of irrational real numbers $u_n\in(1,+\infty)$, $n=0,1,2,\dots$.

The time semi–interval $[\Omega_n,\Omega_{n+1})$ is called the *n–th Kasner era* for the trajectory γ (in [Le13], our eras are called epochs). Within the n–th era, the evolution of a,b,c is approximately described by several consecutive Kasner's formulas. Time intervals where scaling powers (p_i) are constant are called *Kasner's cycles* (in [Le13], our cycles are called eras).

The evolution in the n–th era starts at time Ω_n with a certain value $u=u_n>1$ which determines the sequence of respective scaling powers during the first cycle (1.10):

$$p_1=-\frac{u}{1+u+u^2},\ p_2=\frac{1+u}{1+u+u^2},\ p_3=\frac{u(1+u)}{1+u+u^2}$$

The next cycles inside the same era start with values $u=u_n-1,\ u_n-2,\dots,$ and scaling powers (1.10) corresponding to these numbers, rearranged corresponding to a bijection $(1,2,3)\to(a,b,c)$ which is in turn identical to the previous one, or interchanges b and c (see [MaMar02] or [Le13] for a modular interpretation).

After $k_n := [u_n]$ cycles inside the current era, a jump to the next era comes, with parameter

$$u_{n+1} = \frac{1}{u_n - [u_n]}. \tag{1.11}$$

Moreover, ensuing encoding of γ's and respective sequences (u_i)'s by continued fractions (1.7) of real irrational numbers $x > 1$ is bijective on the set of full measure.

Finally, when we want to include into this picture also the sequence of logarithmic times Ω_n starting new eras, we naturally pass to the two–sided continued fractions and the transformationn T. Here are some details.

(d) Doubly infinite sequences and modular geodesics. Let $H := \{z \in \mathbf{C},\ \mathrm{Im}\, z > 0\}$ be the upper complex half–plane with its Poincaré metric $|dz|^2/|\mathrm{Im}\, z|^2$. Denote also by $\overline{H} := H \cup \{\mathbf{Q} \cup \{\infty\}\}$ this half–plane completed with cusps.

The vertical lines $\mathrm{Re}\, z = n, n \in \mathbf{Z}$, and semicircles in $\overline{H}$ connecting pairs of finite cusps $(p/q, p'/q')$ with $pq' - p'q = \pm 1$, cut $\overline{H}$ into the union of geodesic ideal triangles which is called the *Farey tessellation.*

Following [Ar24], [Se85], consider the set of *oriented* geodesics β's in H with ideal irrational endpoints in $\mathbf{R}$. Let $\beta_{-\infty}$, resp. β_∞ be the initial, resp. the final point of β. Let B be the set of such geodesics with $\beta_{-\infty} \in (-1, 0)$, $\beta_\infty \in (1, \infty)$. Put

$$\beta_{-\infty} = -[0, k_0, k_{-1}, k_{-2}, \dots], \quad \beta_\infty = [k_1, k_2, k_3, \dots], \quad k_i \in \mathbf{Z}_+, \tag{1.12}$$

and encode β by the doubly infinite continued fraction

$$[\dots k_{-2}, k_{-1}, k_0, k_1, k_2, \dots]. \tag{1.13}$$

The geometric meaning of this encoding can be explained as follows. Consider the intersection point $x = x(\beta)$ of β with the imaginary semiaxis in H. Moving along β from x to β_∞, one will intersect an infinite sequence of Farey triangles. Each triangle is entered through a side and left through another side, leaving the ideal intersection point (a cusp) of these sides either to the left, or to the right. Then the infinite word in the alphabet $\{L, R\}$ encoding the consecutive positions of these cusps wrt β will be $L^{k_1} R^{k_2} L^{k_3} R^{k_4} \dots$ Similarly, moving from $\beta_{-\infty}$ to x, we will get the word (infinite to the left) $\dots L^{k_{-1}} R^{k_0}$.

We can enrich the new notation $\dots L^{k_{-1}} R^{k_0} L^{k_1} R^{k_2} L^{k_3} R^{k_4} \dots$ (called *cutting sequence* of our geodesic in [Se85]) by inserting between the consecutive powers of L, R notations for the respective intersection points of β with the sides of Farey triangles. So $x_0 := x = x(\beta)$ will be put between R^{k_0} and L^{k_1}, and generally we can imagine the word

$$\dots L^{k_{-1}} x_{-1} R^{k_0} x_0 L^{k_1} x_1 R^{k_2} x_2 L^{k_3} x_3 R^{k_4} \dots \tag{1.14}$$

Since the Farey tessellation is acted upon by the modular group $PSL(2, \mathbf{Z})$ and its hyperbolic extension including orientation changing isometries of H, we may present another version of the geometric description of geodesic flow. This is an equivalent dynamical system which is the triangular hyperbolic billiard with infinitely distant corners ("pockets"): see [Ar24], [Se85], [Le13], [MaMar15].

Here we use the term "hyperbolic" in order to indicate that sides (boards) of the billiard and trajectories of the ball ("particle") are geodesics with respect to the hyperbolic metric of constant curvature -1 of the billiard table. This is not the standard meaning of the hyperbolicity in this context, where it usually refers to non–vanishing Lyapunov exponents.

(e) **Proposition.** *All hyperbolic triangles of the Farey tessellation of $\overline{H}$ are isomorphic as metric spaces.*

For any two closed triangles having a common side there exists unique metric isomorphism of them identical along this side. It inverts orientation induced by H. Starting with the basic triangle Δ with vertices $\{0, 1, i\infty\}$ and consecutively using these identifications, one can unambiguously define the map $b : \overline{H} \to \Delta$.

Any oriented geodesic on H with irrational end–points in $\mathbf{R}$ is sent by the map b to a billiard ball trajectory on the table Δ never hitting corners.

All this is essentially well known since at least [Ar24].

It is also worth noticing that although all three sides of Δ are of infinite length, this triangle is *equilateral* in the following sense: there exists a group S_6 of hyperbolic isometries of Δ acting on vertices by arbitrary permutations. This group has a unique fixed point $\rho := \exp(\pi i/3)$ in Δ, *the centroid* of Δ.

In fact, this group is generated by two isometries: $z \mapsto 1 - z^{-1}$ and symmetry with respect to the imaginary axis.

Three finite geodesics connecting the centre ρ with points $i, 1 + i, \dfrac{1+i}{2}$ respectively, subdivide Δ into three geodesic quadrangles, each having one infinite (cusp) corner. We will call these points *centroids* of the respective sides of Δ, and the geodesics (ρ, i) etc. *medians* of Δ.

Each quadrangle is the fundamental domain for $PSL(2, \mathbf{Z})$.

(f) Billiard encoding of oriented geodesics. Consider the first stretch of the geodesic β encoded by (1.14) that starts at the point x_0 in $(0, i\infty)$. If $k_0 = 1$,the ball along β reaches the opposite side $(1, i\infty)$ and gets reflected to the third side $(0, 1)$. If $k_0 = 2$, it reaches the opposite side, then returns to the initial side $(0, i\infty)$, and only afterwards gets reflected to $(0, 1)$.

More generally, the ball always spends k_0 unobstructed stretches of its trajectory between $(0, i\infty)$ and $(1, i\infty)$, but then is reflected to $(0, 1)$ either from $(1, i\infty)$ (if k_0 is odd), or from $(0, i\infty)$ (if k_0 is even). We can encode this sequence of stretches by the formal word ∞^{k_0} showing exactly how many times the ball is reflected "in the vicinity" of the pocket $i\infty$, that is, does not cross any of the medians.

A contemplation will convince the reader that this allows one to define an alternative encoding of β by the double infinite word in *three letters* , say a, b, c, serving as names of the vertices $\{0, 1, i\infty\}$.

(g) Kasner's eras in logarithmic time and doubly infinite continued fractions. Now we will explain, how the double infinite continued fractions enter the Mixmaster formalism when we want to mark the consecutive Kasner eras upon the t–axis, or rather upon the Ω–axis, where $\Omega := -\log \int dt/abc$

In the process of construction, these continued fractions will also come with their enrichments.

We start with fixing a "typical" space–time γ whose evolution with $t \to +0$ undergoes (approximately) a series of Kasner's eras described by a continued fraction $[k_0, k_1, k_2, \dots]$, where k_s is the number of Kasner's cycles within s–th era $[\Omega_s, \Omega_{s+1})$. We have enriched this encoding by introducing parameters u_s which determine the Kasner exponents within the first cycle of the era number s by (1.5). A further enrichment comes with putting these eras on the Ω–axis. According to [KLiKhShSi85], [BoNo73], [Bo85], if one defines the sequence of numbers δ_s from the relations

$$\Omega_{s+1} = [1 + \delta_s k_s (u_s + 1/\{u_s\})]\Omega_s,$$

then complete information about these numbers can be encoded by the extension to the left of our initial continued fraction:

$$[\dots, k_{-1}, k_0, k_1, k_2, \dots] \tag{1.15}$$

in such a way that

$$\delta_s = x_s^+/(x_s^+ + x_s^-)$$

where

$$x_s^+ = [0, k_s, k_{s+1}, \dots], \quad x_s^- = [0, k_{s-1}, k_{s-2}, \dots]. \tag{1.16}$$

The following result established in [MaMar15] shows that cosmological time can be approximately measured in terms of geodesic length of path of the billiard ball.

1.3. Theorem. *Let a "typical" Bianchi IX Mixmaster Universe be encoded by the double–sided sequence (1.15). Consider also the respective geodesic in H with its enriched encoding (1.14).*

Then we have "asymptotically" as $s \to \infty$, $s \in \mathbf{Z}_+$:

$$\log \Omega_{2s}/\Omega_0 \simeq 2 \sum_{r=0}^{s-1} \mathrm{dist}\,(x_{2r}, x_{2r+1}), \tag{1.17}$$

where dist *denotes the hyperbolic distance between the consecutive intersection points of the geodesic with sides of the Farey tesselation as in (1.14).*

The formula (1.17) shows that the distance measured along a geodesic can be compared to (doubly) logarithmic cosmological time.

During the stretch of *time/geodesic length* which such a geodesic spends in the vicinity of a vertex of Δ, the respective space–time in a certain sense can be approximated by its degenerate version, corresponding to the vertex itself, and this will justify considering below the respective segments of geodesics as the "instanton Kasner eras".

1.4. Riemannian signature: Bianchi IX models with $SU(2)$–symmetry. Consider the $SU(2)$ Bianchi IX model with metric of the form

$$g = F\left(d\mu^2 + \frac{\sigma_1^2}{W_1^2} + \frac{\sigma_2^2}{W_2^2} + \frac{\sigma_3^2}{W_3^2}\right). \tag{1.18}$$

Here μ is cosmological time, (σ_j) are $SU(2)$–invariant forms along space–sections with $d\sigma_i = \sigma_j \wedge \sigma_k$ for all cyclic permutations of $(1,2,3)$, and F is a conformal factor.

By analogy with the $SO(3)$–case and metric $dt^2 - a(t)^2dx^2 - b(t)^2dy^2 - c(t)^2dz^2$, we may and will treat W_i (as well as some natural monomials in W_i and F) as $SU(2)$*–scaling factors.*

However, contrary to the $SO(3)$–case, generic solutions of Einstein equations in the $SU(2)$–case can be written explicitly in terms of elliptic modular functions, whereas their chaotic behaviour along geodesics in the complex half–plane of time is only a reflection of the chaotic behaviour of the respective billiard ball trajectories.

We will use explicit formulas given in [BaKo98], where they were deduced from the basic results of [Hi95]. The central role in them is played by theta–functions depending on the the complex arguments $i\mu \in H$, $z \in \mathbf{C}$, with parameters (p,q) called theta–characteristics:

$$\vartheta[p,q](z,i\mu) := \sum_{m\in\mathbf{Z}} \exp\{-\pi(m+p)^2\mu + 2\pi i(m+p)(z+q)\}. \tag{1.19}$$

It can be expressed through the theta–function with vanishing characteristics:

$$\vartheta[p,q](z,i\mu) = \exp\{-\pi p^2\mu + 2\pi ipq\}\cdot\vartheta[0,0](z+pi\mu+q,i\mu). \tag{1.20}$$

All these functions satisfy classical automorphy identities with respect to the action of $PGL(2,\mathbf{Z})$.

1.5. Theorem. ([To94], [Hi95], [BaKo98].) *Put*

$$\vartheta[p,q] := \vartheta[p,q](0,i\mu) \tag{1.21}$$

and

$$\vartheta_2 := \vartheta[1/2,0],\ \vartheta_3 := \vartheta[0,0],\ \vartheta_4 := \vartheta[0,1/2]. \tag{1.22}$$

(A) Consider the following scaling factors as functions of μ *with parameters* (p,q)*:*

$$W_1 := \frac{i}{2}\vartheta_3\vartheta_4\frac{\frac{\delta}{\delta q}\vartheta[p,q+1/2]}{e^{\pi ip}\vartheta[p,q]},\ W_2 := \frac{i}{2}\vartheta_2\vartheta_4\frac{\frac{\delta}{\delta q}\vartheta[p+1/2,q+1/2]}{e^{\pi ip}\vartheta[p,q]},$$

$$W_3 := -\frac{1}{2}\vartheta_2\vartheta_3\frac{\frac{\delta}{\delta q}\vartheta[p+1/2,q]}{\vartheta[p,q]}, \tag{1.23}$$

Moreover, define the conformal factor F *with non–zero cosmological constant* Λ *by*

$$F := \frac{2}{\pi\Lambda}\frac{W_1W_2W_3}{(\frac{\delta}{\delta q}\log\vartheta[p,q])^2} \tag{1.24}$$

The metric (1.18) with these scaling factors for real $\mu > 0$ *is real and satisfies the Einstein equations if either*

$$\Lambda < 0,\ p\in\mathbf{R},\ q\in\frac{1}{2}+i\mathbf{R}, \tag{1.25}$$

or

$$\Lambda > 0,\ q \in \mathbf{R},\ p \in \frac{1}{2} + i\mathbf{R}. \tag{1.26}$$

(B) Consider now a different system of scaling factors

$$W_1' := \frac{1}{\mu + q_0} + 2\frac{d}{d\mu}\log\vartheta_2,\ W_2' := \frac{1}{\mu + q_0} + 2\frac{d}{d\mu}\log\vartheta_3,$$

$$W_3' := \frac{1}{\mu + q_0} + 2\frac{d}{d\mu}\log\vartheta_4, \tag{1.27}$$

and

$$F' := -C(\mu + q_0)^2\, W_1' W_2' W_3', \tag{1.28}$$

where $q_0, C \in \mathbf{R}$*,* $C > 0$*.*

The metric (1.18) with these scaling factors for real $\mu > 0$ *is real and satisfies the Einstein equations with vanishing cosmological constant.*

We will now consider values of $i\mu \in \Delta \subset \overline{H}$ in the vicinity of $i\infty$ but not necessarily lying on the imaginary axis. Since we are interested in the instanton analogs of Kasner's solutions, we will collect basic facts about asymptotics of scaling factors for $i\mu \to i\infty$.

For brevity, we will call a number $r \in \mathbf{R}$ *general*, if $r \notin \mathbf{Z} \cup (1/2 + \mathbf{Z})$.

For such r, denote by $\langle r\rangle \in (-1/2, 0) \cup (0, 1/2)$ such real number that $r + m_0 = \langle r\rangle$ for a certain (unique) $m_0 \in \mathbf{Z}$.

1.6. Theorem. *The scaling factors of the Bianchi IX spaces listed in Theorem 1.5 have the following asymptotics near* $\mu = +\infty$*:*

(i) For $\Lambda = 0$*:*

$$W_1' \sim -\frac{\pi}{2}, \quad W_2' \sim W_3' \sim \frac{1}{\mu + q_0}. \tag{1.29}$$

(ii) For $\Lambda < 0$ *and general* p*:*

$$W_1 \sim -\pi\langle p\rangle \exp\{\pi i(\langle p\rangle - p)\}, \quad W_2 \sim \pm W_3,$$

$$W_3 \sim -2\pi i\,\langle p + 1/2\rangle \cdot \exp\{\pi i\,\mathrm{sgn}\,\langle p\rangle q\} \cdot \exp\{\pi\mu(|\langle p\rangle| - 1/2)\}. \tag{1.30}$$

(iii) For $\Lambda > 0$*, real* q *and* $p = 1/2 + ip_0, p_0 \in \mathbf{R}$*:*

$$W_1 \sim \pi p_0\, \tan\{\pi(q - p_0\mu)\} - \frac{1}{2}, \quad W_2 \sim -W_3,$$

$$W_3 \sim 2\pi p_0 \cdot (\cos\, \pi(q - p_0\mu))^{-1}. \tag{1.31}$$

Theorem 1.6 (proved in [MaMar15]) shows that for general members of all solution families from [BaKo98], after eventual sign changes of some W_i's and outside of the pole singularities on the real time axis, we have asymptotically $W_2 = W_3$, $W_1 \neq W_2$.

In the next section, we will show that precisely such a condition allows one to quantize the respective geometric picture in terms of Connes–Landi ([CoLa01]). This gives additional substance to our vision that chaotic Mixmaster evolution along hyperbolic geodesics reflects a certain "dequantization" of the hot quantum early Universe.

1.7. Gravitational instantons and Painlevé VI. Hitchin's classification of gravitational instantons that led to Theorem 1.5 was based upon the reduction of the relevant Einstein equations to a Painlevé VI equation. We will briefly recall basics facts about them; see [Ta01] for a more general context.

Equations of the type Painlevé VI form a four–parametric family. Denote parameters $(\alpha, \beta, \gamma, \delta)$, and the independent variable by t. The corresponding equation for a function $X(t)$ looks as follows:

$$\frac{d^2X}{dt^2} = \frac{1}{2}\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t}\right)\left(\frac{dX}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t}\right)\frac{dX}{dt} +$$

$$+ \frac{X(X-1)(X-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{(X-t)^2}\right].$$

Gravitational instantons correspond to the case

$$(\alpha, \beta, \gamma, \delta) = (\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8}).$$

Solutions in elliptic functions of this equation describe Bianchi IX space–times with SU(2)–symmetry: see [Hi95].

One more interesting case is $(\alpha, \beta, \gamma, \delta) = (\frac{9}{2}, 0, 0, \frac{1}{2})$. According to B. Dubrovin, a specific solution of this equation describes "the mirror of $\mathbf{P}^2$" in a general context of Mirror Symmetry.

In 1907, R. Fuchs has rewritten PVI in the form

$$t(1-t)\left[t(1-t)\frac{d^2}{dt^2} + (1-2t)\frac{d}{dt} - \frac{1}{4}\right]\int_\infty^{(X,Y)} \frac{dx}{\sqrt{x(x-1)(x-t)}} =$$

$$= \alpha Y + \beta\frac{tY}{X^2} + \gamma\frac{(t-1)Y}{(X-1)^2} + (\delta - \frac{1}{2})\frac{t(t-1)Y}{(X-t)^2} \qquad (1.32)$$

Here he enhanced $X := X(t)$ to $(X, Y) := (X(t), Y(t))$ treating the latter pair as a section $P := (X(t), Y(t))$ of the generic elliptic curve $E = E(t): \ Y^2 = X(X-1)(X-t)$.

Up to a simple change of notations, the abelian integral $\int_\infty^{(X,Y)}$ in (1.32) can be directly identified with the abelian integral $\int_0^{R(t)}$ in (1.3) so that this integral is a version of cosmological time. The meaning of the right hand side of (1.32) was clarified in my paper [Ma96]. After having noticed that Painlevé VI can be written on any one–dimensional family of elliptic curves (its dependent variable becoming a (multi)section of such a family), I have applied this remark to the analytic family $E_\tau := \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau) \mapsto \tau \in H$. Denoting by z a fixed coordinate on $\mathbf{C}$, we can rewrite (1.32) in the form

$$\frac{d^2z}{d\tau^2} = \frac{1}{(2\pi i)^2}\sum_{j=0}^{3} \alpha_j \wp_z(z + \frac{T_j}{2}, \tau) \qquad (1.33)$$

Here $(\alpha_0, \dots, \alpha_3) := (\alpha, -\beta, \gamma, \frac{1}{2} - \delta)$, $(T_0, T_1, T_2, T_3) := (0, 1, \tau, 1 + \tau)$, and

$$\wp(z, \tau) := \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z - m\tau - n)^2} - \frac{1}{(m\tau + n)^2}\right).$$

Moreover, equation of the generic elliptic curve becomes

$$\wp_z(z,\tau)^2 = 4(\wp(z,\tau) - e_1(\tau))(\wp(z,\tau) - e_2(\tau))(\wp(z,\tau) - e_3(\tau))$$

where

$$e_i(\tau) = \wp(\frac{T_i}{2}, \tau),$$

so that $e_1 + e_2 + e_3 = 0$.

Since $PGL(2, \mathbf{Z})$ acts on the total space of this family, the "time variable" τ (an abelian integral along closed path on a curve) can be restricted to the fundamental domain of this group or its finite index subgroup, and this leads to the hyperbolic billiard picture.

2 Quantum Big Bang?

2.1. Canonical quantisation of the billiard system and Maass forms. The most straightforward way to produce from the Mixmaster chaotic system its quantum version consists in applying canonical quantisation to the billiard ball moving in one of the version of hyperbolic billiard table discussed above.

This immediately leads to the consideration of Maass wave functions: eigenvectors Ψ of the Laplace–Beltrami operator on the hyperbolic half–plane, invariant with respect to an appropriate subgroup of the (extended) modular group. They play now role of quantum wave–functions of early hot Universe.

We refer to [Le13], sec. VI and VII, for a detailed discussion of this quantisation scheme and relevant references. See also [Fu86].

Below we will discuss a different quantisation scheme, developed in the framework of non commutative geometry (cf. [CoLa01]). We will then connect it with the complex geometry of gravitational instantons, described in subsections 1.4–1.6 above. This was done in our article [MaMar15].

2.2. Theta deformations. In Section 5 of [MaMar14] we showed that the gluing of space–times across the singularity using an algebro-geometric blowup can be made compatible with the idea of spacetime coordinates becoming noncommutative in a neighborhood of the initial singularity where quantum gravity effects begin to dominate.

This compatibility is described there in terms of Connes–Landi theta deformations ([CoLa01]) and Cirio–Landi–Szabo toric deformations ([CiLaSza11–13]) of Grassmannians.

It turns out that the Bianchi IX models with $SU(2)$-symmetry can be made compatible with the hypothesis of noncommutativity at the Planck scale, using isospectral theta deformations.

The metrics on the S^3 sections, in this case, are only left $SU(2)$–invariant. It turns out that among all the $SU(2)$ Bianchi IX spacetime, the only ones that admit isospectral theta–deformations of their spatial S^3–sections are those where the metric tensor

$$g = w_1 w_2 w_3\, d\mu^2 + \frac{w_2 w_3}{w_1}\, \sigma_1^2 + \frac{w_1 w_3}{w_2}\, \sigma_2^2 + \frac{w_1 w_2}{w_3}\, \sigma_3^2 \tag{2.2}$$

is of the special form satisfying $w_1 \neq w_2 = w_3$ (the two directions σ_2 and σ_3 have equal magnitude). In these metrics, the S^3 sections are Berger spheres. This class includes the general Taub-NUT family ([Taub51], [NUT63]), and the Eguchi–Hanson metrics ([EgHa79a], [EgHa79b]). The theta–deformations are obtained, as in the case of the deformations S^3_θ of [CoLa01] of the round 3-sphere, by deforming all the tori of the Hopf fibration to noncommutative tori.

In other words, a Bianchi IX Euclidean spacetime X with $SU(2)$–symmetry admits a noncommutative theta-deformation X_θ, obtained by deforming the tori of the Hopf fibration of each spacial section S^3 to noncommutative tori, if and only if its metric has the $SU(2) \times U(1)$–symmetric form

$$g = w_1 w_3^2 \, d\mu^2 + \frac{w_3^2}{w_1} \, \sigma_1^2 + w_1 \, (\sigma_2^2 + \sigma_3^2). \tag{2.3}$$

(see [MaMar15], Proposition 4.2).

This is in stark contrast with the situation described in [EsMar13], where (Lorentzian and Euclidean) Mixmaster cosmologies of the form

$$\mp dt^2 + a(t)^2 dx^2 + b(t)^2 dy^2 + c(t)^2 dz^2$$

were considered, with T^3-spatial sections, which always admit isospectral theta-deformations (see also [vSuij04], [Mar08]).

We have recalled in the previous section how the general self–dual solutions (with $w_1 \neq w_2 \neq w_3$) can be written explicitly in terms of theta constants [BaKo98], and are obtained from a Darboux–Halphen type system [PeVa12], [Tak92]. In the case of the family of Bianchi IX models with $SU(2) \times U(1)$-symmetry, this system has algebraic solutions that give

$$w_2 = w_3 = \frac{1}{\mu - \mu_0}, \quad w_1 = \frac{\mu - \mu_*}{(\mu - \mu_0)^2}, \tag{2.4}$$

with singularities at μ_* (curvature singularity), μ_0 (Taubian infinity) and ∞ (nut). The condition $\mu_* < \mu_0$ avoids naked singularities, by hiding the curvature singularity at μ_* behind the Taubian infinity, see the discussion in Section 5.2 of [PeVa12].

Consider the operator

$$D_B = -i \begin{pmatrix} \frac{1}{\lambda} X_1 & X_2 + iX_3 \\ X_2 - iX_3 & -\frac{1}{\lambda} X_1 \end{pmatrix} + \frac{\lambda^2 + 2}{2\lambda}, \tag{2.5}$$

where $\{X_1, X_2, X_3\}$ constitute a basis of the Lie algebra orthogonal for the bi–invariant metric. Assume moreover that the left–invariant metric on S^3 is diagonal in this basis, with eigenvalues $\{w^2/w_1, w_1, w_1\}$, with $w = w_2 = w_3$ and $\lambda = w/w_1$, and where the w_i are as in (4.4). Consider also the operator

$$D = \frac{1}{w_1^{1/2} w} \left(\gamma^0 \left(\frac{\partial}{\partial \mu} + \frac{1}{2} \left(\frac{\dot{w}}{w} + \frac{1}{2} \frac{\dot{w}_1}{w_1} \right) \right) + w_1 \, D_B|_{\lambda = \frac{w}{w_1}} \right). \tag{2.6}$$

2.3. Proposition. *The operators D of (2.6) give Dirac operators for isospectral theta deformations of the $SU(2) \times U(1)$–symmetric space–times.*

As in [EsMar13], the Dirac operator of Proposition 2.3 can be seen as involving an anisotropic Hubble parameter H. In the case of the metrics of [EsMar13] this was of the form

$$H = \frac{1}{3}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right)$$

with a, b, c the scaling factors in (2.3).

In the case of the $SU(2)$ Bianchi IX models, the anisotropic Hubble parameter is again of the form $H = \frac{1}{3}(H_1 + H_2 + H_3)$, where now the H_i correspond to the three directions of the vectors dual to the $SU(2)$-forms σ_i in (2.2). For a metric of the form (2.3), or equivalently

$$g = uw\, d\mu^2 + u^2\lambda^2\, \sigma_1^2 + u^2\sigma_2^2 + u^2\sigma_3^2,$$

we take the anisotropic Hubble parameter to be

$$H = \frac{1}{3}\left(\frac{\dot{u}\lambda + u\dot{\lambda}}{u\lambda} + 2\frac{\dot{u}}{u}\right) = \frac{1}{3}\left(3\frac{\dot{u}}{u} + \frac{\dot{\lambda}}{\lambda}\right),$$

where

$$\frac{\dot{u}}{u} = \frac{1}{2}\frac{\dot{w}_1}{w_1}, \qquad \frac{\dot{\lambda}}{\lambda} = \frac{\dot{w}}{w} - \frac{\dot{w}_1}{w_1},$$

so that

$$H = \frac{1}{3}\left(\frac{\dot{w}}{w} + \frac{1}{2}\frac{\dot{w}_1}{w_1}\right),$$

so that we can write the 4–dimensional Dirac operator in the form

$$D = \gamma^0 \frac{1}{uw}\left(\frac{\partial}{\partial\mu} + \frac{3}{2}H\right) + D_B,$$

where $D_B = (w_1^{1/2}/w)\ D_B|_{\lambda = \frac{w}{w_1}}$ is the Dirac operator on the spatial sections S^3 with the left $SU(2)$-invariant metric.

Notice that in the construction above, we have considered the same modulus θ for the noncommutative deformation of all the spatial sections S^3 of the Bianchi IX spacetime, but one could also consider a more general situation where the parameter θ of the deformation is itself a function of the cosmological time μ.

This would allow the dependence of the noncommutativity parameter θ on the energy scale (or on the cosmological timeline), with $\theta = 0$ away from the singularity where classical gravity dominates and noncommutativity only appearing near the singularity. Since a non–constant, continuously varying parameter θ crosses rational and irrational values, this would give rise to a Hofstadter butterfly type picture, with both commutativity (up to Morita equivalence, as in the rational noncommutative tori) and true noncommutativity (irrational noncommutative tori), cf. also [MaMar08].

Another interesting aspect of these noncommutative deformations is the fact that, when we consider a geodesic in the upper half plane encoding Kasner eras in a mixmaster dynamics, the points along the geodesic also determine a family of complex structures on the noncommutative tori T^2_θ of the theta–deformation of the respective spatial section.

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