

## THREE MOST BEAUTIFUL MATHEMATICAL FORMULAS

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**Abstract.** Mathematical formulae are special symbols backed by great Ideas and Forces. The perception of Beauty hidden in mathematical symbols encouraged a harmonization of our world.

“Beauty will save the World”  
*F. Dostojevsky*

Probably is mathematics among all sciences the most close to a discovery of harmony (as music among the arts). I often think about Leonard Euler’s amazing creative life. He was not only a mathematical genius but also sacrificed his life in the name of human evolution. For many years Euler had been working on the theory of music and when he realized that his contemporaries would not accept it (as containing a lot of mathematics), he formulated a thesis that human’s ear oversimplifies the perceiving of harmonies and started working out exercises for the perfection of ear. Euler introduced the term “*Gradus venustatis*” — “Grade of beauty” into the theory of music, which, he believed, was part of mathematics.

People who want to understand the laws that govern the world, take the way of finding World’s Harmony. This way recedes into eternity (for movement is endless) but people, nevertheless, go this way because there is a special pleasure to meet the next idea or a concept.

In the Spring 2002 I have sent the following letter to a number of mathematicians:

“...I think that artists and mathematicians do the same things. They try to discern and explore the World’s Harmony. In my opinion Harmony and Beauty are synonyms.

I hope you might be willing to answer my question: What three mathematical formulas are the most beautiful to you? Perhaps it will be possible to synthesize a new perspective on the meaning of Beauty. In addition this work could be useful for future scientists as a set of Thoughts about the World’s Harmony and Mathematics as a tool of discovering this Beauty.”

Not all of them answered me and from those who answered not everyone sent me formulae. One of the famous mathematicians replied that he does not understand what beautiful mathematical formulae mean, for him there are interesting questions in mathematics and he is working in order to satisfy his

curiosity. I think that this mathematician is on the same way of understanding World's Harmony but he has his own terminology. Another famous mathematician wrote me that beauty in mathematics is a spiritual substance and it cannot be translated into human languages. One could agree with him, but we are not deities but human beings and have to communicate by means of human languages.

In 2002, Stephen Wolfram published his book "A New Kind of Science", where he presented new possibilities in mathematics, in terms of development of computer science. The author presented new features of mathematics, designed to study complex systems, — mathematics should apply experiments, end the numbers obtained from experiments. A number of mathematicians, who studied complex systems in biology, ecology, sociology, . . . argued that the 21st century would focus on beauty of mathematical structures and models than on beautiful formulas, — the new mathematics seeks synthesis, general laws and patterns for differen sciences.

Below, I am providing some responses and comments I have received over the past 12 years.

**Sergio Albeverio (Bonn University, Germany)**

$$e^{i\pi} = -1$$

$$V - E + F = \chi$$

$$dF = 0, \quad -\delta F = J$$

**Sir Michael Atiyah (Edinburgh University, UK)**

$$e^{i\pi} = -1$$

$$V - E + F = 2 \quad (\text{Euler formula for triangulation of polyhedron})$$

$$x^2 + y^2 = z^2 \quad (\text{Pythagoras})$$

*Simplicity + Depth = Beauty.*

I once explained in a general talk that formula  $e^{i\pi} = -1$  was the equivalent of the Hamlet line “To be or not to be” in literature, combining depth with brevity.

**Philippe Blanchard (Bielefeld University, Germany)**

$$3^2 + 4^2 = 5^2$$

$$e^{i\pi} = -1$$

$$i\hbar \frac{\partial \varphi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \varphi + V \varphi \quad (\text{Schrödinger equation})$$

**Ola Bratteli (Oslo University, Norway)**

$$\frac{d^n}{dz^n} f(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} dw \quad (\text{Cauchy's integral formula})$$

$$\int f d\mu = \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n f \circ T^k \quad (\text{Birkhoff ergodic theorem})$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots \quad (\text{Leibniz series})$$

Gianfausto Dell'Antonio (University "La Sapienza", Rome, Italy)

$$\begin{aligned} \partial^\mu F_{\mu\nu} &= 0 \\ \partial^\mu F_{\mu\nu}^* &= \frac{e}{c} j_\nu \end{aligned} \quad (\text{Maxwell equations})$$

$$\delta A(\gamma) = 0, \quad A(\gamma) = \int_\gamma \sum_k p_k dq_k - H dt$$

$$i\hbar \frac{\partial \varphi}{\partial t} = (i\hbar \nabla + eA)^2 \varphi + V\varphi \quad (\text{Schrödinger equation})$$

Comment by N.K.: Understanding of the World's Harmony is removing the boundaries between the sciences. Nowadays, it is difficult to define where the physics ends and the mathematics begins. Of course, in series of works on history and philosophy of science, one can find a proposal to consider the Euclid's geometry as the first physical theory. But I have in mind the development of the science during the last three centuries, when physics was mainly based on experiment.

The evolution goes the way of the Beauty. An excellent example is given by the development of the form of Maxwell equations in time:

Original form

$$\begin{array}{ll} \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 & \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \rho \\ \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\dot{B}_x & \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = j_x + \dot{E}_x \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\dot{B}_y & \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = j_y + \dot{E}_y \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\dot{B}_z & \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = j_z + \dot{E}_z \end{array}$$

End of XIX century

$$\begin{array}{ll} \text{div } B = 0 & \text{div } E = \rho \\ \text{rot } E = -\dot{B} & \text{rot } B = j + \dot{E} \end{array}$$

Beginning of XX century

$$*P_{,\alpha}^{\beta\alpha} = 0 \quad P_{,\alpha}^{\beta\alpha} = j^\beta$$

Modern form

$$dF = 0 \quad -\delta F = J$$

**Yuri Drozd (Kiev National University, Ukraine)**

$$\frac{d}{dx} \int_a^x f(t) dt = f(x) \quad (\text{Barrow-Newton-Leibnitz formula})$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$V - E + F = 2$$

**Gregory Galperin (Eastern Illinois University, USA)**

Three most beautiful ideas:

*Poincaré Recurrence Theorem:* Each point in a bounded dynamical system returns to its almost initial position.

*Geometrization of Motion:* The notion of the configuration space and the phase space (with the increasing the dimension of space)

*Cantor's diagonal:* The number of points on a segment is strictly greater than the number of all natural numbers:

$$c = |\mathbb{R}| > |\mathbb{N}| = \chi_0$$

Comment by N.K.: In the end of XIX century, Henri Poincare has auspicated the modern theory of dynamic systems. Last decades, ideas of geometrization have played a special role in mathematics and physics, and in the study of dynamical systems in particular.

The fractal geometry confirmed a philosophical thesis: “Beauty of Universe is presented in diversity of its unity” and came up to secrets of other great opposites: the finite and the infinite, the order and the disorder. What is born on the border of the order and the disorder? Chaos or self-organizing order of higher level? Maybe, it is at the border of the order and the disorder where our sense of Beauty is born. And the evolution is resolution of more and more global opposites and creation of more and more refined and inspired forms. The way of evolution is the way of Beauty.

Here I would like to cite three quotations from the speech of **Israel Gelfand** (Rutgers University, USA) at the social function at Royal East Research on September 3, 2003 :

1. “In my opinion mathematics is the part of culture as music, poetry and philosophy. I said about this during my lecture at the Conference. There

I mentioned about connection between mathematics, classical music and poetry styles. I was happy when found following four common features: these are — beauty, simplicity, accuracy and crazy ideas. In these four substances — beauty, simplicity, accuracy and crazy ideas combination is the heart of mathematics and classical music. Classical music is not only music of Mozart, Bach or Beethoven. It is also the music of Shostakovich and Schnittke. This is classical music. And I think that all of these four features live in it together. For this reason, as I tried to explain in my lecture, this does not mean that mathematics and classical music are the same thing. They are similar by style of their philosophical arrangement. There is one more similar feature between mathematics and classical music, poetry etc. They all are languages which help us to understand many things”.

2. “I know why Greek philosophers learned geometry! They were philosophers. They learned geometry as philosophy. Great geometricians followed and are following the same tradition — to get over the gap between visible and substance”.

3. “I am sure that in 10-15 years mathematics will be totally different from the one of previous times”.

**Gerald Goldin (Rutgers University, USA)**

$$c^2 = a^2 + b^2 \quad (\text{Pythagoras})$$

$$e^{i\pi} + 1 = 0 \quad (\text{Euler})$$

$$\frac{d^n f}{dz^n}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad (\text{Cauchy})$$

**Alexander Kirillov (University of Pennsylvania, USA)**

$$\int_{\partial M} \omega = \int_M d\omega \quad (\text{Stokes})$$

$$\text{vol}(B_n) = \frac{\pi^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!} \quad (\text{unit ball volume in } \mathbb{R}^n)$$

$$\chi_{\Omega}(\exp X) = \int_{\Omega} e^{i\langle F, X \rangle + \sigma(F)}$$

(the integral formula for a character of an irreducible representation of a Lie group corresponding to the co-adjoint orbit  $\Omega$ ).

**Vladimir Korolyuk (Institute of Mathematics, Kiev, Ukraine)**

$$E = mc^2 \quad (\text{a fundamental law of Nature})$$

$$e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!} \quad (\text{beautiful function})$$

$$\lim_{\varepsilon \rightarrow 0} [\varepsilon^{-1}Q + B]^{-1} = \widehat{B} \quad (\text{via singularity to new properties})$$

**Yuri Manin (MPI, Bonn, Germany)**

$$e^{i\pi} = -1$$

*And also two famous Einstein's formulas:  $E = mc^2$  and the equations of general relativity. But they are, rigorously speaking, rather physical than mathematical ones.*

**Gregory Margulis (Yale University, USA)**

$$\sqrt{\pi} \sum_{n=-\infty}^{\infty} e^{-n^2} = \sum_{n=-\infty}^{\infty} e^{-\frac{n^2}{4}}$$

A special case of so-called "Poisson summation".

$$\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2} \frac{q-1}{2}}$$

Quadratic reciprocity law.

Here

$$\left(\frac{p}{q}\right) = 1 \quad \text{if } q \text{ is a square mod } p$$

and

$$\left(\frac{p}{q}\right) = -1 \quad \text{otherwise,}$$

$p$  and  $q$  are primes not equal to 2

**Robert Minlos (IPPI, Moscow, Russia)**

$$\frac{d\mu_\Lambda}{d\mu^0} = \frac{1}{Z_\Lambda} \exp\{-\beta H_\Lambda\} \quad (\text{Gibbs formula})$$

$$G_t(Q_1, Q_2) = \int e^{-\int_0^t V(\omega(\tau))d\tau} dW_{Q_1, Q_2}(\omega) \quad (\text{Feynman-Kac formula})$$

$$n! = n^n e^{-n} \sqrt{2\pi n} (1 + o(1)) \quad (\text{Stirling formula})$$

**Bernt Øksendal (Oslo, Norway)**

The most beautiful mathematical formula for me is the following

$$e^{i\pi} = -1.$$

This is truly remarkable formula: its beauty lies in its combination of simplicity and power of content. I do not sure what formula to put in second place. One candidate is the following:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots + (-1)^n \frac{1}{2n+1} + \dots,$$

which express a mystical connection between  $\pi$  and the odd numbers.

Another candidate is the Euler formula which express the connection between the number of faces, edges and vertices in a triangulation of a surface...

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David Ruelle (IHES, Bures-sur-Yvette, France)

$$3^2 + 4^2 = 5^2$$

$$e^{i\pi} = -1$$

$$\sum_{n=1}^{\infty} n^{-s} = \prod_p \text{prime} (1 - p^{-s})^{-1}$$

**Yakov Sinai (Princeton University, USA)**

Main beauty is related with *formulations* rather than *formulas*. Some beautiful formulations are the following:

**Pythagorean Theorem:** Let's build up squares on the sides of a right triangle. Then the sum of the areas of two small squares equals the area of the large one.

**Gelfand-Naimark realization:** Every abelian  $C^*$ -algebra is isometrically isomorphic to the algebra of complex continuous functions on the Gelfand spectrum of the algebra.

**The second principle of thermodynamics:** The entropy of a closed system increases.

**Lenin's statement:** An electron is as inexhaustible as an atom.

**Anatoli Skorokhod (MSU, USA)**

$$a^2 + b^2 = c^2 \quad (\text{Pythagoras})$$

$$e^{i\varphi} = \cos \varphi + i \sin \varphi$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \quad \text{where}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

Comment by N.K.: The known Pythagorean formula is mentioned repeatedly, which, I believe, reflects an admiration of the time when people had unified imagination of this world. In the times of Pythagoras, “mathematics” had the same meaning as “knowledge” or “science”, whereas “beauty” meant “cosmos” and was considered as a contrast to “chaos”. Now, at the time of deep differentiation in science, we have a great diverse of definitions of the notions “mathematics” and “beauty” and the arising nostalgia of synthesis. The ancient Greeks regarded the Pythagorean formula as the model of universe development: the two opposites are balanced by a third one, generated by them. The Pythagorean mathematical structure can be considered as one of the first mathematical structures known to us. The mathematical world created by Pythagoras has been cosmical. Its beauty has been incomprehensible to the last degree as well as mysterious.

**Gennady Shipov (Russian ANS, Moscow)**

$$0 \equiv 0 \quad ((1))$$

$$de - e \wedge T = 0 \quad ((2))$$

$$R + dT - T \wedge T = 0 \quad ((3))$$

Equations (2) and (3) are so-called “Cartan structural equations”, where  $R$  is the Riemann tensor. “Nothing happens in the world except for change of curvature and torsion of the space” (due to W. K. Clifford).

Comment by G. Goldin:

$$0 = |\emptyset|$$

Zero (“0”) is the cardinality of the empty set (“ $\emptyset$ ”). It was not easy to understand that the empty set (void, nothing) could be seen as a set, actually existing, its cardinality is a “real” number.

Comment by N.K.: One mathematician sent me a very beautiful formula. Any mathematical formula can be written in this manner. It makes clear that everything stems from the Vacuum-Zero. This formula is:  $0 = \text{Something}$ .

Musical formula by Alfred Shnitke “Voice of Silence” is: long pause and very loud (fff) at one time. This musical formula for me is analogous to  $0 = 0$ .

**Fridrich W. HEHL (Koeln University, Germany)**

In my opinion, very beautiful ones are:

- the Maxwell equations
- the Dirac equations (it is well known fact that the sense of beauty enabled Paul Dirac to foresee the equation of electron).
- the Yang-Mills equations

**Friedrich Hirzebruch (MPI, Bonn, Germany)**

$$e - k + f = 2 \quad (\text{Eulerscher Polyedersatz})$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \quad (\text{Gauß})$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

(*Leibnizsche Reihe: “Gott freut sich der ungeraden Zahlen”.*)

Several people like  $e^{i\pi} + 1 = 0$  because it includes 0, 1, e,  $\pi$ , i all together.

Ludwig Streit (Bielefeld University, Germany)

$$\pi = 3, 141592653589793238462643383279502884197169399375 \dots$$

$$\Theta' = \delta$$

$$(S) \subset (L^2) \subset (S)'$$

Comment by N.K.: There is another number that defines harmony and beauty of forms in our world. It is the golden section  $\Phi$ .  $\Phi = 1, 619339 \dots$  Each formula mentioned here is a reflection of a beautiful Idea. There is no sense in comparing them. One can only enjoy them.

24.03.2004

*P.S.* Lately, many people have appealed to the question about the beauty of mathematical formulas. I would like to thank Leonid Pastur (Paris/Kharkiv) for turning my attention to the book “It must be Beautiful: Great Equations of Modern Science” edited by Graham Farmelo, Granta Publications, London 2002 and Boris Rozovski (Los Angeles) for given information about the article of Kenneth Chang “What Makes an Equation Beautiful” in *The New York Times* on October 24, 2004. In this paper we read:

“Readers of Physics World magazine recently were asked an interesting question: Which equations are the greatest? ... The top vote-getters in the magazine poll were Maxwell’s equations and Euler’s equation  $e^{i\pi} + 1 = 0$ , a purely mathematical construct that finds wide use in theoretical physics”. Some readers of Physics World magazine chose the equation  $1 + 1 = 2$ .

One of my explanations of the growing number of publications about the beauty of mathematical language is that ideas about the Harmony of the Universe oppose (neutralize on the level of mentality) chaos and destructive tendencies which become usual now in our life.

B. Pascal wrote, that in order to understand how a work should be started it is necessary to finish it. The epigraph to the present work maintains “Beauty will save the world”. In response to my question “Will Beauty save the world?”, Errico Pressuti (Rome) answered with a sharp counter-question: “Will the beautiful formula  $e = mc^2$  destroy the world?”

I think the statement “Beauty will save the world” is not exact. It is better to say that “Understanding of and aspiration for Beauty will save the world”, because our moral and esthetics do not allow us to use the Beauty for destructive purposes. The beauty, morals and scientific knowledge are interconnected. On the one hand, “our moral propensity, our aesthetic sense bring their contribution, assisting our power of apprehension to come to its highest achievements” (A. Einstein), on the other hand, “science can turn out to be the assistant of morals” (H. Poincare). For the assertion at the outset of the paper: “Mathematical formulas are special symbols backed by great Ideas and Forces” was found a confirmation. All formulas included in this paper are saying about brilliant ideas. From the Cosmos evolution conception (two contrary origins give birth to the third one, which balances them) to the general principle of relativity. The famous theorem: “The sum of squared cathets (opposition) is equal to the squared hypotenuse”, was considered by Pythagoras and Plato as a mathematical model of the Cosmos evolution concept. Plutarch, in this case, said about the most “beautiful” triangle (3, 4, 5):

$$3^2 + 4^2 = 5^2.$$

At present days, mathematics actively proceeds to translate statements of Philosophy from the level of belief to the concrete knowledge. Sometimes, this way goes through a physical experiment, sometimes through a direct mathematical enlightens.

And here is quotation from the paper “Deterministic and Stochastic Hydrodynamic Equations Arising From Simple Microscopic Model Systems” (by G. Giacomin, J. Lebowitz and E. Presutti): “In fact, one of the basic dogmas of science is that the behavior at any level can be deduced, at least in principle, entirely from the dynamics of the level below it, i.e., there are no new physical laws, only new phenomena, as one goes from atoms to fluids to galaxies”.

*P.P.S.* (After 3 years later of my question addressed to him) **David Elworthy (Warwick University, UK)** sent following comments.

THREE LOVELY FORMULAE

- (1) The most obvious one:

$$(-1) \cdot e^{i\pi} = 1.$$

- (2) One which is less well known:

$$\langle B, B \rangle_t = t,$$

that is the quadratic variation of a one dimensional Brownian motion, up to time  $t$  is  $t$ . A less formal way to write it is as the Ito multiplication rule:

$$dB \cdot dB = dt.$$

This is basic to both Feynman's path integral approach to quantum mechanics and to stochastic analysis. In both directions it shows that there are whole worlds beyond the reach of the standard forms of Newton's mechanics.

- (3) The Poisson summation formula is difficult to beat, and I especially like the version in terms of the Fourier transform of trains of delta-functions:

$$\widehat{\left( \sum_{n=-\infty}^{\infty} \delta_{n\Delta} \right)} = \frac{\sqrt{2\pi}}{\Delta} \sum_{n=-\infty}^{\infty} \delta_{\frac{2n\pi}{\Delta}}.$$

Besides the Poisson summation formula it gives an explanation for why cart wheels in old films appear to go backwards. It is also very satisfying that such expressions as these can be made sense of mathematically.

**Anatoly Vershik** (St. Petersburg Branch of the Steklov Mathematics Institute) (also 3 years later):

The beauty of the mathematical formulas has one peculiarity: beautiful formula must have an element of unexpectedness. Left and right sides must be different, unlike but the beauty is that both of these dissimilarities are the same (in varying degree). The unexpectedness principle must be combine with simplicity and chariness. All this taken together, at my opinion, may serve as formulas beauty criteria.

Of course, the equation known by everybody as the Euler formula is the example of this. It is already more difficult to speak about beauty of mathematical theories and statements, the previous definition of beauty is still present, but there are also many other things. In this respect, the question about the beautiful formula is good, there is concreteness in it. And it is so difficult to argue about the beauty of theories as about the beauty in general.

My first example is following formula (also due to Euler):

$$\prod_{k=0}^{\infty} (1 + x^k) = \sum_{n=0}^{\infty} p(n)x^n,$$

where  $p(n)$  denotes the number of all possible representations of  $n$  as a sum of natural numbers. On the left hand side we have a complicated infinite product and on the right hand side appears a power series with a simple interpretation of the coefficients.

One more example (ergodic theorem):

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n \{f((x + k\sqrt{2}) \bmod 1)\} = \int_0^1 f(x)dx,$$

where  $f$  is any continuous function on  $[0, 1]$ .

My third example is not a formula but an inequality (Cauchy):

$$\frac{1}{n} \sum_{i=1}^n a_i \geq \left( \prod_{i=1}^n a_i \right)^{\frac{1}{n}},$$

$a_i > 0$ . Again, it is a relation between a product and a sum but so simple.

Concerning your epigraph to the paper: I think that to day the beauty is to be saved from the “world”.

Comment by N.K.: Certainly, Beauty and Unexpectedness are correlated. Habitual is not the quality of Beauty. We feel the beauty of the famous Euler formula when we suddenly understand the unity of finiteness and infinite. 1 (unity) is fraught with infinity of spatial principle as well as infinite continued fractions and series organize finite forms and worlds. I think that the science will assert that all oppositions, antagonisms and animosities can be covered by more lofty understanding, since the universe is the unity in its variety.

Will mathematicians study mathematics in XXI century as a part of the World’s Unity philosophy?

**Victor Maslov** (Moscow University, Russia)

Concerning mathematics and philosophy of XXI century, — yes, mathematics in many respects will define philosophy of new time. I shall try to prove that. In my opinion, a philosophical postulate is not the equality or inequality of separate essences, and an equivalence of all possible variants of mappings of one set onto another, for example, all variants of mapping of a set of people onto a set of cities, villages, etc., on a set of surnames, on a set of capitals; mapping of set of particles onto set of energy levels, a set of animals onto a set of species, etc.

As I have found out, the early noticed laws (Tsipf, Pareto, etc.) are laws of the number theory and set theory. The follow from the equivalence of all variant of mappings of these sets. And this is a deep philosophical presumption.

Equality of all variants does cause absolutely precise laws of distribution which are realized in the nature and society.

The duality “randomness—determinancy” which is solved mathematically, also testifies to the philosophy of mathematics. The randomness, according to Kolmogorov, is a very high complexity.

**Anatoly Vershik:** I have an addition to the Kolmogorov’s citation about the randomness (“the randomness is the complexity”), namely, “the randomness is the universality”, this is a mathematical fact too.

**David Ruelle:** I think that mathematics will continue to push both towards unity and no-unity, order and chaos, expected and unexpected, obvious and unobvious. Here is a quotation of A. Grothendieck who says he likes the obvious: “my life’s ambition as a mathematician, or rather my joy and passion, have constantly been to discover obvious things...”.

**Joel Lebowitz** (Rutgers University, USA): I am afraid I don’t have any original ideas. I would certainly include the first formula of Sergio Albeverio and Michael Atiyah among the most beautiful formulas. I also like Sinai’s statement about the second law of thermodynamics except that I would write it “The entropy of a closed macroscopic system never decreases”. Finally I would include the Schrodinger equation for a hydrogen atom. This is the same as the last formula of Philippe Blanchard but with

$$V(r) = -\frac{e^2}{r}.$$

I would also mention Euclid’s proof that there are an infinite number of primes as something very beautiful. Also the proof that the square root of 2 cannot be written as a quotient of two integers.

Comment by N.K.: Professor Lebowitz mentioned also Wigner’s remarkable citation about “unreasonable effectiveness of mathematics in the natural sciences”.

And there I would like to cite a sentence from the mentioned above speech of Israel Gelfand:

“Mathematics is a language. Mathematics is an adequate language in many spheres such as physics, engineering, biology. This is a very important notion – an adequate language... The language of mathematics allows us to organize a lot of things.”

Maxwell has combined mathematics and technique that has lead to the breakthrough in science and changed our understanding of the universe. The marriage of physics and mathematics in the 20 century resulted in the deeper understanding of the relationships that lay in the foundations of the nature of

the world. Today mathematics forms unions with biology, genetics, sociology, psychology etc. This can lead to the General Theory of Life. The study of the consciousness of matter will play an important role there. The notion of consciousness is discussed more and more by the mathematicians. Usually consciousness is linked to the functioning of brain than to the functioning of the heart. Intuition is related to the heart and does not depend on the logic and mind. The heart sounds as the Law of Equilibrium. Mathematical formulae express always one or another side of this Law ( $=$ ). The heart understands what is right (what is harmony) and what is wrong. The Beauty is the harmony (equilibrium) of mind and heart, thoughts and feelings. May be that is what F.Dostojevsky had in mind when he stated that "Beauty will save the World". This thought made profound impact on Albert Einstein who said that Dostojevsky gave him more than Gauss.

In 2007, David Ruelle published his book "The Mathematician's Brain" where he focused on the beauty of mathematics; I am providing some quotes below: "I hope I have convinced you that the love of mathematical beauty is an essential reason why mathematicians do and teach mathematics. But can one say what makes mathematics beautiful? Let me propose one answer to this question: I think that the beauty of mathematics lies in uncovering the hidden simplicity and complexity that coexist in the rigid logical framework that the subject imposes".

Back in the mid-20th century, Eugene Wigner in his work "Remarks on the mind-body question" demonstrated that human consciousness reflects the ability to acquire information, process it and initiate a response in the form of sensations, feelings, thoughts, actions. In case we have the developed consciousness (Superconsciousness), which can contact Plato's world of Ideas, but our brain is incapable of turning perceived information to concrete knowledge (specific thought, formula), then we intuitively feel experience aesthetic pleasure and impression of something unsolved and mysterious. Mysteries are always magnetic. This magnetism is the love of truth, which Apostle Paul refers to in his First Epistle to the Corinthians: "...and tongues will fall silent and knowledge will be abolished. We know just a portion (of the Truth), and in part we prophesy and guess. When what is perfect comes, then what is partial will cease. ... Now we see through a glass dimly, but then face to face". Mathematics is a symbolic language. Development of mathematics is characterized by the growth of people's consciousness. Perhaps symbols will no longer exist, if humanity becomes aware of their common existence and the unity of all things. Mathematics will fulfill its mission. Whilst dissolving in a variety of sciences, and becoming the science itself, it will find the most beautiful and single formula in existence. However, there is a long way to go.

Sir Michael Atiyah, is seeking to confirm his belief that mathematics and art have common roots — Laws or Harmonies of the Universe. Michael Atiyah continues to seek confirmation of his belief that the brain of mathematician can generate the same impulses, - either they perceive a beautiful mathematical formula or they listen to the best of Mozart or Shakespeare, for example. In 2013 he published work "The experience of mathematical beauty and its neural correlates" with Semir Zeki, John Romaya and Dionigi Benincasa.

New in science and art always has a resistance the inertia of the old thinking, perception. Israel Gelfand (in 2003): "Now it is time for a radical

perestroika of the fundamental language of mathematics. During this time, it is especially important to remember the unity of mathematics, to remember its beauty, simplicity and crazy ideas. It is very useful for me to remind myself than when the style of music changed in the 20th century many people said that the modern music lacked harmony, did not follow standard rules, had dissonances, and so on. However, Schoenberg, Stravinsky, Shostakovich and Schnittke were as exact in their music as Bach, Mozart and Beethoven”.

In early 2013, I asked Anatoly Vershik:

— Do you think that mathematical formulas will make way for mathematical models?

Anatoly Vershik:

— Hardly. I think that equations will make way for models. This process started a long time ago, as some scientists might have regretted. The language of equations is no longer enough to describe complex phenomena. A formula is a rare thing, especially when it is beautiful; it is and always be a piece of artwork.

To be continued.