Interdisciplinary Studies of Complex Systems
No. 5 (2014) 5–47

Markov birth-and-death dynamics of populations

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Abstract. Spatial birth-and-death processes are obtained as solutions of a stochastic equation. The processes are required to be finite. Conditions are given for existence and uniqueness of such solutions, as well as for continuous dependence on the initial conditions. The possibility of an explosion and connection with the heuristic generator of the process are discussed.
AMS Subject Classification: Primary: 60K35, 60G55; Secondary: 60H20, 60J25, 82C21.

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Introduction

This article deals with spatial birth-and-death processes which may describe stochastic dynamics of spatial population. Specifically, at each moment of time the population is represented as a collection of motionless points in $\mathbb{R}^d$. We interpret the points as particles, or individuals. Existing particles may die and new particles may appear. Each particle is characterized by its location.

The state space of a spatial birth-and-death Markov process on $\mathbb{R}^d$ with finite number of points is the space of finite configurations over $\mathbb{R}^d$,

$$\Gamma_0(\mathbb{R}^d) = \{\eta \subset \mathbb{R}^d : |\eta| < \infty\},$$

where $|\eta|$ is the number of points of η.

Denote by $\mathscr{B}(\mathbb{R}^d)$ the Borel σ-algebra on $\mathbb{R}^d$. The evolution of a spatial birth-and-death process in $\mathbb{R}^d$ admits the following description. Two functions characterize the development in time, the birth rate coefficient $b : \mathbb{R}^d \times \Gamma_0(\mathbb{R}^d) \to [0;\infty)$ and the death rate coefficient $d : \mathbb{R}^d \times \Gamma_0(\mathbb{R}^d) \to [0;\infty)$. If the system is in state $\eta \in \Gamma_0(\mathbb{R}^d)$ at time t, then the probability that a new particle appears (a "birth") in a bounded set $B \in \mathscr{B}(\mathbb{R}^d)$ over time interval $[t; t + \Delta t]$ is

$$\Delta t \int\limits_B b(x, \eta) dx + o(\Delta t),$$

the probability that a particle $x \in \eta$ is deleted from the configuration (a "death") over time interval $[t; t + \Delta t]$ is

$$d(x, \eta)\Delta t + o(\Delta t),$$

and no two events happen simultaneously. By an event we mean a birth or a death. Using a slightly different terminology, we can say that the rate at which a birth occurs in B is $\int_B b(x, \eta)dx$, the rate at which a particle $x \in \eta$ dies is $d(x, \eta)$, and no two events happen at the same time.

Such processes, in which the birth and death rates depend on the spatial structure of the system as opposed to classical $\mathbb{Z}_+$-valued birth-and-death processes (see e.g. [22], [5], [18, Page 116], [3, Page 109], and references therein), were first studied by Preston in [36]. A heuristic description similar to that above appeared already there. Our description resembles the one in [14].

The (heuristic) generator of a spatial birth-and-death process should be of the form

$$LF(\eta) = \int\limits_{x \in \mathbb{R}^d} b(x, \eta)[F(\eta \cup x) - F(\eta)]dx + \sum_{x \in \eta} d(x, \eta)(F(\eta \setminus x) - F(\eta)), \quad (0.1)$$

for F in an appropriate domain, where $\eta \cup x$ and $\eta \setminus x$ are shorthands for $\eta \cup \{x\}$ and $\eta \setminus \{x\}$, respectively.

Spatial point processes have been used in statistics for simulation purposes, see e.g. [32], [33, chapter 11] and references therein. For application of spatial and stochastic models in biology see e.g. [29], [9], and references therein.

To construct a spatial birth-and-death process with given birth and death rate coefficients, we consider in Section 2 stochastic equations with Poisson type noise

$$\eta_t(B) = \int\limits_{B\times(0;t]\times[0;\infty]} I_{[0;b(x,\eta_{s-})]}(u)dN_1(x,s,u) - \int\limits_{\mathbb{Z}\times(0;t]\times[0;\infty)} I_{\{x_i\in\eta_{r-}\cap B\}} I_{[0;d(x_i,\eta_{r-})]}(v)dN_2(i,r,v) \quad (0.2)$$

where $(\eta_t)_{t\geq 0}$ is a suitable $\Gamma_0(\mathbb{R}^d)$-valued cadlag stochastic process, the "solution" of the equation, I_A is the indicator function of the set A, $B \in \mathscr{B}(\mathbb{R}^d)$ is a Borel set, N_1 is a Poisson point processes on $\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dx \times ds \times du$, N_2 is a Poisson point process on $\mathbb{Z} \times \mathbb{R}_+ \times \mathbb{R}_+$ with intensity $\# \times dr \times dv$, $\#$ is the counting measure on $\mathbb{Z}^d$, η_0 is a (random) initial finite configuration, $b, d : \mathbb{R}^d \times \Gamma_0(\mathbb{R}^d) \to [0;\infty)$ are functions that are measurable with respect to the product σ-algebra $\mathscr{B}(\mathbb{R}) \times \mathscr{B}(\Gamma_0(\mathbb{R}))$ and $\{x_i\}$ is some collection of points satisfying $\eta_s \subset \{x_i\}$ for every moment of time s (the precise definition is given in Section 1.3.1). We require the processes N_1, N_2, η_0 to be independent of each other. Equation (0.2) is understood in the sense that the equality holds a.s. for all bounded $B \in \mathscr{B}(\mathbb{R}^d)$ and $t \geq 0$.

Garcia and Kurtz studied in [14] equations similar to (0.2) for infinite systems. In the earlier work [13] of Garcia another approach was used: birth-and-death processes were obtained as projections of Poisson point processes. A further development of the projection method appears in [15]. Xin [40] formulates and proves functional central limit theorem for spatial birth-and-death processes constructed in [14]. Fournier and Meleard in [11] considered a similar equation for the construction of the Bolker-Pacala-Dieckmann-Law process with finitely many particles.

Holley and Stroock [19] constructed a spatial birth-and-death process as a Markov family of unique solutions to the corresponding martingale problem. For the most part, they consider a process contained in a bounded volume, with bounded birth and death rate coefficients. They also proved the corresponding result for the nearest neighbor model in $\mathbb{R}^1$ with an infinite number of particles.

Kondratiev and Skorokhod [25] constructed a contact process in continuum, with the infinite number of particles. The contact process can be described as a spatial birth-and-death process with

$$b(x,\eta) = \lambda \sum_{y\in\eta} a(x-y), \quad d(x,\eta) \equiv 1,$$

where $\lambda > 0$ and $0 \leq a \in L^1(\mathbb{R}^d)$. Under some additional assumptions, they showed existence of the process for a broad class of initial conditions. Furthermore, if the value of some energy functional on the initial condition is finite, then it stays finite at any point in time.

In the aforementioned references as well as in the present work the evolution of the system in time via Markov process is described. An alternative approach consists in using the concept of statistical dynamics that substitutes the notion of a Markov stochastic process. This approach is based on considering evolutions of measures and their correlation functions. For details see e.g. [7], [8], and references therein.

There is an enormous amount of literature concerning interacting particle systems on lattices and related topics (e.g., [30], [31], [24], [1], [12], [39], etc.) Penrose in [34] gives a general existence result for interacting particle systems on a lattice with local interactions and bounded jump rates (see also [30, Chapter 9]). The spin space is allowed to be non-compact, which gives the opportunity to incorporate spatial birth-and-death processes in continuum. Unfortunately, the assumptions become rather restrictive when applied to continuous space models. More specifically, the birth rate coefficient should be bounded, and for every bounded Borel set B the expression

$$\sum_{x \in \eta \cap B} d(x, \eta)$$

should be bounded uniformly in η, $\eta \in \Gamma(\mathbb{R}^d)$.

Let us briefly describe the contents of the article.

In Section 1 we introduce give some general notions, definitions and results related to Markov processes in configuration spaces. We start with configuration spaces, which are the state spaces for birth-and-death processes, then we introduce and discuss metrical and topological structures thereof. Also, we present some facts and constructions from probability theory, such as integration with respect to a Poisson point process, or a sufficient condition for a functional transformation of a Markov chain to be a Markov chain again.

In the second section we construct a spatial birth-and-death process $(\eta_t)_{t\geq 0}$ as a unique solution to equation (0.2). We prove strong existence and pathwise uniqueness for (0.2). A key condition is that we require b to grow not faster than linearly in the sense that

$$\int_{\mathbb{R}^d} b(x, \eta) dx \leq c_1 |\eta| + c_2. \tag{0.3}$$

The equation is solved pathwisely, "from one jump to another". Also, we prove uniqueness in law for equation (0.2) and the Markov property for the unique solution. Considering (0.2) with a (non-random) initial condition $\alpha \in \Gamma_0(\mathbb{R}^d)$ and denoting corresponding solution by $(\eta(\alpha, t))_{t\geq 0}$, we see that a unique solution induces a Markov family of probability measures on the Skorokhod space $D_{\Gamma_0(\mathbb{R}^d)}[0; \infty)$ (which can be regarded as the canonical space for a solution of (0.2)).

When the birth and death rate coefficients b and d satisfy some continuity assumptions, the solution is expected to have continuous dependence on the initial condition, at least in some proper sense. Realization of this idea and precise formulations are given in Section 2.1. The proof is based on considering a coupling of two birth-and-death processes.

The formal relation of a unique solution to (0.2) and operator L in (0.1) is given via the martingale problem, in Section 2.2, and via some kind of a pointwise convergence, in Section 2.5.

In Section 2.4 we formulate and prove a theorem about coupling of two birth-and-death processes. The idea to compare a spatial birth-and-death process with some "simpler" process goes back to Preston, [36]. In [11] this technique was applied to the study of the probability of extinction.

1 Configuration spaces and Markov processes: miscellaneous

In this section we list some notions and facts we use in this work.

1.1 Some notations and conventions

Sometimes we write ∞ and $+\infty$ interchangeably, so that $f \to \infty$ and $f \to +\infty$, or $a < \infty$ and $a < +\infty$ may have the same meaning. However, $+\infty$ is reserved for the real line only, whereas ∞ have wider range of applications, e.g. for a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ we may write $x_n \to \infty$, $n \to \infty$, which is equivalent to $|x_n| \to +\infty$. On the other hand, we do not assign any meaning to $x_n \to +\infty$.

In all probabilistic constructions we work on a probability space $(\Omega, \mathscr{F}, P)$, sometimes equipped with a filtration of σ-algebras. Elements of Ω are usually denoted as ω.

The set A^c is the complement of the set $A \subset \Omega$: $A^c = \Omega \setminus A$. We write $[a; b]$, $[a; b)$ etc. for the intervals of real numbers. For example, $(a; b] = \{x \in \mathbb{R} \mid a < x \le b\}$, $-\infty \le a < b \le +\infty$. The half line $\mathbb{R}_+$ includes 0: $\mathbb{R}_+ = [0; \infty)$.

1.2 Configuration spaces

In this section we introduce notions and facts about spaces of configurations, in particular, topological and metrical structures on $\Gamma(\mathbb{R}^d)$ as well as a characterization of compact sets of $\Gamma(\mathbb{R}^d)$. We discuss configurations over Euclidean spaces only.

Definition 1.1. For $d \in \mathbb{N}$ and a measurable set $\Lambda \subset \mathbb{R}^d$, the configuration space $\Gamma(\Lambda)$ is defined as

$$\Gamma(\Lambda) = \{\gamma \subset \Lambda : |\gamma \cap K| < +\infty \text{ for any compact } K \subset \mathbb{R}^d\}.$$

We recall that $|A|$ denotes the number of elements of A. We also say that $\Gamma(\Lambda)$ is the space of configurations over Λ. Note that $\varnothing \in \Gamma(\Lambda)$.

Let $\mathbb{Z}_+$ be the set $\{0, 1, 2, \dots\}$. We say that a Radon measure μ on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ is a *counting measure* on $\mathbb{R}^d$ if $\mu(A) \in \mathbb{Z}_+$ for all $A \in \mathscr{B}(\mathbb{R}^d)$. When a counting measure ν satisfies additionally $\nu(\{x\}) \le 1$ for all $x \in \mathbb{R}^d$, we call it a *simple counting measure*.

As long as it does not lead to ambiguities, we identify a configuration with a simple counting Radon measures on $\mathbb{R}^d$: as a measure, a configuration $\gamma \in \Gamma(\mathbb{R}^d)$ maps a set $B \in \mathscr{B}$ into $|\gamma \cap B|$. In other words, $\gamma = \sum\limits_{x\in\gamma} \delta_x$.

One equips $\Gamma(\mathbb{R}^d)$ with the vague topology, i.e., the weakest topology such that for all $f \in C_c(\mathbb{R}^d)$ (the set of continuous functions on $\mathbb{R}^d$ with compact support) the map

$$\Gamma(\mathbb{R}^d) \ni \gamma \mapsto \langle \gamma, f \rangle := \sum_{x \in \gamma} f(x) \in \mathbb{R}$$

is continuous.

Equipped with this topology, $\Gamma(\mathbb{R}^d)$ is a Polish space, i.e., there exists a metric on $\Gamma(\mathbb{R}^d)$ compatible with the vague topology and with respect to which $\Gamma(\mathbb{R}^d)$ is a complete separable metric space, see, e.g., [27], and references therein. We say that a metric is compatible with a given topology if the topology induced by the metric coincides with the given topology.

For a bounded $B \subset \mathbb{R}^d$ and $\gamma \in \Gamma(\mathbb{R}^d)$, we denote $\delta(\gamma, B) = min\{|x - y| : x, y \in \gamma \cap B, x \neq y\}$. Let $B_r(x)$ denote the closed ball in $\mathbb{R}^d$ of the radius r centered at x.

A set is said to be *relatively compact* if its closure is compact. The following theorem gives a characterization of compact sets in $\Gamma(\mathbb{R}^d)$, cf. [27], [19].

Theorem 1.2. *A set $F \subset \Gamma(\mathbb{R}^d)$ is relatively compact in the vague topology if and only if*

$$\sup_{\gamma \in F} \{\gamma(B_n(0)) + \delta^{-1}(\gamma, B_n(0))\} < \infty \tag{1.1}$$

holds for all $n \in \mathbb{N}$.

Proof. Assume that (1.1) is satisfied for some $F \subset \Gamma(\mathbb{R}^d)$. In metric spaces compactness is equivalent to sequential compactness, therefore it is sufficient to show that an arbitrary sequence contains a convergent subsequence in $\Gamma(\mathbb{R}^d)$. To this end, consider an arbitrary sequence $\{\gamma_n\}_{n \in \mathbb{N}} \subset F$. The supremum $\sup_n \gamma_n(B_1(0))$ is finite, consequently, by the Banach–Alaoglu theorem there exists a measure $\alpha_1 \in C(B_1(0))^*$; here $C(B_1(0))^*$ is the dual space of $C(B_1(0))$; and a subsequence $\{\gamma_n^{(1)}\} \subset \{\gamma_n\}$ such that $\gamma_n^{(1)}|_{B_1(0)} \to \alpha_1$ in $C(B_1(0))^*$. Furthermore, one may see that $\alpha_1 \in \Gamma(B_1(0))$ (it is particularly important here that $\sup_{\gamma \in F} \{\delta^{-1}(\gamma, B_1(0))\} < \infty$). Indeed, arguing by contradiction one may get that $\alpha_1(A) \in \mathbb{Z}_+$ for all Borel sets A, and Lemma 1.6 below ensures that α_1 is a simple counting measure.

Similarly, from the sequence $\gamma_n^{(1)}$ we may extract subsequence $\{\gamma_n^{(2)}\} \subset \{\gamma_n^{(1)}\}$ in such a way that $\gamma_n^{(2)}$ converges to some $\alpha_2 \in \Gamma(B_2(0))$. Continuing in the same way, we will find a sequence of sequences $\{\gamma_n^{(m)}\}$ such that $\gamma_n^{(m)} \to \alpha_m \in \Gamma(B_m(0))$ and $\{\gamma_n^{(m+1)}\} \subset \{\gamma_n^{(m)}\}$. Consider now the sequence $\{\gamma_n^{(n)}\}_{n \in \mathbb{N}}$. For any m, restrictions of its elements to $B_m(0)$ converge to α_m in $\Gamma(B_m(0))$, Therefore, $\gamma_n^{(n)} \to \alpha$ in $\Gamma(\mathbb{R}^d)$, where $\alpha = \bigcup_n \alpha_n$.

Conversely, if (1.1) is not fulfilled for some $n_0 \in \mathbb{N}$, then we can construct a sequence $\{\gamma_n\}_{n \in \mathbb{N}} \subset F$ such that either the first summand in (1.1) tends to infinity:

$$\gamma_j(B_{n_0}(0)) \to \infty, \quad j \to \infty$$

in which case, of course, there is no convergent subsequence, or the second summand in (1.1) tends to infinity. In the latter case, a subsequence of the sequence $\{\gamma_n|_{B_{n_0}(0)}\}_{n \in \mathbb{N}}$ may converge to a counting measure (when all γ_n are considered as measures). However, the limit measure can not be a simple counting measure. Thus, the sequence $\{\gamma_n\}_{n\in\mathbb{N}} \subset F$ does not contain a convergent subsequence in $\Gamma(\mathbb{R}^d)$. □

We denote by $CS(\Gamma(\mathbb{R}^d))$ the space of all compact subsets of $\Gamma(\mathbb{R}^d)$.

Proposition 1.3. *The topological space* $\Gamma(\mathbb{R}^d)$ *is not* σ*-compact.*

Proof. Let $\{K_m\}_{n\in\mathbb{N}}$ be an arbitrary sequence from $CS(\Gamma(\mathbb{R}^d))$. We will show that $\bigcup_n K_n \neq \Gamma(\mathbb{R}^d)$. To each compact K_m we may assign a sequence $q_1^{(m)}$, $q_2^{(m)}, \ldots$ of positive numbers such that

$$\sup_{\gamma\in K_m} \{\gamma(B_n(0)) + \delta^{-1}(\gamma, B_n(0))\} < q_n^{(m)}.$$

There exists a configuration whose intersection with $B_n(0)$ contains at least $q_n^{(n)} + 1$ points, for each $n \in \mathbb{N}$. This configuration does not belong to any of the sets $\{K_m\}_{m\in\mathbb{N}}$, hence it can not belong to the union $\bigcup_m K_m$. □

Remark 1.4. Since $\Gamma(\mathbb{R}^d)$ is a separable metrizable space, Proposition 1.3 implies that $\Gamma(\mathbb{R}^d)$ is *not locally compact*.

For another description of all compact sets in $\Gamma(\mathbb{R}^d)$ we will use the set $\Phi \subset C(\mathbb{R}^d)$ of all positive continuous functions ϕ satisfying the following conditions:

1) $\phi(x) = \phi(y)$ whenever $|x| = |y|$, $x, y \in \mathbb{R}^d$,

2) $\lim_{|x|\to\infty} \phi(x) = 0$.

For $\phi \in \Phi$ we denote

$$\Psi = \Psi_\phi(x, y) := \phi(x)\phi(y)\frac{|x-y|+1}{|x-y|} I\{x \neq y\}.$$

Proposition 1.5. *(i) For all* $c > 0$ *and* $\phi \in \Phi$

$$K_c := \left\{\gamma : \iint_{\mathbb{R}^d\times\mathbb{R}^d} \Psi_\phi(x, y)\gamma(dx)\gamma(dy) \leqslant c\right\} \in CS(\Gamma(\mathbb{R}^d));$$

(ii) For all $K \in CS(\Gamma(\mathbb{R}^d))$ *there exist* $\phi \in \Phi$ *such that*

$$\sup_{\gamma\in K} \{\iint_{\mathbb{R}^d\times\mathbb{R}^d} \Psi_\phi(x, y)\gamma(dx)\gamma(dy)\} \leqslant 1.$$

Proof. (i) Denote $\theta_n = \min_{x\in B_n(0)} \phi(x) > 0$. For $\gamma \in K_c$ we have

$$\begin{aligned} c &\geqslant \iint_{B_n(0)\times B_n(0)} \Psi(x, y)\gamma(dx)\gamma(dy) \\ &\geqslant \iint_{B_n(0)\times B_n(0)} \phi(x)\phi(y)I\{x \neq y\}\gamma(dx)\gamma(dy) \geqslant \theta_n^2\gamma(B_n(0))(\gamma(B_n(0)) - 1) \end{aligned}$$

and

$$c \geqslant \iint\limits_{B_n(0)\times B_n(0)} \Psi(x,y)\gamma(dx)\gamma(dy) \geqslant \theta_n^2 \frac{\delta^{-1}(\gamma, B_n(0))+1}{\delta^{-1}(\gamma, B_n(0))} \geqslant \theta_n^2 \delta^{-1}(\gamma, B_n(0)).$$

Consequently,

$$\sup_{\gamma\in K_c} \gamma(B_n(0)) \leqslant \theta_n\sqrt{c}+1,$$

and

$$\sup_{\gamma\in K_c} \delta^{-1}(\gamma, B_n(0)) \leqslant \frac{c}{\theta_n^2}.$$

It remains to show that K_c is closed, in which case Theorem 1.2 will imply compactness of K_c. The space $\Gamma(\mathbb{R}^d)$ is metrizable, therefore sequential closedness will suffice. Take $\gamma_k \in K_c$, $\gamma_k \to \gamma$ in $\Gamma(\mathbb{R}^d)$, $k\to\infty$. For $n\in\mathbb{N}$, let $\Psi_n \in C_c(\mathbb{R}^d\times\mathbb{R}^d)$ be an increasing sequence of functions such that $\Psi_n \leqslant \Psi$, $\Psi_n(x,y)=\Psi(x,y)$ for $x,y\in\mathbb{R}^d$ satisfying $|x|,|y|\leq n$, $|x-y|\geq \dfrac{1}{n}$. For such a sequence we have $\Psi_n(x,y)\uparrow\Psi(x,y)$ for all $x,y\in\mathbb{R}^d$, $x\neq y$. For each $f\in C_c(\mathbb{R}^d\times\mathbb{R}^d)$, the map

$$\eta\mapsto\langle\eta\times\eta, f\rangle := \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} f(x,y)\eta(dx)\eta(dy)$$

is continuous in the vague topology. Thus for all $n\in\mathbb{N}$, $\langle\gamma_k\times\gamma_k,\Psi_n\rangle\to\langle\gamma\times\gamma,\Psi_n\rangle$. Consequently, $\langle\gamma\times\gamma,\Psi_n\rangle\leq c$, $n\in\mathbb{N}$, and by Fatou's Lemma

$$\langle\gamma\times\gamma,\Psi\rangle = \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \Psi(x,y)\gamma(dx)\gamma(dy)$$

$$= \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \liminf_n \Psi_n(x,y)\gamma(dx)\gamma(dy) \leq \liminf_n \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \Psi_n(x,y)\gamma(dx)\gamma(dy)\leq c.$$

To prove (ii), for a given compact set $K\subset\Gamma(\mathbb{R}^d)$ and a given function $\phi\in\Phi$, denote

$$a_n(K) := \sup_{\gamma\in K}\{\gamma(B_n(0))+\delta^{-1}(\gamma,B_n(0))\}$$

and

$$b_n(\phi) := \sup_{|x|>n}|\phi(x)|.$$

Theorem 1.2 implies $a_n(K)<\infty$, and we can estimate

$$\iint\limits_{\left(B_{n+1}(0)\setminus B_n(0)\right)\times\left(B_{n+1}(0)\setminus B_n(0)\right)} \Psi(x,y)\gamma(dx)\gamma(dy)$$

$$= \iint\limits_{\left(B_{n+1}(0)\setminus B_n(0)\right)\times\left(B_{n+1}(0)\setminus B_n(0)\right)} \phi(x)\phi(y)\frac{|x-y|+1}{|x-y|}I\{x\neq y\}\gamma(dx)\gamma(dy)$$

$$\leqslant \iint\limits_{\left(B_{n+1}(0)\setminus B_n(0)\right)\times\left(B_{n+1}(0)\setminus B_n(0)\right)} b_n^2(a_n+1)\gamma(dx)\gamma(dy) \leqslant b_n^2(a_n+1)^3.$$

Taking a function $\phi \in \Phi$ such that

$$3b_n^2(\phi)(a_n+1)^3 < \frac{6}{\pi^2}\frac{1}{(n+1)^2},$$

we get

$$\sup_{\gamma\in K}\left\{\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\Psi(x,y)\gamma(dx)\gamma(dy)\right\}\leqslant 1. \qquad \square$$

1.2.1 The space of finite configurations

For $\Lambda \subset \mathbb{R}^d$, the space $\Gamma_0(\Lambda)$ is defined as

$$\Gamma_0(\Lambda) := \{\eta \subset \Lambda : |\eta| < \infty\}.$$

We see that $\Gamma_0(\Lambda)$ is the collection of all finite subsets of Λ. We denote the space of n-point configurations as $\Gamma_0^{(n)}(\Lambda)$:

$$\Gamma_0^{(n)}(\Lambda) := \{\eta \in \Gamma_0(\Lambda) \mid |\eta| = n\}, \quad n \in \mathbb{N},$$

and $\Gamma_0^{(0)}(\Lambda) := \{\varnothing\}$. Sometimes we will write Γ_0 instead of $\Gamma_0(\mathbb{R}^d)$. Recall that we occasionally write $\eta \setminus x$ instead of $\eta \setminus \{x\}$, $\eta \cup x$ instead of $\eta \cup \{x\}$.

To define a topological structure on $\Gamma_0(\mathbb{R}^d)$, we introduce the following surjections (see, e.g., [26] and references therein)

$$\begin{aligned} sym : \bigsqcup_{n=0}^{\infty}\widetilde{(\mathbb{R}^d)^n} &\to \Gamma_0(\mathbb{R}^d)\\ sym((x_1,\dots,x_n)) &= \{x_1,\dots,x_n\}, \end{aligned} \tag{1.2}$$

where

$$\widetilde{(\mathbb{R}^d)^n} := \{(x_1,\dots,x_n) \in (\mathbb{R}^d)^n \mid x_j \in \mathbb{R}^d, j=1,\dots,n, x_i \neq x_j, i\neq j\}, \tag{1.3}$$

and, by convention, $\widetilde{(\mathbb{R}^d)^0} = \{\varnothing\}$.

The map *sym* produces a one-to-one correspondence between $\Gamma_0^{(n)}(\mathbb{R}^d)$, $n \geq 1$, and the quotient space $\widetilde{(\mathbb{R}^d)^n}/\sim_n$, where $\sim_n$ is the equivalence relation on $(\mathbb{R}^d)^n$,

$$(x_1,\dots,x_n) \sim_n (y_1,\dots,y_n)$$

when there exist a permutation $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$ such that

$$(x_{\sigma(1)},\dots,x_{\sigma(n)}) = (y_1,\dots,y_n).$$

We endow $\Gamma_0^{(n)}(\mathbb{R}^d)$ with the topology induced by this one-to-one correspondence. Equivalently, a set $A \subset \Gamma_0^{(n)}(\mathbb{R}^d)$ is open iff $sym^{-1}(A)$ is open in $\widetilde{(\mathbb{R}^d)^n}$. The space $\widetilde{(\mathbb{R}^d)^n} \subset (\mathbb{R}^d)^n$ we consider, of course, with the relative, or subspace, topology. As far as $\Gamma_0^{(0)}(\mathbb{R}^d) = \{\varnothing\}$ is concerned, we regard it as an open set.

Having defined topological structures on $\Gamma_0^{(n)}(\mathbb{R}^d)$, $n \geq 0$, we endow $\Gamma_0(\mathbb{R}^d)$ with the topology of disjoint union,

$$\Gamma_0(\mathbb{R}^d) = \bigsqcup_{n=0}^{\infty} \Gamma_0^{(n)}(\mathbb{R}^d). \tag{1.4}$$

In this topology, a set $K \subset \Gamma_0(\mathbb{R}^d)$ is compact iff $K \subset \bigsqcup_{n=0}^{N} \Gamma_0^{(n)}(\mathbb{R}^d)$ for some $N \in \mathbb{N}$ and for each $n \leq N$ the set $K \cap \Gamma_0^{(n)}(\mathbb{R}^d)$ is compact in $\Gamma_0^{(n)}(\mathbb{R}^d)$. A set $K_n \subset \Gamma_0^{(n)}(\mathbb{R}^d)$ is compact iff $sym^{-1}(K_n)$ is compact in $\widetilde{(\mathbb{R}^d)^n}$. We note that in order for K_n to be compact, the set $sym^{-1}K_n$, regarded as a subset of $(\mathbb{R}^d)^n$, should not have limit points on the diagonals, i.e. limit points from the set $(\mathbb{R}^d)^n \setminus \widetilde{(\mathbb{R}^d)^n}$.

Let us introduce a metric compatible with the described topology on $\Gamma_0(\mathbb{R}^d)$. We set

$$dist(\zeta,\eta) := \begin{cases} 1 \wedge d_{Eucl}(\zeta,\eta), & |\zeta| = |\eta|, \\ 1, & \text{otherwise.} \end{cases}$$

Here $d_{Eucl}(\zeta,\eta)$ is the metric induced by the Euclidean metric and the map *sym*:

$$d_{Eucl}(\zeta,\eta) = \inf\{|x-y| : x \in sym^{-1}\zeta, y \in sym^{-1}\eta\}, \tag{1.5}$$

where $|x-y|$ is the Euclidean distance between x and y, $sym^{-1}\eta = sym^{-1}(\{\eta\})$. In many aspects, this metric resembles the Wasserstein type distance in [37]. The differences are, *dist* is bounded by 1 and it is defined on $\Gamma_0(\mathbb{R}^d)$ only.

Note that the metric *dist* satisfies equalities

$$dist(\zeta \cup x, \eta \cup x) = dist(\zeta,\eta) \tag{1.6}$$

for $\zeta,\eta \in \Gamma_0(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $x \notin \zeta,\eta$, and

$$dist(\zeta \setminus x, \eta \setminus x) = dist(\zeta,\eta), \tag{1.7}$$

$x \in \zeta,\eta$. We note that the space $\Gamma_0(\mathbb{R}^d)$ equipped with this metric is not complete. Nevertheless, $\Gamma_0(\mathbb{R}^d)$ is a Polish space, i.e., $\Gamma_0(\mathbb{R}^d)$ is separable and there exists a metric $\tilde{\rho}$ which induces the same topology as *dist* does and such that $\Gamma_0(\mathbb{R}^d)$ equipped with $\tilde{\rho}$ is a complete metric space. To prove this, we embed $\Gamma_0^{(n)}(\mathbb{R}^d)$ into the space $\ddot{\Gamma}_0^{(n)}(\mathbb{R}^d)$ of n-point multiple configurations, which we define as the space of all counting measures η on $\mathbb{R}^d$ with $\eta(\mathbb{R}^d) = n$. Abusing notation, we may represent each $\eta \in \ddot{\Gamma}_0^{(n)}(\mathbb{R}^d)$ as a set $\{x_1, \dots, x_n\}$, where some points among $x_j \in \mathbb{R}^d$ may be equal (recall our convention on identifying a configuration with a measure; as a measure, $\eta = \sum\limits_{j=1}^{n} \delta_{x_j}$). One should keep in mind that $\{x_1, \dots, x_n\}$ is not really a set here, since it is possible that $x_i = x_j$ for $i \neq j$, $i, j \in \{1, \dots, n\}$. The representation allows us to extend *sym* to the map

$$\begin{aligned} \overline{sym} : \bigsqcup_{m=0}^{\infty} (\mathbb{R}^d)^n &\to \ddot{\Gamma}_0^{(n)}(\mathbb{R}^d) \\ \overline{sym}((x_1, \dots, x_n)) &:= \{x_1, \dots, x_n\}, \end{aligned} \tag{1.8}$$

and define a metric on $\ddot{\Gamma}_0^{(n)}(\mathbb{R}^d)$: for $\zeta, \eta \in \ddot{\Gamma}_0^{(n)}(\mathbb{R}^d)$ we set $\overline{dist}(\zeta, \eta) = 1 \wedge \overline{d_{Eucl}}(\zeta, \eta)$, $\overline{d_{Eucl}}(\zeta, \eta)$ is the metric induced by the Euclidean metric and the map $\overline{sym}$:

$$\overline{d_{Eucl}}(\zeta, \eta) = \inf\{|x - y| : x \in \overline{sym}^{-1}\zeta, y \in \overline{sym}^{-1}\eta\}, \tag{1.9}$$

The metrics *dist* and $\overline{dist}$ coincide on $\Gamma_0^{(n)}(\mathbb{R}^d) \times \Gamma_0^{(n)}(\mathbb{R}^d)$ (as functions). Furthermore, one can see that $(\ddot{\Gamma}_0^{(n)}(\mathbb{R}^d), \overline{dist})$ is a complete separable metric space, and thus a Polish space. The next lemma describes convergence in $\ddot{\Gamma}_0^{(n)}(\mathbb{R}^d)$ (compare with Lemma 3.3 in [27]).

Lemma 1.6. *Assume that $\eta^m \to \eta$ in $\ddot{\Gamma}_0^{(n)}(\mathbb{R}^d)$, and let $\eta = \{x_1, \dots, x_n\}$. Then η^m, $m \in \mathbb{N}$, may be numbered, $\eta^m = \{x_1^m, \dots, x_n^m\}$, in such a way that*

$$x_i^m \to x_i, \quad m \to \infty$$

in $\mathbb{R}^d$.

Proof. The inequality $\overline{dist}(\eta^m, \eta^m) < \varepsilon$ implies existence of a point from η^m in the ball $B_\varepsilon(x_i)$ of radius ε centered at x_i, $i \in \{1, \dots, n\}$. Furthermore, in the case when x_i is a multiple point, i.e., if $x_j = x_i$ for some $j \neq i$, then there are at least as many points from η^m in $B_\varepsilon(x_i)$ as $\eta(\{x_i\})$. Observe that, for $\varepsilon < \frac{1}{2} \inf\{|x - y| : \eta(\{x\}), \eta(\{x\}) \geq 1\} \wedge 1$, we have in the previous sentence "exactly as many" instead of "at least as many", because otherwise there would not be enough points in η^m. The statement of the lemma follows by letting $\varepsilon \to 0$. □

Lemma 1.7. $\Gamma_0(\mathbb{R}^d)$ *is a Polish space.*

Proof. Since $\Gamma_0(\mathbb{R}^d)$ is a disjoint union of countably many spaces $\Gamma_0^{(n)}(\mathbb{R}^d)$, it suffices to establish that each of them is a Polish space. To prove that $\Gamma_0^{(n)}(\mathbb{R}^d)$ is a Polish space, $n \in \mathbb{N}$, we will show that it is a countable intersection of open sets in a Polish space $\ddot{\Gamma}_0^{(n)}(\mathbb{R}^d)$. Then we may apply Alexandrov's theorem: any G_δ subset of a Polish space is a Polish space, see §33, VI in [28].

To do so, denote by $\mathbf{B}_m$ the closed ball of radius m in $\mathbb{R}^d$, with the center at the origin. Define $F_m := \{\eta \in \ddot{\Gamma}_0^{(n)}(\mathbb{R}^d) \mid \eta(\{x\}) \geq 2 \textit{ for some } x \in \mathbf{B}_m\}$ and note that

$$\Gamma_0^{(n)}(\mathbb{R}^d) = \bigcap_{m=1}^{\infty} [\ddot{\Gamma}_0^{(n)}(\mathbb{R}^d) \setminus F_m]$$

Since $\ddot{\Gamma}_0^{(n)}(\mathbb{R}^d)$ is Polish, it only remains to show that F_m is closed in $\ddot{\Gamma}_0^{(n)}(\mathbb{R}^d)$. This is an immediate consequence of the previous lemma. □

1.2.2 Lebesgue-Poisson measures

Here we define the Lebesgue-Poisson measure on $\Gamma_0(\mathbb{R}^d)$, corresponding to a non-atomic Radon measure σ on $\mathbb{R}^d$. Our prime example for σ will be the Lebesgue measure on $\mathbb{R}^d$. For any $n \in \mathbb{N}$ the product measure $\sigma^{\otimes n}$ can be considered by restriction as a measure on $\widetilde{(\mathbb{R}^d)^n}$. The projection of this measure on $\Gamma_0^{(n)}$ via *sym* we denote by $\sigma^{(n)}$, so that

$$\sigma^{(n)}(A) = \sigma^{\otimes n}(sym^{-1}A), \qquad A \in \mathscr{B}(\Gamma_0^{(n)}).$$

On $\Gamma_0^{(0)}$ the measure $\sigma^{(0)}$ is given by $\sigma^{(0)}(\{\varnothing\}) = 1$. The *Lebesgue-Poisson measure* on $(\Gamma_0(\mathbb{R}^d), \mathscr{B}(\Gamma_0(\mathbb{R}^d)))$ is defined as

$$\lambda_\sigma := \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{(n)}. \tag{1.10}$$

The measure λ_σ is finite iff σ is finite. We say that σ is the *intensity measure* of λ_σ.

1.2.3 The Skorokhod space

For a complete separable metric space (E, ρ) the space D_E of all cadlag E-valued functions equipped with the Skorokhod topology is a Polish space; for this statement and related definitions, see, e.g., Theorem 5.6, Chapter 3 in [6]. Let ρ_D be a metric on D_E compatible with the Skorokhod topology and such that (D_E, ρ_D) is a complete separable metric space. Denote by $(\mathcal{P}(D_E), \rho_p)$ the metric space of probability measures on $\mathscr{B}(D_E)$, the Borel σ - algebra of D_E, with the Prohorov metric, i.e. for $P, Q \in \mathcal{P}(D_E)$

$$\rho_p(P, Q) = \inf\{\varepsilon > 0 : P(F) \leq Q(F^\varepsilon) + \varepsilon \text{ for all } F \in \mathscr{B}(D_E)\} \tag{1.11}$$

where

$$F^\varepsilon = \{x \in D_E : \rho_D(x, F) < \varepsilon\}.$$

Then $(\mathcal{P}(D_E), \rho_p)$ is separable and complete; see, e.g., [6], Section 1, Chapter 3, and Theorem 1.7, Chapter 3. The Borel σ-algebra $\mathscr{B}(D_E)$ coincides with the one generated by the coordinate mappings; see Theorem 7.1, Chapter 3 in [6]. In this work, we mostly consider $D_{\Gamma_0(\mathbb{R}^d)}[0; T]$ and $D_{\Gamma(\mathbb{R}^d)}[0; T]$ endowed with the Skorokhod topology.

1.3 Integration with respect to Poisson point processes

We give a short introduction to the theory of integration with respect to Poisson point processes. For construction of Poisson point processes with given intensity, see e.g. [21, Chapter 12], [23], [38, Chapter 12, § 1] or [20, Chapter 1, § 8,9]. All definitions, constructions and statements about integration given here may be found in [20, Chapter 2, § 3]. See also [17, Chapter 1] for the theory of integration with respect to an orthogonal martingale measure.

On some filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}\}_{t\geq 0}, P)$, consider a Poisson point process N on $\mathbb{R}_+ \times \mathbf{X} \times \mathbb{R}_+$ with intensity measure $dt \times \beta(dx) \times du$, where $\mathbf{X} = \mathbb{R}^d$ or $\mathbf{X} = \mathbb{Z}^d$. We require the filtration $\{\mathscr{F}\}_{t\geq 0}$ to be increasing and right-continuous, and we assume that $\mathscr{F}_0$ is complete under P. We interpret the argument from the first space $\mathbb{R}_+$ as time. For $\mathbf{X} = \mathbb{R}^d$ the intensity measure β will be the Lebesgue measure on $\mathbb{R}^d$, for $\mathbf{X} = \mathbb{Z}^d$ we set $\beta = \#$, where

$$\#A = |A|, \quad A \in \mathscr{B}(\mathbb{Z}^d).$$

The Borel σ-algebra over $\mathbb{Z}^d$ is the collection of all subsets of $\mathbb{Z}^d$, i.e. $\mathscr{B}(\mathbb{Z}^d) = 2^{\mathbb{Z}^d}$. Again, as is the case with configurations, for $X = \mathbb{R}^d$ we treat a point process as a random collection of points as well as a random measure.

We say that the process N is called *compatible* with $(\mathscr{F}_t, t \geq 0)$ if N is adapted, that is, all random variables of the type $N(\bar{T}_1, U)$, $\bar{T}_1 \in \mathscr{B}([0;t])$, $U \in \mathscr{B}(\mathbf{X} \times \mathbb{R}_+)$, are $\mathscr{F}_t$-measurable, and all random variables of the type $N(t+h, U) - N(t, U)$, $h \geq 0, U \in \mathscr{B}(\mathbf{X} \times \mathbb{R}_+)$, are independent of $\mathscr{F}_t$, $N(t, U) = N([0;t], U)$. For any $U \in \mathscr{B}(\mathbf{X} \times \mathbb{R}_+)$ with $(\beta \times l)(U) < \infty$, l is the Lebesgue measure on $\mathbb{R}^d$, the process $(N([0;t], U) - t\beta \times l(U), t \geq 0)$ is a martingale (with respect to $(\mathscr{F}_t, t \geq 0)$; see [20, Lemma 3.1, Page 60]).

Definition 1.8. A process $f : \mathbb{R}_+ \times \mathbf{X} \times \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is *predictable*, if it is measurable with respect to the smallest σ - algebra generated by all g having the following properties:

(i) for each $t > 0$, $(x, u, \omega) \mapsto g(t, x, u, \omega))$ is $\mathscr{B}(\mathbf{X} \times \mathbb{R}_+) \times \mathscr{F}_t$ measurable;

(ii) for each (x, u, ω), the map $t \mapsto g(t, x, u, \omega))$ is left continuous.

For a predictable process $f \in L^1([0;T] \times \mathbf{X} \times \mathbb{R}_+ \times \Omega)$, $t \in [0;T]$ and $U \in \mathscr{B}(\mathbf{X} \times \mathbb{R}_+)$ we define the integral $I_t(f) = \int\limits_{[0;t]\times U} f(s, x, u, \omega) dN(s, x, u)$ as the Lebesgue-Stieltjes integral with respect to the measure N:

$$\int\limits_{[0;t]\times U} f(s, x, u, \omega) dN(s, x, u) = \sum_{s\leq t, (s,x,u)\in N} f(s, x, u, \omega).$$

This sum is well defined, since

$$E \sum_{s\leq t, (s,x,u)\in N} |f(s, x, u, \omega)| = \int\limits_{[0;t]\times U} |f(s, x, u, \omega)| ds \beta(dx) du < \infty$$

We use $dN(s, x, u)$ and $N(ds, dx, du)$ interchangeably when we integrate over all variables. The process $I_t(f)$ is right-continuous as a function of t, and adapted. Moreover, the process

$$\tilde{I}_t(f) = \int\limits_{[0;t]\times U} f(s,x,u,\omega)[dN(s,x,u) - ds\beta(dx)du]$$

is a martingale with respect to $(\mathscr{F}_t, t \geq 0)$, [20, Page 62]. Thus,

$$E \int\limits_{[0;t]\times U} f(s,x,u,\omega)dN(s,x,u) = E \int\limits_{[0;t]\times U} f(s,x,u,\omega)ds\beta(dx)du. \qquad (1.12)$$

This equality will be used several times throughout this work.

Remark 1.9. We can extend the collection of integrands, in particular, we can define $\int\limits_{[0;t]\times U} f(s,x,u,\omega)dN(s,x,u)$ for f satisfying

$$E \int\limits_{[0;t]\times U} (|f(s,x,u,\omega)| \wedge 1)ds\beta(dx)du < \infty.$$

However, we do not use such integrands.

The Lebesgue-Stieltjes integral is defined ω-wisely and it is a function of an integrand and an integrator. As a result, we have the following statement. The sign $\stackrel{d}{=}$ means equality in distribution.

Statement 1.10. *Let M_k be Poisson point processes defined on some, possibly different, probability spaces, and let α_k be integrands, $k = 1,2$, such that integrals $\int \alpha_k dM_k$ are well defined. If $(\alpha_1, M_1) \stackrel{d}{=} (\alpha_2, M_2)$, then*

$$\int \alpha_1 dM_1 \stackrel{d}{=} \int \alpha_2 dM_2.$$

The proof is straightforward.

1.3.1 An auxiliary construction

Let $\tilde{\#}$ be the counting measure on $[0,1]$, i.e.

$$\tilde{\#}C = |C|, \qquad C \in \mathscr{B}([0;1]).$$

The measure $\tilde{\#}$ is not σ-finite. For a cadlag $\Gamma_0(\mathbb{R}^d)$-valued process $(\eta_t)_{t\in[0;\infty]}$, adapted to $\{\mathscr{F}_t\}_{t\in[0;\infty]}$, we would like to define integrals of the form

$$\int\limits_{\mathbb{R}^d\times[0;\infty]\times[0;\infty)} I_{\{x\in B\cap\eta_{r-}\}} f(x,r,v,\omega) d\tilde{N}_2(x,r,v) \qquad (1.13)$$

where B is a bounded Borel subset of $\mathbb{R}^d$, f is a bounded predictable process and $\tilde{N}_2$ is a Poisson point process on $\mathbb{R}^d \times [0;T] \times [0;\infty)$ with intensity $\tilde{\#} \times dr \times dv$, compatible with $\{\mathscr{F}_t\}_{t\in[0;\infty]}$.

We can not hope to give a meaningful definition for an integral of the type (1.13), because of the measurability issues. For example, the map

$$\Omega \to \mathbb{R},$$
$$\omega \mapsto \tilde{N}_2(u(\omega), [0;1], [0;1]),$$

where u is an independent of $\tilde{N}_2$ uniformly distributed on $[0;1]$ random variable, does not have to be a random variable. Even if it were a random variable, some undesirable phenomena would appear, see, e.g., [35].

To avoid this difficulty, we employ another construction. A similar approach was used in [11]. If we could give meaningful definition to the integrals of the type (1.13), we would expect

$$\int\limits_{\mathbb{R}^d \times [0;t] \times [0;\infty)} I_{\{x \in B \cap \eta_{r-}\}} f(x, r, v, \omega) d\tilde{N}_2(x, r, v) - \int\limits_{\mathbb{R}^d \times [0;t] \times [0;\infty)} I_{\{x \in B \cap \eta_{r-}\}} f(x, r, v, \omega) \tilde{\#}(dx) dr dv$$

to be a martingale (under some conditions on f and B).

Having this in mind, consider a Poisson point process N_2 on $\mathbb{Z} \times \mathbb{R}_+ \times \mathbb{R}_+$ with intensity $\# \times dr \times dv$, defined on $(\Omega, \mathscr{F}, \{\mathscr{F}\}_{t \geq 0}, P)$ (here $\#$ denotes the counting measure on $\mathbb{Z}$. This measure is σ-finite). We require N_2 to be compatible with $\{\mathscr{F}\}_{t \geq 0}$. Let $(\eta_t)_{t \in [0,\infty]}$ be an adapted cadlag process in $\Gamma_0(\mathbb{R}^d)$, satisfying the following condition: for any $T < \infty$,

$$R_T = |\bigcup_{t \in [0;T]} \eta_t| < \infty \quad \text{a.s.} \tag{1.14}$$

The set $R_\infty := \bigcup_{t \in [0;\infty]} \eta_t$ is at most countable, provided (1.14). Let $\preccurlyeq$ be the lexicographical order on $\mathbb{R}^d$. We can label the points of η_0,

$$\eta_0 = \{x_0, x_{-1}, \dots, x_{-q}\}, \quad x_0 \preccurlyeq x_{-1} \preccurlyeq \dots \preccurlyeq x_{-q}.$$

There exists an a.s. unique representation

$$R_\infty \setminus \eta_0 = \{x_1, x_2, \dots\}$$

such that for any $n, m \in \mathbb{N}$, $n < m$, either $\inf\limits_{s \geq 0}\{s : x_n \in \eta_s\} < \inf\limits_{s \geq 0}\{s : x_m \in \eta_s\}$, or $\inf\limits_{s \geq 0}\{s : x_n \in \eta_s\} = \inf\limits_{s \geq 0}\{s : x_m \in \eta_s\}$ and $x_n \preccurlyeq x_m$. In other words, as time goes on, appearing points are added to $\{x_1, x_2, \dots\}$ in the order in which they appear. If several points appear simultaneously, we add them in the lexicographical order.

For the sake of convenience, we set $x_{-i} = \Delta$, $i \leq -q-1$, where $\Delta \notin \mathbb{Z}$. We say that the sequence $\{\dots, x_{-1}, x_1, x_2, \dots\}$ is *related* to $(\eta_t)_{t \in [0;\infty]}$.

For a predictable process $f \in L^1(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega)$ and $B \in \mathscr{B}(\mathbb{R}^d)$, consider

$$\int\limits_{\mathbb{Z} \times (t_1;t_2] \times [0;\infty)} I_{\{x_i \in \eta_{r-} \cap B\}} f(x_i, r, v, \omega) dN_2(i, r, v). \tag{1.15}$$

Assume that R_T is bounded for some $T > 0$. Then, for a bounded predictable $f \in L^1(R^d \times \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega)$ and $B \in \mathscr{B}(\mathbb{R}^d)$, the process

$$\int\limits_{\mathbb{Z}\times(0;t]\times[0;\infty)} I_{\{x_i \in \eta_{r-} \cap B\}} f(x_i, r, v, \omega) dN_2(i, r, v) - \int\limits_{\mathbb{Z}\times(0;t]\times[0;\infty)} I_{\{x_i \in \eta_{r-} \cap B\}} f(x_i, r, v, \omega) \#(di) dr dv$$

is a martingale, cf. [20, Page 62].

1.3.2 The strong Markov property of a Poisson point process

We will need the strong Markov property of a Poisson point process. To simplify notations, assume that N is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^d$ with intensity measure $dt \times dx$. Let N be compatible with a right-continuous complete filtration $\{\mathscr{F}_t\}_{t\ge 0}$, and τ be a finite a.s. $\{\mathscr{F}_t\}_{t\ge 0}$-stopping time (stopping time with respect to $\{\mathscr{F}_t\}_{t\ge 0}$). Introduce another Point process $\overline{N}$ on $\mathbb{R}_+ \times \mathbb{R}^d$,

$$\overline{N}([0;s] \times U) = N((\tau; \tau + s] \times U), \quad U \in \mathscr{B}(\mathbb{R}^d).$$

Proposition 1.11. *The process $\overline{N}$ is a Poisson point process with intensity $dt \times dx$, independent of $\mathscr{F}_\tau$.*

Proof. To prove the proposition, it is enough to show that

(i) for any $b > a > 0$ and open bounded $U \subset \mathbb{R}^d$, $\overline{N}((a;b), U)$ is a Poisson random variable with mean $(b-a)\beta(U)$, and

(ii) for any $b_k > a_k > 0$, $k = 1, \ldots, m$, and any open bounded $U_k \subset \mathbb{R}^d$, such that $((a_i; b_i) \times U_i) \cap ((a_j; b_j) \times U_j) = \varnothing$, $i \neq j$, the collection $\{\overline{N}((a_k; b_k) \times U_k)\}_{k=1,m}$ is a sequence of independent random variables, independent of $\mathscr{F}_\tau$.

Indeed, $\overline{N}$ is determined completely by values on sets of type $(b-a)\beta(U)$, a, b, U as in (i), therefore it must be an independent of $\mathscr{F}_\tau$ Poisson point process if (i) and (ii) hold.

Let τ_n be the sequence of $\{\mathscr{F}_t\}_{t\ge 0}$-stopping times, $\tau_n = \frac{k}{2^n}$ on $\{\tau \in (\frac{k-1}{2^n}; \frac{k}{2^n}]\}$, $k \in \mathbb{N}$. Then $\tau_n \downarrow \tau$ and $\tau_n - \tau \le \frac{1}{2^n}$. The stopping times τ_n take only countably many values. The process N satisfies the strong Markov property for τ_n: the processes $\overline{N}_n$, defined by

$$\overline{N}_n([0;s] \times U) := N((\tau_n; \tau_n + s] \times U),$$

are Poisson point processes, independent of $\mathscr{F}_{\tau_n}$. To prove this, take k with $P\{\tau_n = \frac{k}{2^n}\} > 0$ and note that on $\{\tau_n = \frac{k}{2^n}\}$, $\overline{N}_n$ coincides with process the Poisson point process $\tilde{N}_{\frac{k}{2^n}}$ given by

$$\tilde{N}_{\frac{k}{2^n}}([0;s] \times U) := N\bigg((\frac{k}{2^n}; \frac{k}{2^n} + s] \times U) \bigg), \quad U \in \mathscr{B}(\mathbb{R}^d).$$

Conditionally on $\{\tau_n = \frac{k}{2^n}\}$, $\tilde{N}_{\frac{k}{2^n}}$ is again a Poisson point process, with the same intensity. Furthermore, conditionally on $\{\tau_n = \frac{k}{2^n}\}$, $\tilde{N}_{\frac{k}{2^n}}$ is independent of $\mathscr{F}_{\frac{k}{2^n}}$, hence it is independent of $\mathscr{F}_\tau \subset \mathscr{F}_{\frac{k}{2^n}}$.

To prove (i), note that $\overline{N}_n((a;b)\times U) \to \overline{N}((a;b)\times U)$ a.s. and all random variables $\overline{N}_n((a;b)\times U)$ have the same distribution, therefore $\overline{N}((a;b)\times U)$ is a Poisson random variable with mean $(b-a)\lambda(U)$. The random variables $\overline{N}_n((a;b)\times U)$ are independent of $\mathscr{F}_\tau$, hence $\overline{N}((a;b)\times U)$ is independent of $\mathscr{F}_\tau$, too. Similarly, (ii) follows. □

Analogously, the strong Markov property for a Poisson point process on $\mathbb{R}_+ \times \mathbb{N}$ with intensity $dt \times \#$ may be formulated and proven.

Remark 1.12. We assumed in Proposition 1.11 that the filtration $\{\mathscr{F}_t\}_{t\geq 0}$, compatible with N, is right-continuous and complete. To be able to apply Proposition 1.11, we should show that such filtrations exist.

Introduce the natural filtration of N,

$$\mathscr{F}_t^0 = \sigma\{N_k(C,B), B \in \mathscr{B}(\mathbb{R}^d), C \in \mathscr{B}([0;t])\},$$

and let $\mathscr{F}_t$ be the completion of $\mathscr{F}_t^0$ under P. Then N is compatible with $\{\mathscr{F}_t\}$. We claim that $\{\mathscr{F}_t\}_{t\geq 0}$, defined in such a way, is right-continuous (this may be regarded as an analog of Blumenthal $0-1$ law). Indeed, as in the proof of Proposition 1.11, one may check that $\tilde{N}_a$ is independent of $\mathscr{F}_{a+}$. Since $\mathscr{F}_\infty = \sigma(\tilde{N}_a) \vee \mathscr{F}_a$, $\sigma(\tilde{N}_a)$ and $\mathscr{F}_a$ are independent and $\mathscr{F}_{a+} \subset \mathscr{F}_\infty$, one sees that $\mathscr{F}_{a+} \subset \mathscr{F}_a$. Thus, $\mathscr{F}_{a+} = \mathscr{F}_a$.

Remark 1.13. We prefer to work with right-continuous complete filtrations, because we want to ensure that there is no problem with conditional probabilities, and that the hitting times we will consider are stopping times.

1.4 Miscellaneous

When we write $\xi \sim Exp(\lambda)$, we mean that the random variable ξ is exponentially distributed with parameter λ.

Lemma 1.14. *If α and β are exponentially distributed random variables with parameters a and b respectively (notation: $\alpha \sim Exp(a)$, $\beta \sim Exp(b)$) and they are independent, then*

$$P\{\alpha < \beta\} = \frac{a}{a+b}.$$

Proof. Indeed,

$$P\{\alpha < \beta\} = \int_0^\infty aP\{x < \beta\}e^{-ax} = a\int_0^\infty e^{-(a+b)x} = \frac{a}{a+b}. \qquad \square$$

Here are few other properties of exponential distributions. If $\xi_1, \xi_2, \ldots, \xi_n$ are independent exponentially distributed random variables with parameters

$c_1, \ldots, c_n$ respectively, then $\min\limits_{k \in \{1,\ldots,n\}} \xi_k$ is exponentially distributed with parameter $c_1 + \ldots + c_n$. Again, the proof may be done by direct computation. If $\xi_1, \xi_2, \ldots$ are independent exponentially distributed random variables with parameter c and $\alpha_1, \alpha_2, \ldots$ is an independent sequence of independent Bernoulli random variables with parameter $p \in (0;1)$, then the random variable

$$\xi = \sum_{i=1}^{\theta} \xi_i, \quad \theta = \min\{k \in \mathbb{N} : \alpha_k = 1\}$$

is exponentially distributed with parameter $\frac{c}{p}$. The random variable ξ is the time of the first jump of a thinned Poisson point process with intensity c. The statement about the distribution of ξ is a consequence of the property that the independent thinning of a Poisson point process with intensity λ is a Poisson point process with intensity $p\lambda$, see [21, Theorem 12.2,(iv)].

We will also need the result about finiteness of the expectation of the Yule process. A Yule process $(Z_t)_{t\geq 0}$ is a pure birth Markov process in $\mathbb{Z}_+$ with birth rate μn, $\mu > 0$, $n \in \mathbb{Z}_+$. That is, if $Z_t = n$, then a birth occur at rate μn, i.e.

$$P\{Z_{t+\Delta t} - Z_t = 1 \mid Z_t = n\} = \mu n + o(\Delta t).$$

For more details about Yule processes see e.g. [3, Chapter 3], [18, Chapter 5], [2] and references therein. Let $(Z_t(n))_{t\geq 0}$ be a Yule process started at n. The process $(Z_t(n))_{t\geq 0}$ can be considered as a sum of n independent Yule processes started from 1, see e.g. [2]. The expectation of $Z_t(1)$ is finite and $EZ_t(1) = e^{\mu t}$, see e.g. [3, Chapter 3, Section 6] or [18, Chapter 5, Sections 6,7]. Consequently, if $(Z_t)_{t\geq 0}$ is a Yule process with $EZ_0 < \infty$, then $EZ_t < \infty$ and $EZ_t = EZ_0 e^{\mu t}$.

Here are some other properties of Poisson point processes which are used throughout in the article. If N is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+$ with intensity $ds \times dx \times du$, then a.s.

$$\forall x \in \mathbb{R}^d : N(\mathbb{R}_+ \times \{x\} \times \mathbb{R}_+) \leq 1. \tag{1.16}$$

Put differently, no plane of the form $\mathbb{R}_+ \times \{x\} \times \mathbb{R}_+$ contains more than 1 point of N. Using the σ-additivity of the probability measure, one can deduce (1.16) from

$$\forall x \in \mathbb{R}^d : N([0;1] \times \{x\} \times [0;1]) \leq 1. \tag{1.17}$$

We can write

$$\Big\{\forall x \in \mathbb{R}^d : N([0;1] \times \{x\} \times [0;1]) \leq 1\Big\}$$
$$\supset \Big\{\forall k \in \{0,1,\ldots,n-1\} : N([0;1] \times [\frac{k}{n}; \frac{k+1}{n}] \times [0;1]) \leq 1\Big\},$$

and then we can compute

$$P\Big\{\forall k\in\{0,1,\dots,n-1\}: N([0;1]\times[\frac{k}{n};\frac{k+1}{n}]\times[0;1])\le 1\Big\}$$
$$=\Big(P\{N([0;1]\times[0;\frac{1}{n}]\times[0;1])\le 1\}\Big)^n=\Big(\exp(-\frac{1}{n})[1+\frac{1}{n}]\Big)^n$$
$$=\Big(1-o(\frac{1}{n})\Big)^n=1-o(\frac{1}{n}).$$

Thus, (1.17) holds.

Let $\psi\in L^1(\mathbb{R}^d)$, $\psi\ge 0$. Consider the time until the first arrival

$$\tau=\inf\{t>0:\int\limits_{[0;t]\times\mathbb{R}^d\times\mathbb{R}_+} I_{[0;\psi(x)]}(u)N(ds,dx,du)>0\}. \tag{1.18}$$

The random variable τ is distributed exponentially with the parameter $||\psi||_{L^1}$. From (1.16) we know that a.s.

$$N(\{\tau\}\times\mathbb{R}^d\times\mathbb{R}_+)=N\big(\{(\tau,x,u)\mid x\in\mathbb{R}^d,u\in[0;\psi(x)]\}\big)=1$$

Let x_τ be the unique element of $\mathbb{R}^d$ defined by

$$N(\{\tau\}\times\{x_\tau\}\times\mathbb{R}_+)=1.$$

Then

$$P\{x_\tau\in B\}=\frac{\int_B\psi(x)dx}{\int_{\mathbb{R}^d}\psi(x)dx},\quad B\in\mathscr{B}(\mathbb{R}^d). \tag{1.19}$$

1.5 Pure jump type Markov processes

In this section we give a very concise treatment of pure jump type Markov processes. Most of the definitions and facts given here can be found in [21, Chapter 12]; see also, e.g., [16, Chapter 3, § 1].

We say that a process $X=(X_t)_{t\ge 0}$ in some measurable space $(S,\mathcal{S})$ is of *pure jump type* if its paths are a.s. right-continuous and constant apart from isolated jumps. In that case we may denote the jump times of X by $\tau_1,\tau_2,\dots$, with understanding that $\tau_n=\infty$ if there are fewer that n jumps. The times τ_n are stopping times with respect to the right-continuous filtration induced by X. For convenience we may choose X to be the identity mapping on the canonical path space $(\Omega,\mathscr{F})=(S^{[0;\infty)},\mathcal{S}^{[0;\infty)})$. When X is a Markov process, the distribution with initial state x is denoted by P_x, and we note that the mapping $x\mapsto P_x(A)$ is measurable in x, $A\in\Omega$.

Theorem. [21, Theorem 12.14] (strong Markov property, Doob). *A pure jump type Markov process satisfies strong Markov property at every stopping time.*

We say that a state $x\in S$ is *absorbing* if $P_x\{X\equiv x\}=1$.

Theorem. [21, Lemma 12.16]. *If x is non-absorbing, then under P_x the time τ_1 until the first jump is exponentially distributed and independent of $\theta_{\tau_1}X$.*

Here θ_t is a shift, and $\theta_{\tau_1} X$ defines a new process,

$$\theta_{\tau_1} X(s) = X(s + \tau_1).$$

For a non-absorbing state x, we may define the *rate function* $c(x)$ and *jump transition kernel* $\mu(x, B)$ by

$$c(x) = (E_x \tau_1)^{-1}, \quad \mu(x, B) = P_x\{X_{\tau_1} \in B\}, \qquad x \in S, \ B \in \mathcal{S}.$$

In the sequel, $c(x)$ will also be referred to as *jump rate*. The kernel $c\mu$ is called a *rate kernel*.

The following theorem gives an explicit representation of the process in terms of a discrete-time Markov chain and a sequence of exponentially distributed random variables. This result shows in particular that the distribution P_x is uniquely determined by the rate kernel $c\mu$. We assume existence of the required randomization variables (so that the underlying probability space is "rich enough").

Theorem. [21, Theorem 12.17] (embedded Markov chain). *Let X be a pure jump type Markov process with rate kernel $c\mu$. Then there exists a Markov process Y on $\mathbb{Z}_+$ with transition kernel μ and an independent sequence of i.i.d., exponentially distributed random variables $\gamma_1, \gamma_2, \dots$ with mean* 1 *such that a.s.*

$$X_t = Y_n, \quad t \in [\tau_n, \tau_{n+1}), \ n \in \mathbb{Z}_+, \tag{1.20}$$

where

$$\tau_n = \sum_{k=1}^{n} \frac{\gamma_k}{c(Y_{k-1})}, \quad n \in \mathbb{Z}_+. \tag{1.21}$$

In particular, the differences between the moments of jumps $\tau_{n+1} - \tau_n$ of a pure jump type Markov process are exponentially distributed given the embedded chain Y, with parameter $c(Y_n)$. If $c(Y_k) = 0$ for some (random) k, we set $\tau_n = \infty$ for $n \geq k+1$, while Y_n are not defined, $n \geq k+1$.

Theorem. [21, Theorem 12.18] (synthesis). *For any rate kernel $c\mu$ on S with $\mu(x, \{x\}) \equiv 0$, consider a Markov chain Y with transition kernel μ and a sequence $\gamma_1, \gamma_2, \dots$ of independent exponentially distributed random variables with mean 1, independent of Y. Assume that $\sum_n \frac{\gamma_n}{c(Y_n)} = \infty$ a.s. for every initial distribution for Y. Then* (1.20) *and* (1.21) *define a pure jump type Markov process with rate kernel $c\mu$.*

Next proposition gives a convenient criterion for non-explosion.

Theorem. [21, Theorem 12.19] (explosion). *For any rate kernel $c\mu$ and initial state x, let (Y_n) and (τ_n) be such as in Theorem 12.17. Then a.s.*

$$\tau_n \to \infty \quad \text{iff} \quad \sum_n \frac{1}{c(Y_n)} = \infty. \tag{1.22}$$

In particular, $\tau_n \to \infty$ a.s. when x is recurrent for (Y_n).

1.6 Markovian functions of a Markov chain

Let $(S, \mathscr{B}(S))$ be a Polish (state) space. Consider a (homogeneous) Markov chain on $(S, \mathscr{B}(S))$ as a family of probability measures on S^∞. Namely, on the measurable space $(\Omega, \mathscr{F}) = (S^\infty, \mathscr{B}(S^\infty))$ consider a family of probability measures $\{P_s\}_{s \in S}$ such that for the coordinate mappings

$$X_n : \Omega \to S,$$
$$X_n(s_1, s_2, \ldots) = s_n$$

the process $X = \{X_n\}_{n \in \mathbb{Z}_+}$ is a Markov chain, and for all $s \in S$

$$P_s\{X_0 = s\} = 1,$$
$$P_s\{X_{n+m_j} \in A_j, j = 1, \ldots, k_1 \mid \mathscr{F}_n\} = P_{X_n}\{X_{m_j} \in A_j, j = 1, \ldots, k_1\}.$$

Here $A_j \in \mathscr{B}(S)$, $m_j \in \mathbb{N}$, $k_1 \in \mathbb{N}$, $\mathscr{F}_n = \sigma\{X_1, \ldots, X_n\}$. The space S is separable, hence there exists a transition probability kernel $Q : S \times \mathscr{B}(S) \to [0; 1]$ such that

$$Q(s, A) = P_s\{X_1 \in A\}, \quad s \in S, \ A \in \mathscr{B}(S).$$

Consider a transformation of the chain X, $Y_n = f(X_n)$, where $f : S \to \mathbb{Z}_+$ is a Borel-measurable function, with convention $\mathscr{B}(\mathbb{Z}_+) = 2^{\mathbb{Z}_+}$. In the future we will need to know when the process $Y = \{Y_n\}_{\mathbb{Z}_+}$ is a Markov chain. A similar question appeared for the first time in [4].

A sufficient condition for Y to be a Markov chain is given in the next lemma.

Lemma 1.15. *Assume that for any bounded Borel function $h : S \to S$*

$$E_s h(X_1) = E_q h(X_1) \text{ whenever } f(s) = f(q), \tag{1.23}$$

Then Y is a Markov chain.

Remark 1.16. Condition (1.23) is the equality of distributions of X_1 under two different measures, P_s and P_q.

Proof. For the natural filtrations of the processes X and Y we have an inclusion

$$\mathscr{F}_n^X \supset \mathscr{F}_n^Y, \quad n \in \mathbb{N}, \tag{1.24}$$

since Y is a function of X. For $k \in \mathbb{N}$ and bounded Borel functions $h_j : \mathbb{Z}_+ \to \mathbb{R}$, $j = 1, 2, \ldots, k$ (any function on $\mathbb{Z}_+$ is a Borel function),

$$\begin{aligned} E_s \left[\prod_{j=1}^k h_j(Y_{n+j}) \mid \mathscr{F}_n^X \right] &= E_{X_n} \prod_{j=1}^k h_j(f(X_j)) \\ &= \int_S Q(x_0, dx_1) h_1(f(x_1)) \int_S Q(x_1, dx_2) h_2(f(x_2)) \ldots \\ &\quad \times \int_S Q(x_{n-1}, dx_n) h_n(f(x_n)) \Big|_{x_0 = X_n} \end{aligned} \tag{1.25}$$

To transform the last integral, we introduce a new kernel: for $y \in f(S)$ chose $x \in S$ with $f(x) = y$, and then for $B \subset \mathbb{Z}_+$ define

$$\overline{Q}(y, B) = Q(x, f^{-1}(B)); \tag{1.26}$$

The expression on the right-hand side does not depend on the choice of x because of (1.23). To make the kernel $\overline{Q}$ defined on $\mathbb{Z}_+ \times \mathscr{B}(\mathbb{Z}_+)$, we set

$$\overline{Q}(y, B) = I_{\{0 \in B\}}, \ y \notin f(S).$$

Then from the change of variables formula for the Lebesgue integral it follows that the last integral in (1.25) allows the representation

$$\int_S Q(x_{n-1}, dx_n) h_n(f(x_n)) = \int_{\mathbb{Z}_+} \overline{Q}(f(x_{n-1}), dz_n) h_n(z_n).$$

Likewise, we set $z_{n-1} = f(x_{n-1})$ in the next to last integral:

$$\begin{aligned} &\int_S Q(x_{n-2}, dx_{n-1}) h_n(f(x_{n-1})) \int_S Q(x_{n-1}, dx_n) h_n(f(x_n)) \\ &= \int_S Q(x_{n-2}, dx_{n-1}) h_n(f(x_{n-1})) \int_{\mathbb{Z}_+} \overline{Q}(f(x_{n-1}), dz_n) h_n(z_n) \\ &= \int_{\mathbb{Z}_+} \overline{Q}(f(x_{n-2}), dz_{n-1}) h_n(z_{n-1}) \int_{\mathbb{Z}_+} \overline{Q}(z_{n-1}, dz_n) h_n(z_n). \end{aligned}$$

Further proceeding, we get

$$\begin{aligned} &\int_S Q(x_0, dx_1) h_1(f(x_1)) \int_S Q(x_1, dx_2) h_2(f(x_2)) \dots \int_S Q(x_{n-1}, dx_n) h_n(f(x_n)) \\ &= \int_{\mathbb{Z}_+} \overline{Q}(z_0, dz_1) h_1(z_1) \int_{\mathbb{Z}_+} \overline{Q}(z_1, dz_2) h_2(z_2) \dots \int_{\mathbb{Z}_+} \overline{Q}(z_{n-1}, dz_n) h_n(z_n), \end{aligned}$$

where $z_0 = f(x_0)$.

Thus,

$$\begin{aligned} &E_s \left[\prod_{j=1}^{k} h_j(Y_{n+j}) \mid \mathscr{F}_n^X \right] \\ &= \int_{\mathbb{Z}_+} \overline{Q}(f(X_0), dz_1) h_1(z_1) \int_{\mathbb{Z}_+} \overline{Q}(z_1, dz_2) h_2(z_2) \dots \int_{\mathbb{Z}_+} \overline{Q}(z_{n-1}, dz_n) h_n(z_n). \end{aligned}$$

This equality and (1.24) imply that Y is a Markov chain. □

Remark 1.17. The kernel $\overline{Q}$ and the chain $f(X_n)$ are related: for all $s \in S$, $n, m \in \mathbb{N}$ and $M \subset \mathbb{N}$,

$$P_s\{f(X_{n+1}) \in M \mid f(X_n) = m\} = \overline{Q}(m, M)$$

whenever $P_s\{f(X_{n+1}) = m\} > 0$. Informally, one may say that $\overline{Q}$ is the transition probability kernel for the chain $\{f(X_n)\}_{n \in \mathbb{Z}_+}$.

Remark 1.18. Clearly, this result holds for a Markov chain which is not necessarily defined on a canonical state space, because the property of a process to be a Markov chain depends on its distribution only.

2 A birth-and-death process in the space of finite configurations: construction and basic properties

We would like to construct a Markov process in the space of finite configurations $\Gamma_0(\mathbb{R}^d)$, with a heuristic generator of the form

$$LF(\eta) = \int\limits_{x\in\mathbb{R}^d} b(x,\eta)[F(\eta\cup x) - F(\eta)]dx \\ + \sum_{x\in\eta} d(x,\eta)(F(\eta\setminus x) - F(\eta)). \quad (2.1)$$

for F in an appropriate domain. We call the functions $b : \mathbb{R}^d \times \Gamma_0(\mathbb{R}^d) \to [0;\infty)$ and $d : \mathbb{R}^d \times \Gamma_0(\mathbb{R}^d) \to [0;\infty)$ the *birth rate coefficient* and the *death rate coefficient*, respectively. Theorem 2.16 summarizes the main results obtained in this section.

To construct a spatial birth-and-death process, we consider the stochastic equation with Poisson noise

$$\eta_t(B) = \int\limits_{B\times(0;t]\times[0;\infty]} I_{[0;b(x,\eta_{s-})]}(u)dN_1(x,s,u) \\ - \int\limits_{\mathbb{Z}\times(0;t]\times[0;\infty)} I_{\{x_i\in\eta_{r-}\cap B\}} I_{[0;d(x_i,\eta_{r-})]}(v)dN_2(i,r,v) + \eta_0(B), \quad (2.2)$$

where $(\eta_t)_{t\geq 0}$ is a suitable cadlag $\Gamma_0(\mathbb{R}^d)$-valued stochastic process, the "solution" of the equation, $B \in \mathscr{B}(\mathbb{R}^d)$ is a Borel set, N_1 is a Poisson point process on $\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dx \times ds \times du$, N_2 is a Poisson point process on $\mathbb{Z}\times\mathbb{R}_+\times\mathbb{R}_+$ with intensity $\#\times dr\times dv$; η_0 is a (random) finite initial configuration, $b, d : \mathbb{R}^d \times \Gamma_0(R^d) \to [0;\infty)$ are functions measurable with respect to the product σ-algebra $\mathscr{B}(\mathbb{R}) \times \mathscr{B}(\Gamma_0(\mathbb{R}))$, and the sequence $\{\ldots, x_{-1}, x_0, x_1, \ldots\}$ is related to $(\eta_t)_{t\in[0;\infty]}$, as described in Section 1.3.1. We require the processes N_1, N_2, η_0 to be independent of each other. Equation (2.2) is understood in the sense that the equality holds a.s. for every bounded $B \in \mathscr{B}(\mathbb{R}^d)$ and $t \geq 0$.

As it was said in the preliminaries on Page 9, we identify a finite configuration with a finite simple counting measure, so that a configuration γ acts as a measure in the following way:

$$\gamma(A) = |\gamma \cap A|, \quad A \in \mathscr{B}(\mathbb{R}^d).$$

We will treat an element of $\Gamma_0(\mathbb{R}^d)$ both as a set and as a counting measure, as long as this does not lead to ambiguity. An appearing of a new point will be interpreted as a birth, and a disappearing will be interpreted as a death. We will refer to points of η_t as particles.

Some authors write $\tilde{d}(x, \eta \setminus x)$ where we write $d(x, \eta)$, so that (2.1) translates to

$$LF(\eta) = \int\limits_{x\in\mathbb{R}^d} b(x,\eta)[F(\eta\cup x) - F(\eta)]dx$$
$$+ \sum_{x\in\eta} \tilde{d}(x,\eta\setminus x)(F(\eta\setminus x) - F(\eta)), \quad (2.3)$$

see e.g. [36], [10].

These settings are formally equivalent: the relation between d and $\tilde{d}$ is given by

$$d(x,\eta) = \tilde{d}(x,\eta\setminus x), \quad \eta\in\Gamma_0(\mathbb{R}^d), x\in\eta,$$

or, equivalently,

$$d(x,\xi\cup x) = \tilde{d}(x,\xi), \quad \xi\in\Gamma_0(\mathbb{R}^d), x\in\mathbb{R}^d\setminus\xi.$$

The settings used here appeared in [19], [14], etc.

We define the *cumulative death rate* at ζ by

$$D(\zeta) = \sum_{x\in\zeta} d(x,\zeta), \quad (2.4)$$

and the *cumulative birth rate* by

$$B(\zeta) = \int\limits_{x\in\mathbb{R}^d} b(x,\zeta)dx. \quad (2.5)$$

Definition 2.1. A *(weak) solution* of equation (2.2) is a triple $((\eta_t)_{t\geq 0}, N_1, N_2)$, $(\Omega, \mathscr{F}, P)$, $(\{\mathscr{F}_t\}_{t\geq 0})$, where

(i) $(\Omega, \mathscr{F}, P)$ is a probability space, and $\{\mathscr{F}_t\}_{t\geq 0}$ is an increasing, right-continuous and complete filtration of sub - σ - algebras of $\mathscr{F}$,

(ii) N_1 is a Poisson point process on $\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}_+$ with intensity $dx\times ds\times du$,

(iii) N_2 is a Poisson point process on $\mathbb{Z}\times\mathbb{R}_+\times\mathbb{R}_+$ with intensity $\#\times ds\times du$,

(iv) η_0 is a random $\mathscr{F}_0$-measurable element in $\Gamma_0(\mathbb{R}^d)$,

(v) the processes N_1, N_2 and η_0 are independent, the processes N_1 and N_2 are compatible with $\{\mathscr{F}_t\}_{t\geq 0}$,

(vi) $(\eta_t)_{t\geq 0}$ is a cadlag $\Gamma_0(\mathbb{R}^d)$-valued process adapted to $\{\mathscr{F}_t\}_{t\geq 0}$, $\eta_t\big|_{t=0} = \eta_0$,

(vii) all integrals in (2.2) are well-defined, and

(viii) equality (2.2) holds a.s. for all $t\in[0;\infty]$ and all bounded Borel sets B, with $\{x_m\}_{m\in\mathbb{Z}}$ being the sequence related to $(\eta_t)_{t\geq 0}$.

Note that due to Statement 1.10 item (viii) of this definition is a statement about the joint distribution of $(\eta_t), N_1, N_2$.

Let

$$\mathscr{C}_t^0 = \sigma\big\{\eta_0, N_1(B,[0;q],C), N_2(i,[0;q],C);$$
$$B\in\mathscr{B}(\mathbb{R}^d), C\in\mathscr{B}(\mathbb{R}_+), q\in[0;t], i\in\mathbb{Z}\big\},$$

and let $\mathscr{C}_t$ be the completion of $\mathscr{C}_t^0$ under P. Note that $\{\mathscr{C}_t\}_{t\geq 0}$ is a right-continuous filtration, see Remark 1.12.

Definition 2.2. A solution of (2.2) is called *strong* if $(\eta_t)_{t\geq 0}$ is adapted to $(\mathscr{C}_t, t \geq 0)$.

Remark 2.3. In the definition above we considered solutions as processes indexed by $t \in [0;\infty)$. The reformulations for the case $t \in [0;T]$, $0 < T < \infty$, are straightforward. This remark applies to the results below, too.

Sometimes only the solution process (that is, $(\eta_t)_{t\geq 0}$) will be referred to as a (strong or weak) solution, when all the other structures are clear from the context.

We will say that the *existence of strong solution* holds, if on any probability space with given N_1, N_2, η_0, satisfying (i)-(v) of Definition (2.1), there exists a strong solution.

Definition 2.4. We say that pathwise uniqueness holds for equation (2.2) and an initial distribution ν if, whenever the triples $((\eta_t)_{t\geq 0}, N_1, N_2)$, $(\Omega, \mathscr{F}, P)$, $(\{\mathscr{F}_t\}_{t\geq 0})$ and $((\bar{\eta}_t)_{t\geq 0}, N_1, N_2)$, $(\Omega, \mathscr{F}, P)$, $(\{\bar{\mathscr{F}}_t\}_{t\geq 0})$ are weak solutions of (2.2) with $P\{\eta_0 = \bar{\eta}_0\} = 1$ and $Law(\eta) = \nu$, we have $P\{\eta_t = \bar{\eta}_t, t \in [0;T]\} = 1$ (that is, the processes $\eta, \bar{\eta}$ are indistinguishable).

We assume that the birth rate b satisfies the following conditions: sublinear growth on the second variable in the sense that

$$\int_{\mathbb{R}^d} b(x,\eta)dx \leq c_1|\eta| + c_2, \tag{2.6}$$

and let d satisfy

$$\forall m \in \mathbb{N} : \sup_{x\in\mathbb{R}^d, |\eta|\leq m} d(x,\eta) < \infty. \tag{2.7}$$

We also assume that

$$E|\eta_0| < \infty. \tag{2.8}$$

By a non-random initial condition we understand an initial condition with a distribution, concentrated at one point: for some $\eta' \in \Gamma_0(\mathbb{R}^d)$,

$$P\{\eta_0 = \eta'\} = 1.$$

From now on, we work on a filtered probability space $(\Omega, \mathscr{F}, (\{\mathscr{F}_t\}_{t\geq 0}), P)$. On this probability space, the Poisson point processes N_1, N_2 and η_0 are defined, so that the whole set-up satisfies (i)-(v) of Definition 2.1.

Let us now consider the equation

$$\overline{\eta}_t(B) = \int_{B\times(0;t]\times[0;\infty]} I_{[0;\bar{b}(x,\overline{\eta}_s)]} dN(x,s,u) + \eta_0(B), \tag{2.9}$$

where $\bar{b}(x,\eta) := \sup_{\xi\subset\eta} b(x,\xi)$. Note that $\bar{b}$ satisfies sublinear growth condition (2.6), if b satisfies it.

This equation is of the type (2.2) (with $\bar{b}$ being the birth rate coefficient, and the zero function being the death rate coefficient), and all definitions of existence and uniqueness of solution are applicable here. Later a unique solution of (2.9) will be used as a majorant of a solution to (2.2).

Proposition 2.5. *Under assumptions* (2.6) *and* (2.8)*, strong existence and pathwise uniqueness hold for equation* (2.9)*. The unique solution* $(\overline{\eta}_t)_{t\geq 0}$ *satisfies*

$$E|\overline{\eta}_t| < \infty, \quad t \geq 0. \tag{2.10}$$

Proof. For $\omega \in \{\int\limits_{\mathbb{R}^d} \overline{b}(x, \eta_0)dx = 0\}$, set $\zeta_t \equiv \eta_0$, $\sigma_n = \infty$, $n \in \mathbb{N}$.

For $\omega \in F := \{\int\limits_{\mathbb{R}^d} \overline{b}(x, \eta_0)dx > 0\}$, we define the sequence of random pairs $\{(\sigma_n, \zeta_{\sigma_n})\}$, where $\sigma_0 = 0$,

$$\sigma_{n+1} = \inf\left\{t > 0 : \int\limits_{\mathbb{R}^d \times (\sigma_n; \sigma_n + t] \times [0;\infty)} I_{[0;\overline{b}(x,\zeta_{\sigma_n})]}(u) dN_1(x, s, u) > 0\right\} + \sigma_n,$$

and

$$\zeta_0 = \eta_0, \quad \zeta_{\sigma_{n+1}} = \zeta_{\sigma_n} \cup \{z_{n+1}\}$$

for $z_{n+1} = \{x \in \mathbb{R}^d : N_1(x, \sigma_{n+1}, [0; \overline{b}(x, \zeta_{\sigma_n})]) > 0\}$. From (1.16) it follows that the points z_n are uniquely determined almost surely on F. Moreover, $\sigma_{n+1} > \sigma_n$ a.s., and σ_n are finite a.s. on F (particularly because $\overline{b}(x, \zeta_{\sigma_n}) \geq \overline{b}(x, \eta_0)$). For $\omega \in F$, we define $\zeta_t = \zeta_{\sigma_n}$ for $t \in [\sigma_n; \sigma_{n+1})$. Then by induction on n it follows that σ_n is a stopping time for each $n \in \mathbb{N}$, and ζ_{σ_n} is $\mathscr{F}_{\sigma_n} \cap F$-measurable. By direct substitution we see that $(\zeta_t)_{t\geq 0}$ is a strong solution for (2.9) on the time interval $t \in [0; \lim\limits_{n\to\infty} \sigma_n)$. Although we have not defined what is a solution, or a strong solution, on a random time interval, we do not discuss it here. Instead we are going to show that

$$\lim_{n\to\infty} \sigma_n = \infty \quad \text{a.s.} \tag{2.11}$$

This relation is evidently true on the complement of F. If $P(F) = 0$, then (2.11) is proven.

If $P(F) > 0$, define a probability measure on F, $Q(A) = \dfrac{P(A)}{P(F)}$, $A \in \mathscr{S} := \mathscr{F} \cap F$, and define $\mathscr{S}_t = \mathscr{F}_t \cap F$.

The process N_1 is independent of F, therefore it is a Poisson point process on $(F, \mathscr{S}, Q)$ with the same intensity, compatible with $\{\mathscr{S}_t\}_{t\geq 0}$. From now on and until other is specified, we work on the filtered probability space $(F, \mathscr{S}, \{\mathscr{S}_t\}_{t\geq 0}, Q)$. We use the same symbols for random processes and random variables, having in mind that we consider their restrictions to F.

The process $(\zeta_t)_{t\in[0;\lim\limits_{n\to\infty}\sigma_n)}$ has the Markov property, because the process N_1 has the strong Markov property and independent increments. Indeed, conditioning on $\mathscr{S}_{\sigma_n}$,

$$E\big[I_{\{\zeta_{\sigma_{n+1}} = \zeta_{\sigma_n} \cup x \text{ for some } x\in B\}} \mid \mathscr{S}_{\sigma_n}\big] = \frac{\int\limits_B \overline{b}(x, \zeta_{\sigma_n})dx}{\int\limits_{\mathbb{R}^d} \overline{b}(x, \zeta_{\sigma_n})dx},$$

thus the chain $\{\zeta_{\sigma_n}\}_{n\in Z_+}$ is a Markov chain, and, given $\{\zeta_{\sigma_n}\}_{n\in Z_+}$, $\sigma_{n+1} - \sigma_n$ are distributed exponentially:

$$E\{I_{\{\sigma_{n+1}-\sigma_n > a\}} \mid \{\zeta_{\sigma_n}\}_{n\in Z_+}\} = \exp\left\{-a \int\limits_{\mathbb{R}^d} \overline{b}(x, \zeta_{\sigma_n})dx\right\}.$$

Therefore, the random variables $\gamma_n = (\sigma_n - \sigma_{n-1})(\int\limits_{\mathbb{R}^d} \bar{b}(x, \zeta_{\sigma_n})dx)$ constitute a sequence of independent random variables exponentially distributed with parameter 1, independent of $\{\zeta_{\sigma_n}\}_{n \in Z_+}$. Theorem 12.18 in [21] (see above) implies that $(\zeta_t)_{t \in [0; \lim\limits_{n\to\infty} \sigma_n)}$ is a pure jump type Markov process.

The jump rate of $(\zeta_t)_{t \in [0; \lim\limits_{n\to\infty} \sigma_n)}$ is given by

$$c(\alpha) = \int\limits_{\mathbb{R}^d} \bar{b}(x, \alpha)dx.$$

Condition (2.6) implies that $c(\alpha) \leq c_1|\alpha| + c_2$. Consequently,

$$c(\zeta_{\sigma_n}) \leq c_1|\zeta_{\sigma_n}| + c_2 = c_1|\zeta_0| + c_1 n + c_2.$$

We see that $\sum_n \dfrac{1}{c(\zeta_{\sigma_n})} = \infty$ a.s., hence Proposition 12.19 in [21] (given in Section 1.5) implies that $\sigma_n \to \infty$.

Now, we return again to our initial probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$.

Thus, we have existence of a strong solution. Uniqueness follows by induction on jumps of the process. Indeed, let $(\tilde{\zeta}_t)_{t\geq 0}$ be another solution of (2.9). From (viii) of Definition 2.1 and equality

$$\int\limits_{\mathbb{R}^d \times (0;\sigma_1) \times [0;\infty]} I_{[0;\bar{b}(x,\eta_0)]} dN_1(x, s, u) = 0,$$

one can see that $P\{\tilde{\zeta}$ has a birth before $\sigma_1\} = 0$. At the same time, equality

$$\int\limits_{\mathbb{R}^d \times \{\sigma_1\} \times [0;\infty]} I_{[0;\bar{b}(x,\eta_0)]} dN_1(x, s, u) = 1,$$

which holds a.s., yields that $\tilde{\zeta}$ has a birth at the moment σ_1, and in the same point of space at that. Therefore, $\tilde{\zeta}$ coincides with ζ up to σ_1 a.s. Similar reasoning shows that they coincide up to σ_n a.s., and, because $\sigma_n \to \infty$ a.s.,

$$P\{\tilde{\zeta}_t = \zeta_t \text{ for all } t \geq 0\} = 1.$$

Thus, pathwise uniqueness holds. The constructed solution is strong.

Now we turn our attention to (2.10). We can write

$$|\zeta_t| = |\eta_0| + \sum_{n=1}^{\infty} I\{|\zeta_t| - |\eta_0| \geq n\} = |\eta_0| + \sum_{n=1}^{\infty} I\{\sigma_n \leq t\}. \tag{2.12}$$

Since

$$\sigma_n = \sum_{i=1}^{n} \frac{\gamma_i}{\int\limits_{\mathbb{R}^d} \bar{b}(x, \zeta_{\sigma_i})dx},$$

we have

$$\{\sigma_n \le t\} = \{\sum_{i=1}^{n} \frac{\gamma_i}{\int\limits_{\mathbb{R}^d} \bar{b}(x, \zeta_{\sigma_i})dx} \le t\} \subset \{\sum_{i=1}^{n} \frac{\gamma_i}{c_1|\zeta_{\sigma_i}| + c_2} \le t\}$$

$$\subset \{\sum_{i=1}^{n} \frac{\gamma_i}{(c_1 + c_2)(|\eta_0| + i)} \le t\} = \{Z_t - Z_0 \ge n\},$$

where (Z_t) is the Yule process (see Page 22) with birth rate defined as follows: $Z_t - Z_0 = n$ when

$$\sum_{i=1}^{n} \frac{\gamma_i}{(c_1 + c_2)(|\eta_0| + i)} \le t < \sum_{i=1}^{n+1} \frac{\gamma_i}{(c_1 + c_2)(|\eta_0| + i)},$$

and $Z_0 = |\eta_0|$. Thus, we have $|\zeta_t| \le Z_t$ a.s., hence $E|\zeta_t| \le EZ_t < \infty$. □

Theorem 2.6. *Under assumptions* (2.6)-(2.8)*, pathwise uniqueness and strong existence hold for equation* (2.2)*. The unique solution* (η_t) *is a pure jump type process satisfying*

$$E|\eta_t| < \infty, \quad t \ge 0. \tag{2.13}$$

Proof. Let us define stopping times with respect to $\{\mathscr{F}_t, t \ge 0\}$, $0 = \theta_0 \le \theta_1 \le \theta_2 \le \theta_3 \le \ldots$, and the sequence of (random) configurations $\{\eta_{\theta_j}\}_{j\in\mathbb{N}}$ as follows: as long as

$$B(\eta_{\theta_n}) + D(\eta_{\theta_n}) > 0,$$

we set

$$\theta_{n+1} = \theta^b_{n+1} \wedge \theta^d_{n+1} + \theta_n,$$

$$\theta^b_{n+1} = \inf\{t > 0 : \int\limits_{\mathbb{R}^d \times (\theta_n;\theta_n+t]\times[0;\infty)} I_{[0;b(x,\eta_{\theta_n})]}(u)dN_1(x,s,u) > 0\},$$

$$\theta^d_{n+1} = \inf\{t > 0 : \int\limits_{(\theta_n;\theta_n+t]\times[0;\infty)} I_{\{x_i\in\eta_{\theta_n}\}} I_{[0;d(x_i,\eta_{\theta_n})]}(v)dN_2(i,r,v) > 0\},$$

$\eta_{\theta_{n+1}} = \eta_{\theta_n} \cup \{z_{n+1}\}$ if $\theta^b_{n+1} \le \theta^d_{n+1}$, where $\{z_{n+1}\} = \{z \in \mathbb{R}^d : N_1(z, \theta_n + \theta^b_{n+1}, \mathbb{R}_+) > 0\}$; $\eta_{\theta_{n+1}} = \eta_{\theta_n} \setminus \{z_{n+1}\}$ if $\theta^b_{n+1} > \theta^d_{n+1}$, where $\{z_{n+1}\} = \{x_i \in \eta_{\theta_n} : N_2(i, \theta_n + \theta^d_{n+1}, \mathbb{R}_+) > 0\}$; the configuration $\eta_{\theta_0} = \eta_0$ is the initial condition of (2.2), $\eta_t = \eta_{\theta_n}$ for $t \in [\theta_n; \theta_{n+1})$, $\{x_i\}$ is the sequence related to $(\eta_t)_{t\ge 0}$. Note that

$$P\{\theta^b_{n+1} = \theta^d_{n+1} \text{ for some } n \mid B(\eta_{\theta_n}) + D(\eta_{\theta_n}) > 0\} = 0,$$

the points z_n are a.s. uniquely determined, and

$$P\{z_{n+1} \in \eta_{\theta_n} \mid \theta^b_{n+1} \le \theta^d_{n+1}\} = 0.$$

If for some n

$$B(\eta_{\theta_n}) + D(\eta_{\theta_n}) = 0,$$

then we set $\theta_{n+k} = \infty$, $k \in \mathbb{N}$, and $\eta_t = \eta_{\theta_n}$, $t \ge \theta_n$.

As in the proof of Proposition 2.5, (η_t) is a strong solution of (2.2), $t \in [0; \lim_n \theta_n)$.

Random variables $\theta_n, n \in \mathbb{N}$, are stopping times with respect to the filtration $\{\mathscr{F}_t, t \geq 0\}$. Using the strong Markov property of a Poisson point process, we see that, on $\{\theta_n < \infty\}$, the conditional distribution of θ^b_{n+1} given $\mathscr{F}_{\theta_n}$ is $\exp(\int\limits_{\mathbb{R}^d} b(x, \eta_{\theta_n})dx)$, and the conditional distribution of θ^d_{n+1} given $\mathscr{F}_{\theta_n}$ is $\exp(\sum\limits_{x \in \eta_{\theta_n}} d(x, \eta_{\theta_n}))$. In particular, $\theta^b_n, \theta^d_n > 0$, $n \in \mathbb{N}$, and the process (η_t) is of pure jump type.

Similarly to the proof of Proposition 2.5, one can show by induction on n that equation (2.2) has a unique solution on $[0; \theta_n]$. Namely, each two solutions coincide on $[0; \theta_n]$ a.s. Thus, any solution coincides with (η_t) a.s. for all $t \in [0; \theta_n]$.

Now we will show that $\theta_n \to \infty$ a.s. as $n \to \infty$. Denote by θ'_k the moment of the k-th birth. It is sufficient to show that $\theta'_k \to \infty$, $k \to \infty$, because only finitely many deaths may occur between any two births, since there are only finitely particles. By induction on k' one may see that $\{\theta'_k\}_{k' \in \mathbb{N}} \subset \{\sigma_i\}_{i \in \mathbb{N}}$, where σ_i are the moments of births of $(\overline{\eta}_t)_{t \geq 0}$, the solution of (2.9), and $\eta_t \subset \overline{\eta}_t$ for all $t \in [0; \lim_n \theta_n)$. For instance, let us show that $(\overline{\eta}_t)_{t \geq 0}$ has a birth at θ'_1. We have $\overline{\eta}_{\theta'_1 -} \supset \overline{\eta}_0 = \eta_0$, and $\eta_{\theta'_1 -} \subset \eta_t \mid_{t=0} = \eta_0$, hence for all $x \in \mathbb{R}^d$

$$\overline{b}(x, \overline{\eta}_{\theta'_1 -}) \geq \overline{b}(x, \eta_{\theta'_1 -}) \geq b(x, \eta_{\theta'_1 -})$$

The latter implies that at time moment θ'_1 a birth occurs for the process $(\overline{\eta}_t)_{t \geq 0}$ in the same point. Hence, $\eta_{\theta'_1} \subset \overline{\eta}_{\theta'_1}$, and we can go on. Since $\sigma_k \to \infty$ as $k \to \infty$, we also have $\theta'_k \to \infty$, and therefore $\theta_n \to \infty$, $n \to \infty$.

Since $\eta_t \subset \overline{\eta}_t$ a.s., Proposition 2.5 implies (2.13). □

In particular, for any time t the integral

$$\int\limits_{\mathbb{R}^d \times (0;t] \times [0;\infty]} I_{[0; b(x, \eta_{s-})]}(u) dN_1(x, s, u)$$

is finite a.s.

Remark 2.7. Let η_0 be a non-random initial condition, $\eta_0 \equiv \alpha$, $\alpha \in \Gamma_0(\mathbb{R}^d)$. The solution of (2.2) with $\eta_0 \equiv \alpha$ will be denoted as $(\eta(\alpha, t))_{t \geq 0}$. Let P_α be the push-forward of P under the mapping

$$\Omega \ni \omega \mapsto (\eta(\alpha, \cdot)) \in D_{\Gamma_0(\mathbb{R}^d)}[0; T]. \tag{2.14}$$

From the proof one may derive that, for fixed $\omega \in \Omega$, constructed unique solution is jointly measurable in (t, α). Thus, the family $\{P_\alpha\}$ of probability measures on $D_{\Gamma_0(\mathbb{R}^d)}[0; T]$ is measurable in α. We will often use formulations related to the probability space $(D_{\Gamma_0(\mathbb{R}^d)}[0; T], \mathscr{B}(D_{\Gamma_0(\mathbb{R}^d)}[0; T]), P_\alpha)$; in this case, coordinate mappings will be denoted by η_t,

$$\eta_t(x) = x(t), \quad x \in D_{\Gamma_0(\mathbb{R}^d)}[0; T].$$

The processes $(\eta_t)_{t \in [0;T]}$ and $(\eta(\alpha, \cdot))_{t \in [0;T]}$ have the same law (under P_α and P, respectively). As one would expect, the family of measures $\{P_\alpha, \alpha \in$

$\Gamma_0(\mathbb{R}^d)\}$ is a Markov process, or a Markov family of probability measures; see Theorem 2.15 below. For a measure μ on $\Gamma_0(\mathbb{R}^d)$, we define

$$P_\mu = \int P_\alpha \mu(d\alpha).$$

We denote by E_μ the expectation under P_μ.

Remark 2.8. Let b_1, d_1 be another pair of birth and death coefficients, satisfying all conditions imposed on b and d. Consider a unique solution $(\tilde{\eta}_t)$ of (2.2) with coefficients b_1, d_1 instead of b, d, but with the same initial condition η_0 and all the other underlying structures. *If for all* $\zeta \in D$, *where* $D \in \mathscr{B}(\Gamma_0(\mathbb{R}^d))$, $b_1(\cdot,\zeta) \equiv b(\cdot,\zeta)$, $d_1(\cdot,\zeta) \equiv d(\cdot,\zeta)$, *then* $\tilde{\eta}_t = \eta_t$ *for all* $t \le \inf\{s \ge 0 : \eta_s \notin D\} = \inf\{s \ge 0 : \tilde{\eta}_s \notin D\}$. This may be proven in the same way as the theorem above.

Remark 2.9. Assume that all the conditions of Theorem 2.6 are fulfilled except Condition (2.8). Then we could not claim that (2.13) holds. However, other conclusions of the theorem would hold. We are mostly interested in the case of a non-random initial condition, therefore we do not discuss the case when (2.13) is not satisfied.

Remark 2.10. We solved equation (2.2) ω-wisely. As a consequence, there is a functional dependence of the solution process and the "input": the process $(\eta_t)_{t\ge 0}$ is some function of η_0, N_1 and N_2. Note that θ_n and z_n from the proof of Theorem 2.6 are measurable functions of η_0, N_1 and N_2 in the sense that, e.g., $\theta_1 = F_1(\eta_0, N_1, N_2)$ a.s. for a measurable $F_1 : \Gamma_0(\mathbb{R}^d) \times \Gamma(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+) \times \Gamma(\mathbb{Z}^d \times \mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}_+$.

Proposition 2.11. *If* $(\eta_t)_{t\ge 0}$ *is a solution to equation* (2.2), *then the inequality*

$$E|\eta_t| < (c_2 t + E|\eta_0|)e^{c_1 t}$$

holds for all $t > 0$.

Proof. We already know that $E|\eta_t|$ is finite. Since η_t satisfies equation (2.2) we have

$$\begin{aligned}\eta_t(B) = & \int\limits_{B\times(0;t]\times[0;\infty]} I_{[0;b(x,\eta_{s-})]}(u)dN_1(x,s,u) \\ & - \int\limits_{\mathbb{Z}\times(0;t]\times[0;\infty)} I_{\{x_i\in\eta_{r-}\cap B\}} I_{[0;d(x_i,\eta_{r-})]}(v)dN_2(i,r,v) \\ & \le \int\limits_{B\times(0;t]\times[0;\infty]} I_{[0;b(x,\eta_{s-})]}(u)dN_1(x,s,u) + \eta_0(B).\end{aligned}$$

For $B = \mathbb{R}^d$, taking expectation in the last inequality, we obtain

$$E|\eta_t| = E\eta_t(\mathbb{R}^d) \le E \int\limits_{\mathbb{R}^d\times(0;t]\times[0;\infty]} I_{[0;b(x,\eta_{s-})]}(u)dN_1(x,s,u) + E\eta_0(\mathbb{R}^d)$$

$$= E \int\limits_{\mathbb{R}^d \times (0;t] \times [0;\infty]} I_{[0;b(x,\eta_{s-})]}(u) dx ds du + E\eta_0(\mathbb{R}^d)$$

$$= E \int\limits_{\mathbb{R}^d \times (0;t]} b(x, \eta_{s-}) dx ds + E\eta_0(\mathbb{R}^d).$$

Since η is a solution of (2.2), we have for all $s \in [0;t]$ almost surely $\eta_{s-} = \eta_s$. Consequently, $E|\eta_{s-}| = E|\eta_s|$. Applying this and (2.6), we see that

$$E\eta_t(\mathbb{R}^d) \leq E \int\limits_{(0;t]} (c_1|\eta_{s-}| + c_2) ds + E\eta_0(\mathbb{R}^d) = c_1 \int\limits_{(0;t]} E|\eta_s| ds + c_2 t + E\eta_0(\mathbb{R}^d),$$

so the statement of the lemma follows from (2.8) and Gronwall's inequality. □

Definition 2.12. We say that *joint uniqueness in law* holds for equation (2.2) with an initial distribution ν if any two (weak) solutions $((\eta_t), N_1, N_2)$ and $((\eta_t)', N_1', N_2')$ of (2.2), $Law(\eta_0) = Law((\eta_0)') = \nu$, have the same joint distribution:

$$Law((\eta_t), N_1, N_2) = Law((\eta_t)', N_1', N_2').$$

The following corollary is a consequence of Theorem 2.6 and Remark 2.10

.

Corollary 2.13. *Joint uniqueness in law holds for equation* (2.2) *with initial distribution* ν *satisfying*

$$\int_{\Gamma_0(\mathbb{R}^d)} |\gamma| \nu(d\gamma) < \infty.$$

Remark 2.14. We note here that altering the order of the initial configuration does not change the law of the solution. We could replace the lexicographical order with any other. To see this, note that if ς is a permutation of $\mathbb{Z}$ (that is, $\varsigma : \mathbb{Z} \to \mathbb{Z}$ is a bijection), then the process $\tilde{N}_2$ defined by

$$\tilde{N}_2(K, R, V) = N_2(\varsigma K, R, V), \quad K \subset \mathbb{Z}, R, V \in \mathscr{B}(\mathbb{R}_+), \tag{2.15}$$

has the same law as N_2, and is adapted to $\{\mathscr{F}_t\}_{t \geq 0}$, too. Therefore, solutions of (2.2) and of (2.2) with N_2 being replaced by $\tilde{N}_2$ have the same law. But replacing N_2 with $\tilde{N}_2$ in equation (2.2) is equivalent to replacing $\{x_{-|\eta_0|+1}, \ldots, x_0, x_1, \ldots\}$ with

$$\{x_{\varsigma^{-1}(-|\eta_0|+1)}, \ldots, x_{\varsigma^{-1}(0)}, x_{\varsigma^{-1}(1)}, \ldots\}.$$

Let ν be a distribution on $\Gamma_0(\mathbb{R}^d)$, and let $T > 0$. Denote by $\mathscr{L}(\nu, b, d, T)$ the law of the restriction $(\eta_t)_{t \in [0;T]}$ of the unique solution $(\eta_t)_{t \geq 0}$ to (2.2) with an initial condition distributed according to ν. Note that $\mathscr{L}(\nu, b, d, T)$ is a distribution on $D_{\Gamma_0(\mathbb{R}^d)}([0;T])$. As usually, the Markov property of a solution follows from uniqueness.

Theorem 2.15. *The unique solution* $(\eta_t)_{t \in [0;T]}$ *of* (2.2) *is a Markov process.*

Proof. Take arbitrary $t' < t$, $t', t \in [0;T]$. Consider the equation

$$\xi_t(B) = \int\limits_{B\times(t';t]\times[0;\infty]} I_{[0;b(x,\xi_{s-})]}(u) dN_1(x,s,u) - \int\limits_{\mathbb{Z}\times(t';t]\times[0;\infty)} I_{\{x'_i \in \xi_{r-}\cap B\}} I_{[0;d(x'_i,\xi_{r-})]} dN_2(i,r,v) + \eta_{t'}(B), \quad (2.16)$$

where the sequence $\{x'_i\}$ is related to the process $(\xi_s)_{s\in[0;t]}$, $\xi_s = \eta_s$. The unique solution of (2.16) is $(\eta_s)_{s\in[t';t]}$. As in the proof of Theorem 2.6 we can see that $(\eta_s)_{s\in[t';t]}$ is measurable with respect to the filtration generated by the random variables $N_1(B,[s;q],U)$, $N_2(i,[s;q],U)$, and $\eta_{t'}(B)$, where $B \in \mathscr{B}(\mathbb{R}^d)$, $i \in \mathbb{Z}$, $t' \le s \le q \le t$, $U \in \mathscr{B}(\mathbb{R}_+)$. Poisson point process have independent increments, hence

$$P\{(\eta_t)_{t\in[s;T]} \in U \mid \mathscr{F}_s\} = P\{(\eta_t)_{t\in[s;T]} \in U \mid \eta_s\}$$

almost surely. Furthermore, using arguments similar to those in Remark 2.14, we can conclude that $(\eta_s)_{s\in[t';t]}$ is distributed according to $\mathscr{L}(\nu_{t'}, b, d, t-t')$, where $\nu_{t'}$ is the distribution of $\eta_{t'}$. □

The following theorem sums up the results we have obtained so far.

Theorem 2.16. *Under assumptions* (2.6), (2.7), (2.8), *equation* (2.2) *has a unique solution. This solution is a pure jump type Markov process. The family of push-forward measures* $\{P_\alpha, \alpha \in \Gamma_0(\mathbb{R}^d)\}$ *defined in Remark 2.7 forms a Markov process, or a Markov family of probability measures, on* $D_{\Gamma_0(\mathbb{R}^d)}[0;\infty)$.

Proof. The statement is a consequence of Theorem 2.6, Remark 2.7 and Theorem 2.15. In particular, the Markov property of $\{P_\alpha, \alpha \in \Gamma_0(\mathbb{R}^d)\}$ follows from the statement given in the last sentence of the proof of Theorem 2.15. □

We call the unique solution of (2.2) (or, sometimes, the corresponding family of measures on $D_{\Gamma_0(\mathbb{R}^d)}[0;\infty)$) a *(spatial) birth-and-death Markov process.*

Remark 2.17. We note that d does not need to be defined on the whole space $\mathbb{R}^d \times \Gamma_0(\mathbb{R}^d)$. The equation makes sense even if $d(x,\eta)$ is defined on $\{(x,\eta) \mid x \in \eta\}$. Of course, any such function may be extended to a function on $\mathbb{R}^d \times \Gamma_0(\mathbb{R}^d)$.

2.1 Continuous dependence on initial conditions

In order to prove the continuity of the distribution of the solution of (2.2) with respect to initial conditions, we make the following continuity assumptions on b and d.

Continuity assumptions 2.18. Let b, d be continuous with respect to both arguments. Furthermore, let the map

$$\Gamma_0(\mathbb{R}^d) \ni \eta \mapsto b(\cdot,\eta) \in L^1(\mathbb{R}^d).$$

be continuous.

In light of Remark 2.17, let us explain what we understand by continuity of d when $d(x,\eta)$ is defined only on $\{(x,\eta) \mid x \in \eta\}$. We require that, whenever $\eta_n \to \eta$ and $\eta_n \ni z_n \to x \in \eta$, we also have $d(z_n,\eta_n) \to d(x,\eta)$. Similar condition appeared in [19, Theorem 3.1].

Theorem 2.19. *Let the birth and death coefficients b and d satisfy the above continuity assumptions 2.18. Then for every $T > 0$ the map*

$$\Gamma_0(\mathbb{R}^d) \ni \alpha \mapsto Law\{\eta(\alpha,\cdot).,\cdot \in (0;T]\},$$

which assigns to a non-random initial condition $\eta_0 = \alpha$ the law of the solution of equation (2.2) *stopped at time T, is continuous.*

Remark 2.20. We mean continuity in the space of measures on $D_{\Gamma_0(\mathbb{R}^d)}[0;T]$; see Page 16.

Proof. Denote by $\eta(\alpha,\cdot)$ the solution of (2.2), started from α. Let $\alpha_n \to \alpha$, $\alpha_n, \alpha \in \Gamma_0(\mathbb{R}^d)$, $\alpha = \{x_0, x_{-1}, \dots, x_{-|\alpha|+1}\}$, $x_0 \preccurlyeq x_{-1} \preccurlyeq \dots \preccurlyeq x_{-|\alpha|+1}$. With no loss in generality we assume that $|\alpha_n| = |\alpha|$, $n \in \mathbb{N}$. By Lemma 1.6 we can label elements of α_n, $\alpha_n = \{x_0^{(n)}, x_{-1}^{(n)}, \dots, x_{-|\alpha|+1}^{(n)}\}$, so that $x_{-i}^{(n)} \to x_{-i}$, $i = 0, \dots, |\alpha| - 1$. Taking into account Remark 2.14, we can assume

$$x_0^{(n)} \preccurlyeq x_{-1}^{(n)} \preccurlyeq \dots \preccurlyeq x_{-|\alpha|+1}^{(n)} \tag{2.17}$$

without loss of generality (in the sense that we do not have to use lexicographical order; not in the sense that we can make $x_0^{(n)}, x_{-1}^{(n)}, \dots$ satisfy (2.17) with the lexicographical order).

We will show that

$$\sup_{t\in[0;T]} dist(\eta(\alpha,t), \eta(\alpha_n,t)) \overset{p}{\to} 0, \quad n \to \infty. \tag{2.18}$$

Let $\{\theta_i\}_{i\in\mathbb{N}}$ be the moments of jumps of process $\eta(\alpha,\cdot)$. Without loss of generality, assume that $d(x,\alpha) > 0$, $x \in \alpha$, and $||b(\cdot,\alpha)||_{L^1} > 0$, $L^1 := L^1(\mathbb{R}^d)$ (if some of these inequalities are not fulfilled, the following reasonings should be changed insignificantly).

Depending on whether a birth or a death occurs at θ_1, we have either

$$N_1(\{x_1\} \times \{\theta_1\} \times [0; b(x_1,\eta_0)]) = 1 \tag{2.19}$$

or for some $x_{-k} \in \alpha$

$$N_2(\{-k\} \times \{\theta_1\} \times [0; d(x_{-k},\alpha)]) = 1.$$

The probability of last two equalities holding simultaneously is zero, hence we can neglect this event. In both cases $N_1(x_1, \{\theta_1\}, \{b(x_1,\alpha)\}) = 0$ and $N_2(-k, \{\theta_1\}, \{d(x_{-k},\alpha)\}) = 0$ a.s. We also have

$$N_1(\mathbb{R}^d \times [0;\theta_1) \times [0; b(x,\alpha)]) = 0,$$

and for all $j \in 0, 1, \dots, |\alpha| - 1$

$$N_2(\{-j\} \times [0;\theta_1) \times [0; d(x_{-j},\alpha)]) = 0.$$

Denote

$$m := b(x_1, \alpha) \wedge \min\{d(x, \alpha) : x \in \alpha\} \wedge ||b(\cdot, \alpha)||_{L^1} \wedge 1$$

and fix $\varepsilon > 0$. Let $\delta_1 > 0$ be so small that for $\nu \in \Gamma_0(\mathbb{R}^d)$, $\nu = \{x'_0, x'_{-1}, \dots, x'_{-|\alpha|+1}\}$, $|x_{-j} - x'_{-j}| \le \delta_1$ the inequalities

$$|d(x'_{-j}, \nu) - d(x_{-j}, \alpha)| < \varepsilon m, \qquad ||b(\cdot, \nu) - b(\cdot, \alpha)||_{L^1} < \varepsilon m$$

hold. Then we may estimate

$$P\Big\{ \int\limits_{\mathbb{R}^d \times [0;\theta_1) \times [0;\infty]} I_{[0;b(x,\nu)]}(u) dN_1(x, s, u) \ge 1 \Big\} < \varepsilon. \tag{2.20}$$

and

$$P\Big\{ \int\limits_{\mathbb{Z} \times [0;\theta_1) \times [0;\infty]} I_{\{x'_{-i} \in \nu\}} I_{[0;d(x'_{-i}, \nu)]}(v) dN_2(i, r, v) \ge 1 \Big\} < \varepsilon |\alpha|. \tag{2.21}$$

Indeed, the random variable

$$\tilde{\theta} := \inf_{t>0} \Big\{ \int\limits_{\mathbb{R}^d \times [0;t) \times [0;\infty]} I_{[0;0 \vee \{b(x,\nu) - b(x,\alpha\})]}(u) dN_1(x, s, u) \ge 1 \Big\} \tag{2.22}$$

is exponentially distributed with parameter $||(b(\cdot, \nu) - b(\cdot, \alpha))_+||_{L^1} < \varepsilon ||b(\cdot, \alpha)||_{L^1}$. By Lemma 1.14,

$$P\{\tilde{\theta} < \theta_1\} < \frac{\varepsilon ||b(\cdot, \alpha)||_{L^1}}{||b(\cdot, \alpha)||_{L^1}} = \varepsilon, \tag{2.23}$$

which is exactly (2.20). Likewise, (2.21) follows.

Similarly, the probability that the same event as for $\eta(\alpha, \cdot)$ occurs at time θ_1 for $\eta(\nu, \cdot)$ is high. Indeed, assume, for example, that a birth occurs at θ_1, that is to say that (2.19) holds. Once more using Lemma 1.14 we get

$$P\{N_1(\{x_1\} \times \{\theta_1\} \times [0; b(x_1, \nu)]) = 0\} \le \frac{||(b(\cdot, \nu) - b(\cdot, \alpha))_+||_{L^1}}{||b(\cdot, \alpha)||_{L^1}} \le \varepsilon.$$

The case of death occurring at θ_1 may be analyzed in the same way.

From inequalities (1.6) and (1.7) we may deduce

$$\sup_{t \in (0;\theta_1]} dist(\eta(\alpha, t), \eta(\alpha_n, t)) \xrightarrow{p} 0, n \to \infty. \tag{2.24}$$

Proceeding in the same manner we may extend this to

$$\sup_{t \in (0;\theta_n]} dist(\eta(\alpha, t), \eta(\alpha_n, t)) \xrightarrow{p} 0, n \to \infty, \tag{2.25}$$

particularly because of the strong Markov property of a Poisson point process. In fact, with high probability the processes $\eta(\alpha_n, \cdot)$ and $\eta(\alpha, \cdot)$ change up to time θ_n in the same way in the following sense: births occur in the same places at the same time moments. Deaths occur at the same time moments, and when a point is deleted from $\eta(\alpha, \cdot)$, then its counterpart is deleted from $\eta(\alpha_n, \cdot)$. Since $\theta_n \to \infty$, we get (2.18). □

Remark 2.21. In fact, we have proved an even stronger statement. Namely, take $\alpha_n \to \alpha$. Then there exist processes $(\xi_t^{(n)})_{t\in[0;T]}$ such that

$$(\xi_t^{(n)})_{t\in[0;T]} \stackrel{d}{=} (\eta(\alpha_n, t))_{t\in[0;T]}$$

and

$$\sup_{t\in[0;T]} dist(\eta(\alpha, t), \xi_t^{(n)}) \xrightarrow{p} 0, \quad n \to \infty.$$

Thus, $Law\{\eta(\alpha, \cdot), \cdot \in (0; T]\}$ and $Law\{\eta(\alpha_n, \cdot), \cdot \in (0; T]\}$ are close in the space of measures over D_{Γ_0}, even when D_{Γ_0} is considered as topological space equipped with the *uniform* topology (induced by metric *dist*), and not with the Skorokhod topology.

2.2 The martingale problem

Now we briefly discuss the martingale problem associated with L defined in (2.1). Let $C_b(\Gamma_0(\mathbb{R}^d))$ be the space of all bounded continuous functions on $\Gamma_0(\mathbb{R}^d)$. We equip $C_b(\Gamma_0(\mathbb{R}^d))$ with the supremum norm.

Definition 2.22. A probability measure Q on $(D_{\Gamma_0}[0;\infty), \mathscr{B}(D_{\Gamma_0}[0;\infty)))$ is called a solution to the local martingale problem associated with L if

$$M_t^f = f(y(t)) - f(y(0)) - \int_0^t Lf(y(s-))ds, \quad \mathscr{I}_t, \quad 0 \le t < \infty,$$

is a local martingale for every $f \in C_b(\Gamma_0)$. Here y is the coordinate mapping, $y(t)(\omega) = \omega(t)$, $\omega \in D_{\Gamma_0}[0;\infty)$, $\mathscr{I}_t$ is the completion of $\sigma(y(s), 0 \le s \le t)$ under Q.

Thus, we require M^f to be a local martingale under Q with respect to $\{\mathscr{I}_t\}_{t\ge 0}$. Note that L can be considered as a bounded operator on $C_b(\Gamma_0(\mathbb{R}^d))$.

Proposition 2.23. *Let* $(\eta(\alpha, t))_{t\ge 0}$ *be a solution to* (2.2). *Then for every* $f \in C(\Gamma_0)$ *the process*

$$M_t^f = f(\eta(\alpha, t)) - f(\eta(\alpha, t)) - \int_0^t Lf(\eta(\alpha, s-))ds \tag{2.26}$$

is a local martingale under P *with respect to* $\{\mathscr{F}_t\}_{t\ge 0}$.

Proof. In this proof ζ_t will stand for $\eta(\alpha, t)$. Denote $\tau_n = \inf\{t \ge 0 : |\zeta_t| > n \text{ or } \zeta_t \nsubseteq [-n; n]^d\}$. Clearly, τ_n, $n \in \mathbb{N}$, is a stopping time and $\tau_n \to \infty$ a.s. Let $\zeta_t^n = \zeta_{t\wedge\tau_n}$. We want to show that $({}^{(n)}M_t^f)_{t\ge 0}$ is a martingale, where

$${}^{(n)}M_t^f = f(\zeta_t^n) - f(\zeta_t^n) - \int_0^t Lf(\zeta_{s-}^n)ds. \tag{2.27}$$

The process $(\zeta_t)_{t\ge 0}$ satisfies

$$\zeta_t = \sum_{s\le t, \zeta_s \ne \zeta_{s-}} [\zeta_s - \zeta_{s-}] + \zeta_0. \tag{2.28}$$

In the above equality as well as in few other places throughout this proof we treat elements of $\Gamma_0(\mathbb{R}^d)$ as measures rather than as configurations. Since (ζ_t) is of the pure jump type, the sum on the right-hand side of (2.28) is a.s. finite. Consequently we have

$$f(\zeta_t^n) - f(\zeta_0^n) = \sum_{s \le t, \zeta_s \ne \zeta_{s-}} [f(\zeta_s^n) - f(\zeta_{s-}^n)] \tag{2.29}$$

$$= \int\limits_{B\times(0;t]\times[0;\infty]} [f(\zeta_s) - f(\zeta_{s-})] I_{\{s\le\tau_n\}} I_{[0;b(x,\zeta_{s-})]}(u) dN_1(x,s,u)$$

$$- \int\limits_{\mathbb{Z}\times(0;t]\times[0;\infty]} I_{\{x_i\in\zeta_{s-}\}} [f(\zeta_s) - f(\zeta_{s-})] I_{\{s\le\tau_n\}} I_{[0;d(x_i,\zeta_{s-})]}(v) dN_2(i,s,v).$$

Note that $\zeta_s = \zeta_{s-} \cup x$ a.s. in the first summand on the right-hand side of (2.29), and $\zeta_s = \zeta_{s-} \setminus x_i$ a.s. in the second summand. Now, we may write

$$\int_0^t I_{\{s\le\tau_n\}} Lf(\zeta_s) ds \tag{2.30}$$

$$= \int_0^t \int\limits_{x\in\mathbb{R}^d, u\ge 0} I_{\{s\le\tau_n\}} I_{[0;b(x,\zeta_{s-})]}(u) [f(\zeta_{s-}\cup x) - f(\zeta_{s-})] dx du ds$$

$$- \int_0^t \int\limits_{x\in\mathbb{R}^d, u\ge 0} I_{\{s\le\tau_n\}} I_{[0;d(x,\zeta_{s-}))]}(v) [f(\zeta_{s-}\setminus x) - f(\zeta_{s-})] \zeta_{s-}(dx) dv ds.$$

Functions $b, d(\cdot,\cdot)$ and f are bounded on $\mathbb{R}^d \times \{\alpha : |\alpha| \le n \text{ and } \alpha \subset [-n;n]^d\}$ and $\{\alpha : |\alpha| \le n \text{ and } \alpha \subset [-n;n]^d\}$ respectively by a constant $C > 0$. Now, for a predictable bounded processes $(\gamma_s(x,u))_{0\le s\le t}$ and $(\beta_s(x,v))_{0\le s\le t}$, the processes

$$\int\limits_{B\times(0;t]\times[0;C]} I_{\{s\le\tau_n\}} \gamma_s(x,u) [dN_1(x,s,u) - dx ds du],$$

$$\int\limits_{\mathbb{Z}\times(0;t]\times[0;C]} I_{\{s\le\tau_n\}} I_{\{x_i\in\zeta_{s-}\}} \beta_s(x_i,v) [dN_2(i,s,v) - \#(di) ds dv].$$

are martingales. Observe that

$$\int\limits_{\mathbb{Z}\times(0;t]\times[0;C]} I_{\{s\le\tau_n\}} I_{\{x_i\in\zeta_{s-}\}} \beta_s(x_i,v) \#(di) ds dv$$

$$= \int\limits_{\mathbb{Z}\times(0;t]\times[0;C]} I_{\{s\le\tau_n\}} \beta_s(x,v) \zeta_{s-}(dx) ds dv$$

Taking

$$\gamma_s(x,u) = I_{[0;b(x,\zeta_{s-})]}(u) [f(\zeta_{s-}\cup x) - f(\zeta_{s-})],$$

$$\beta_s(x,v)) = I_{[0;d(x,\zeta_{s-})]}(v) [f(\zeta_{s-}\setminus x) - f(\zeta_{s-})],$$

we see that the difference on the right hand side of (2.27) is a martingale because of (2.29) and (2.30). □

Corollary 2.24. *The unique solution of* (2.2) *induces a solution of the martingale problem 2.22.*

Remark 2.25. Since $y(s) = y(s-)$ P_α-a.s., the process

$$f(y(t)) - f(y(0)) - \int_0^t Lf(y(s))ds, \ 0 \le t < \infty,$$

is a local martingale, too.

2.3 Birth rate without sublinear growth condition

In this section we will consider equation (2.2) with the a birth rate coefficient that does not satisfy the sublinear growth condition (2.6).

Instead, we assume only that

$$\sup_{x\in\mathbb{R}^d, |\eta|\le m} b(x,\eta) < \infty. \tag{2.31}$$

Under this assumption we can not guarantee existence of solution on the whole line $[0;\infty)$ or even on a finite interval $[0;T]$. It is possible that infinitely many points appear in finite time.

We would like to show that a unique solution exists up to an explosion time, maybe finite. Consider birth and death coefficients

$$b_n(x,\eta) = b(x,\eta)I_{\{|\eta|\le n\}}, \qquad d_n(x,\eta) = d(x,\eta)I_{\{|\eta|\le n\}}. \tag{2.32}$$

Functions b_n, d_n are bounded, so equation (2.2) with birth rate coefficient b_n and death rate coefficient d_n has a unique solution by Theorem 2.6. Remark 2.8 provides the existence and uniqueness of solution to (2.2) (with birth and death rate coefficients b and d, respectively) up to the (random stopping) time $\tau_n = \inf\{s \ge 0 : |\eta_s| > n\}$. Clearly, $\tau_{n+1} \ge \tau_n$; if $\tau_n \to \infty$ a.s., then we have existence and uniqueness for (2.2); if $\tau_n \uparrow \tau < \infty$ with positive probability, then we have an *explosion*. However, existence and uniqueness hold up to explosion time τ. When we have an explosion we say that the solution blows up.

2.4 Coupling

Here we discuss the coupling of two birth-and-death processes. The theorem we prove here will be used in the sequel. As a matter of fact, we have already used the coupling technique in the proof of Theorem 2.6.

Consider two equations of the form (2.2),

$$\begin{aligned} \xi_t^{(k)}(B) = & \int_{B\times(0;t]\times[0;\infty)} I_{[0;b_k(x,\xi_{s-}^{(k)})]}(u)dN_1(x,s,u) \\ & - \int_{\mathbb{Z}\times(0;t]\times[0;\infty)} I_{\{x_i^{(k)}\in\xi_{r-}^{(k)}\cap B\}} I_{[0;d(x_i^{(k)},\eta_{r-})]}(v)dN_2(i,r,v) \\ & + \xi_0^{(k)}(B), \quad k=1,2, \end{aligned} \tag{2.33}$$

where $t \in [0;T]$ and $\{x_i^{(k)}\}$ is the sequence related to $(\xi_t^{(k)})_{t\in[0;T]}$.

Assume that initial conditions $\xi_0^{(k)}$ and coefficients b_k, d_k satisfy the conditions of Theorem 2.6. Let $(\xi_t^{(k)})_{t\in[0;T]}$ be the unique strong solutions.

Theorem 2.26. *Assume that almost surely $\xi_0^{(1)} \subset \xi_0^{(2)}$, and for any two finite configurations $\eta^1 \subset \eta^2$,*

$$b_1(x,\eta^1) \le b_2(x,\eta^2), \quad x \in \mathbb{R}^d \tag{2.34}$$

and

$$d_1(x,\eta^1) \ge d_2(x,\eta^2), \quad x \in \eta^1.$$

Then there exists a cadlag $\Gamma_0(\mathbb{R}^d)$-valued process $(\eta_t)_{t\in[0;T]}$ such that $(\eta_t)_{t\in[0;T]}$ and $(\xi_t^{(1)})_{t\in[0;T]}$ have the same law and

$$\eta_t \subset \xi_t^{(2)}, \quad t \in [0;T]. \tag{2.35}$$

Proof. Let $\{\dots, x_{-1}^{(2)}, x_0^{(2)}, x_1^{(2)}, \dots\}$ be the sequence related to $(\xi_t^{(2)})_{t\in[0;T]}$. Consider the equation

$$\begin{aligned} \eta_t(B) = \int\limits_{B\times(0;t]\times[0;\infty]} & I_{[0;b_k(x,\eta_{s-})]}(u)dN_1(x,s,u) \\ - \int\limits_{\mathbb{Z}\times(0;t]\times[0;\infty)} & I_{\{x_i^{(2)}\in\eta_{r-}\cap B\}} I_{[0;d(x_i^{(2)},\eta_{r-})]}(v)dN_2(i,r,v) \\ & + \xi_0^{(1)}(B), \quad k = 1,2. \end{aligned} \tag{2.36}$$

Note that here $\{x_i^{(2)}\}$ is related to $(\xi_t^{(2)})_{t\in[0;T]}$ and not to $(\eta_t)_{t\in[0;T]}$. Thus (2.36) is not an equation of form (2.2). Nonetheless, the existence of a unique solution can be shown in the same way as in the proof of Theorem 2.6. Denote the unique strong solution of (2.36) by $(\eta_t)_{t\in[0;T]}$.

Denote by $\{\tau_m\}_{m\in\mathbb{N}}$ the moments of jumps of $(\eta_t)_{t\in[0;T]}$ and $(\xi_t^{(2)})_{t\in[0;T]}$, $0 < \tau_1 < \tau_2 < \tau_3 < \dots$. More precisely, a time $t \in \{\tau_m\}_{m\in\mathbb{N}}$ iff at least one of the processes $(\eta_t)_{t\in[0;T]}$ and $(\xi_t^{(2)})_{t\in[0;T]}$ jumps at time t.

We will show by induction that each moment of birth for $(\eta_t)_{t\in[0;T]}$ is a moment of birth for $(\xi_t^{(2)})_{t\in[0;T]}$ too, and each moment of death for $(\xi_t^{(2)})_{t\in[0;T]}$ is a moment of death for $(\eta_t)_{t\in[0;T]}$ if the dying point is in $(\eta_t)_{t\in[0;T]}$. Moreover, in both cases the birth or the death occurs at exactly the same point. Here a moment of birth is a random time at which a new point appears, a moment of death is a random time at which a point disappears from the configuration. The statement formulated above is in fact equivalent to (2.35).

Here we deal only with the base case, the induction step is done in the same way. We have nothing to show if τ_1 is a moment of a birth of $(\xi_t^{(2)})_{t\in[0;T]}$ or a moment of death of $(\eta_t)_{t\in[0;T]}$. Assume that a new point is born for $(\eta_t)_{t\in[0;T]}$ at τ_1,

$$\eta_{\tau_1} \setminus \eta_{\tau_1-} = \{x_1\}.$$

The process $(\eta_t)_{t\in[0;T]}$ satisfies (2.36), therefore $N_1(\{x\},\{\tau_1\},[0;b_k(x_1,\eta_{\tau_1-})]) = 1$. Since

$$\eta_{\tau_1-} = \xi_0^{(1)} \subset \xi_0^{(2)} = \xi_{\tau_1-}^{(2)},$$

by (2.34)

$$N_1(\{x\},\{\tau_1\},[0;b_k(x_1,\xi_{\tau_1-}^{(2)})]) = 1,$$

hence

$$\xi_{\tau_1}^{(2)} \setminus \xi_{\tau_1-}^{(2)} = \{x_1\}.$$

The case when τ_2 is a moment of death for $(\xi_t^{(2)})_{t\in[0;T]}$ is analyzed analogously.

It remains to show that $(\eta_t)_{t\in[0;T]}$ and $(\xi_t^{(1)})_{t\in[0;T]}$ have the same law. We mentioned above that formally equation (2.36) is not of the form (2.2), so we can not directly apply the uniqueness in law result. However, since $\eta_t \in \xi_t^{(2)}$ a.s., $t \in [0;T]$, we can still consider (2.36) as an equation of the form (2.2). Indeed, let $\{\dots,y_{-1},y_0,y_1,\dots\}$ be the sequence related to η_t. We have $\{y_{-|\xi_0^{(1)}|+1},\dots,y_{-1},y_0,y_1,\dots\} \subset \{x_{-|\xi_0^{(2)}|+1},\dots,x_{-1}^{(2)},x_0^{(2)},x_1^{(2)},\dots\}$. There exists an injection $\varsigma : \{-|\xi_0^{(1)}|+1,\dots,0,1,\dots\} \to \{-|\xi_0^{(2)}|+1,\dots,0,1,\dots\}$ such that $y_{\varsigma(i)} = x_i$. Denote $\theta_i = \inf\{s \geq 0 : y_i \in \eta_s\}$. Note that θ_i is a stopping time with respect to $\{\mathscr{F}_t\}$. Define a Poisson point process $\bar{N}_2$ by

$$\bar{N}_2(\{i\}\times R\times V) = N_2(\{i\}\times R\times V), \quad i\in\mathbb{Z}, R\subset[0;\theta_i], V\subset\mathbb{R}_+,$$

and

$$\bar{N}_2(\{i\}\times R\times V) = N_2(\{\varsigma(i)\}\times R\times V), \quad i\in\mathbb{Z}, R\subset(\theta_i;\infty), V\subset\mathbb{R}_+.$$

The process $\bar{N}_2$ is $\{\mathscr{F}_t\}$-adapted. One can see that $(\eta_t)_{t\in[0;T]}$ is the unique solution of equation (2.2) with N_2 replaced by $\bar{N}_2$. Hence

$$(\eta_t)_{t\in[0;T]} \overset{d}{=} (\xi_t^{(1)})_{t\in[0;T]}. \qquad \square$$

2.5 Related semigroup of operators

We say now a few words about the semigroup of operators related to the unique solution of (2.2). We write $\eta(\alpha,t)$ for a unique solution of (2.2), started from $\alpha\in\Gamma_0(\mathbb{R}^d)$. We want to define an operator S_t by

$$S_t f(\alpha) = Ef(\eta(\alpha,t)) \quad (= E_\alpha f(\eta(t))) \tag{2.37}$$

for an appropriate class of functions. Unfortunately, it seems difficult to make S_t a C_0-semigroup on some functional Banach space for general b, d satisfying (2.6) and (2.7).

We start with the case when the cumulative birth and death rates are bounded. Let $C_b = C_b(\Gamma_0(\mathbb{R}^d))$ be the space of all bounded continuous functions on $\Gamma_0(\mathbb{R}^d)$. It becomes a Banach space once it is equipped with the supremum norm. We assume the existence of a constant $C>0$ such that for all $\zeta\in\Gamma_0(\mathbb{R}^d)$

$$|B(\zeta)| + |D(\zeta)| < C, \tag{2.38}$$

where B and D are defined in (2.4) and (2.5). Formula (2.1) defines then a bounded operator $L : C_b \to C_b$, and we will show that S_t coincides with e^{tL}. For $f \in C_b$, the function $S_t f$ is bounded and continuous. Boundedness is a consequence of the boundedness of f, and continuity of $S_t f$ follows from Remark 2.21. Indeed, let $\alpha_n \to \alpha$, $\xi_t^{(n)} \overset{d}{=} \eta(\alpha_n, t)$ and

$$dist(\eta(\alpha, t), \xi_t^{(n)}) \overset{p}{\to} 0, \quad n \to \infty.$$

Unlike $\Gamma(\mathbb{R}^d)$, the space $\Gamma_0(\mathbb{R}^d)$ is a σ-compact space. Consequently, for all $\varepsilon > 0$ there exists a compact $K_\varepsilon \subset \Gamma_0(\mathbb{R}^d)$ such that for large enough n

$$P\{\eta(\alpha, t) \in K_\varepsilon,\ \xi_t^{(n)} \in K_\varepsilon\} \geq 1 - \varepsilon.$$

Also, for fixed $\delta > 0$ and for large enough n

$$P\{dist(\eta(\alpha, t), \xi_t^{(n)}) \leq \delta\} \geq 1 - \delta.$$

Fix $\varepsilon > 0$. There exists $\delta_\varepsilon \in (0; \varepsilon)$ such that $|f(\beta) - f(\gamma)| \leq \varepsilon$ whenever $dist(\beta, \gamma) \leq \delta_\varepsilon$, $\beta, \gamma \in K_\varepsilon$. We have for large enough n

$$|E[f(\eta(\alpha, t)) - f(\xi_t^{(n)})]|$$
$$\leq E|f(\eta(\alpha, t)) - f(\xi_t^{(n)})| I\{\eta(\alpha, t) \in K_\varepsilon,\ \xi_t^{(n)} \in K_\varepsilon, dist(\eta(\alpha, t), \xi_t^{(n)}) \leq \delta_\varepsilon\}$$
$$+2(\delta_\varepsilon + \varepsilon)||f|| \leq \varepsilon + 2(\delta_\varepsilon + \varepsilon)||f||,$$

where $||f|| = \sup_{\zeta \in \Gamma_0(\mathbb{R}^d)} |f(\zeta)|$. Letting $\varepsilon \to 0$, we see that

$$Ef(\eta(\alpha_n, t)) = Ef(\xi_t^{(n)}) \to Ef(\eta(\alpha, t)).$$

Thus, $S_t f$ is continuous (note that the continuity of $S_t f$ does not follow from Theorem 2.19 alone, since for a fixed $t \in [0; T]$ the functional $D_{\Gamma_0(\mathbb{R}^d)}[0; T] \ni x \mapsto x(t) \in \mathbb{R}$ is not continuous in the Skorokhod topology). Furthermore, since for small t and for all $A \in \mathscr{B}(\mathbb{R}^d)$,

$$P\{\eta(\alpha, t) = \alpha\} = 1 - t[B(\alpha) + D(\alpha)] + o(t), \tag{2.39}$$

$$P\{\eta(\alpha, t) = \alpha \cup \{y\} \text{ for some } y \in A\} = t \int\limits_{y \in A} b(y, \alpha) dy + o(t), \tag{2.40}$$

and for $x \in \alpha$

$$P\{\eta(\alpha, t) = \alpha \setminus \{x\}\} = td(x, \alpha) + o(t), \tag{2.41}$$

we may estimate

$$|S_t f(\alpha) - f(\alpha)| \leq t\,[B(\alpha) + D(\alpha)]\,||f|| + o(t)||f|| \leq C||f||t + o(t).$$

Therefore, (2.37) defines a C_0 semigroup on C_b. Its generator

$$\tilde{L} f(\alpha) = \lim_{t \to 0+} \frac{S_t f(\alpha)}{t}$$

$$= \lim_{t \to 0+} \left[\int\limits_{x \in \mathbb{R}^d} b(x, \alpha)[f(\alpha \cup x) - f(\alpha)] dx + \sum_{x \in \alpha} d(x, \alpha)(f(\alpha \setminus x) - f(\alpha)) + o(t) \right]$$

$$= Lf(\alpha).$$

Thus, $S_t = e^{tL}$, and we have proved the following

Proposition 2.27. *Assume that* (2.38) *is fulfilled. Then the family of operators* $(S_t, t \geq 0)$ *on* C_b *defined in* (2.37) *constitutes a* C_0*-semigroup. Its generator coincides with* L *given in* (2.1).

Now we turn out attention to general b, d satisfying (2.6) and (2.7) but not necessarily (2.38). The family of operators $(S_t)_{t\geq 0}$ still constitutes a semigroup, however it does not have to be strongly continuous anymore. Consider truncated birth and death coefficients (2.32) and corresponding process $\eta^n(\alpha, t)$. Remark 2.8 implies that $\eta^n(\alpha, t) = \eta(\alpha, t)$ for all $t \in [0; \tau_n]$, where

$$\tau_n = \inf\{s \geq 0 : |\eta(\alpha, s)| > n\}. \tag{2.42}$$

Growth condition (2.6) implies that $\tau_n \to \infty$ for any $\alpha \in \Gamma_0(\mathbb{R}^d)$.

Truncated coefficients b_n, d_n satisfy (2.38) and

$$S_t^{(n)} f(\alpha) = Ef(\eta^{(n)}(\alpha, t)) \tag{2.43}$$

defines a C_0 - semigroup on C_b. In particular, for all $\alpha \in \Gamma_0(\mathbb{R}^d)$

$$L^{(n)} f(\alpha) = \lim_{t\to 0+} \frac{Ef(\eta^{(n)}(\alpha, t)) - f(\alpha)}{t},$$

where $L^{(n)}$ is operator defined as in (2.1) but with b_n, d_n instead of b, d. Letting $n \to \infty$ we get, for fixed α and f,

$$Lf(\alpha) = \lim_{t\to 0+} \frac{Ef(\eta(\alpha, t)) - f(\alpha)}{t} = \lim_{t\to 0+} \frac{S_t f(\alpha) - f(\alpha)}{t}. \tag{2.44}$$

Taking limit by n is possible: for $n \geq |\alpha| + 2$, $\eta^{(n)}(\alpha, t)$ satisfies (2.39), (2.40) and (2.41), therefore $\eta(\alpha, t)$ satisfies (2.39), (2.40) and (2.41), too. Thus, we have

Proposition 2.28. *Let* b *and* d *satisfy* (2.6) *and* (2.7) *but not necessarily* (2.38). *Then the family of operators* $(S_t, t \geq 0)$ *constitutes a semigroup on* C_b *which does not have to be strongly continuous. However, for every* $\alpha \in \Gamma_0(\mathbb{R}^d)$ *and* $f \in C_b$ *we have* (2.44).

Formula (2.44) gives us the formal relation of $(\eta(\alpha, t))_{t\geq 0}$ to the operator L. Of course, for fixed f the convergence in (2.44) does not have to be uniform in α.

Remark 2.29. The question about the construction of a semigroup acting on some class of probability measures on $\Gamma_0(\mathbb{R}^d)$ is yet to be studied.

Acknowledgements

This paper is part of the author's PhD thesis written under the supervision of Professor Yuri Kondratiev, whom the author would like to thank for suggesting the problem and stimulating discussions. The financial support of the German science foundation through the IGK is gratefully appreciated.

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